

**LECTURES ON MAASS FORMS**  
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1. INTRODUCTION

1.1. **The aim of these lecture series.** The aim of these lecture series is to explain the basic concepts and ideas of

- Maass forms;
- their automorphic  $L$ -functions;
- the Kuznetsov trace formula;
- subconvexity of Rankin-Selberg  $L$ -functions with applications.

For simplicity, only the theory for

$$\Gamma = SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\} \quad (1.1)$$

will be treated; the general theory for congruence subgroups  $\Gamma_0(N)$  is similar in philosophy but more involved in details. For detailed treatment of Maass forms in book form, the reader is referred to e.g. Borel [2], Bump [3], Iwaniec [4], and Ye [20].

Even for the simplest case (1.1), there are already big amount of materials in the literature. To explain these materials in such a short time, I have to omit most of the proofs, and content myself just with illustrations and explanations.

The contents are as follows:

§2. Maass forms for  $SL_2(\mathbb{Z})$

§3. Fourier expansion for Maass forms

§4. Spectral decomposition of non-Euclidean Laplacian  $\Delta$

§5. Hecke theory for Maass forms

§6. The Kuznetsov trace formula

§7. Automorphic  $L$ -functions

§8. Number theoretical background

§9. Subconvexity bounds for Rankin-Selberg  $L$ -functions

§10. An application of subconvexity bounds

§11. A proof of subconvexity

**1.2. Notation.** Throughout the lecture series,  $\Gamma$  is always  $SL_2(\mathbb{Z})$ , as in (1.1). We use  $s = \sigma + it$  and  $z = x + iy$  to denote complex variables. The Vinogradov symbol  $A \ll B$  means that  $A = O(B)$ , and as usual  $e(x) = e^{2\pi ix}$ .

## 2. MAASS FORMS FOR $SL_2(\mathbb{Z})$

The standard Laplace operator on the complex plane  $\mathbb{C}$  is defined by

$$\Delta^e = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

but on the upper half-plane  $\mathbb{H}$ , the non-Euclidean Laplace operator is given by

$$\Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

The operator  $\Delta$  is invariant under the action of  $SL_2(\mathbb{R})$ ; that is, for any smooth function  $f$  on  $\mathbb{H}$ , any element  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ ,

$$(\Delta f) \circ g = \Delta(f \circ g). \tag{2.1}$$

The last equation means that, for any smooth function  $f$  on  $\mathbb{H}$ , any element  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ , and any  $z \in \mathbb{H}$ ,

$$(\Delta f)(gz) = \Delta(f(gz)), \tag{2.2}$$

where we recall that

$$gz = \frac{az + b}{cz + d}.$$

It is well known that  $SL_2(\mathbb{R})$  is generated by  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  and  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ . Thus in order to show (2.1), it is sufficient to check it for the generators.

**Definition 2.1.** A smooth function  $f \neq 0$  on  $\mathbb{H}$  is called a Maass form for the group  $\Gamma$  if it satisfies the following properties:

- For all  $g \in \Gamma$  and all  $z \in \mathbb{H}$ ,

$$f(gz) = f(z);$$

- $f$  is an eigenfunction of the non-Euclidean Laplacian,

$$\Delta f = \lambda f,$$

where  $\lambda$  is an eigenvalue of  $\Delta$ ;

- There exists a positive integer  $N$  such that

$$f(z) \ll y^N, \quad y \rightarrow +\infty.$$

**Definition 2.2.** A Maass form  $f$  is said to be a cusp form if

$$\int_0^1 f\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} z\right) db = \int_0^1 f(z+b) db = 0$$

holds for all  $z \in \mathbb{H}$ .

As an example of Maass forms, we consider non-analytic Eisenstein series

$$E^*(z, s) = \frac{\Gamma(s)}{2\pi^s} \sum_{\substack{m, n \in \mathbb{Z} \\ (m, n) \neq (0, 0)}} \frac{y^s}{|mz + n|^{2s}}. \quad (2.3)$$

**Proposition 2.3.** Let  $s = \sigma + it$ . Then

- The Eisenstein series  $E^*(z, s)$  is absolutely convergent for  $\sigma > 1$ ;
- Although  $E^*(s, z)$  is not analytic as a function of  $z \in \mathbb{H}$ , it is strictly automorphic, i.e. for  $\sigma > 1$ ,

$$E^*(gz, s) = E^*(z, s), \quad g \in \Gamma. \quad (2.4)$$

- For  $\sigma > 1$ , the Eisenstein series  $E^*(s, z)$  is an eigenfunction of  $\Delta$ ,

$$\Delta E^*(z, s) = s(1-s)E^*(z, s), \quad (2.5)$$

*Proof.* We have

$$\frac{y^s}{|mz + n|^{2s}} \ll \frac{|y|^\sigma}{\min(|my|, |n|)^{2\sigma}}.$$

Therefore, for  $\sigma > 1$ ,

$$\begin{aligned} \sum_{\substack{m, n \in \mathbb{Z} \\ (m, n) \neq (0, 0)}} \frac{y^s}{|mz + n|^{2s}} &\ll |y|^\sigma + |y|^\sigma \sum_{m \geq 1} \sum_{n \leq m|y|} \frac{1}{n^{2\sigma}} + |y|^{-\sigma} \sum_{n \geq 1} \sum_{m \leq n/|y|} \frac{1}{m^{2\sigma}} \\ &\ll |y|^\sigma + \frac{|y|^{1-\sigma}}{2\sigma-2} + \frac{|y|^{\sigma-1}}{2\sigma-2}. \end{aligned}$$

And hence the series in (2.3) is absolutely convergent in  $\sigma > 1$ . It is not analytic as a function of  $z \in \mathbb{H}$ .

To see  $E^*(s, z)$  is strictly automorphic, we introduce  $E(z, s)$  by the following formula

$$E^*(z, s) = E(z, s)\xi(s), \quad (2.6)$$

where  $\xi(s) = \pi^{-s}\Gamma(s)\zeta(2s)$  and  $\zeta(s)$  is the Riemann zeta-function. Since

$$\mathfrak{S}gz = \frac{y}{|cz + d|^2}, \quad \text{for } g = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma,$$

we have

$$E(z, s) = y^s + \frac{1}{2} \sum_{\substack{(c,d)=1 \\ c \neq 0}} \frac{y^s}{|cz + d|^{2s}} = \frac{1}{2} \sum_{g \in \Gamma_\infty \setminus \Gamma} (\mathfrak{S}gz)^s, \quad (2.7)$$

where  $\Gamma_\infty$  is the subgroup of  $\Gamma$  generated by  $\begin{pmatrix} 1 & \\ 0 & 1 \end{pmatrix}$ . From (2.7), it is easy to see that  $E(z, s)$  (and also  $E^*(z, s)$ ) is invariant under  $\Gamma$ , hence (2.4).

That  $E(z, s)$  is an eigenfunction of  $\Delta$  in  $\sigma > 1$  can be checked term by term in the expansion (2.7). In fact, we have

$$\Delta y^s = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (y^s) = s(1-s)y^s,$$

so that the first term  $y^s$  in (2.7) is an eigenfunction of  $\Delta$  with eigenvalue  $\lambda = s(1-s)$ . The general term  $\frac{y^s}{|cz+d|^{2s}}$  in (2.7) is also an eigenfunction of  $\Delta$  with eigenvalue  $\lambda = s(1-s)$ , since  $\Delta$  is invariant under  $\Gamma$ , and

$$(\mathfrak{S}gz)^s = \frac{y^s}{|cz + d|^{2s}} \quad \text{for } g = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma.$$

Therefore,

$$\Delta E(z, s) = s(1-s)E(z, s), \quad (2.8)$$

and hence (2.7). □

We remark that Proposition 2.3 does not prove that  $E^*(z, s)$  is a Maass form, since Proposition 2.3 holds only for  $\sigma > 1$ , and the growth condition in Definition 2.1 is to be checked. These will be done via the analytic continuation of  $E^*(z, s)$ .

### 3. FOURIER EXPANSION FOR MAASS FORMS

**3.1. Fourier expansion of Maass forms.** We will firstly give the Fourier expansion of a Maass form  $f$ .

**Theorem 3.1.** *Let  $f(z)$  be a Maass form for  $\Gamma$ .*

- *Then it has the Fourier expansion*

$$f(z) = \sum_{n \in \mathbb{Z}} a_n \sqrt{y} K_{i\nu}(2\pi|n|y) e(nx). \quad (3.1)$$

where  $K_{i\nu}$  is the modified Bessel function of the third kind.

- *For  $\lambda \geq 1/4$ , a square-integrable Maass form  $f(z)$  must be a cusp form, and its Fourier expansion takes the form*

$$f(z) = \sum_{n \neq 0} a_n \sqrt{y} K_{i\nu}(2\pi|n|y) e(nx).$$

*Proof.* Since

$$f(gz) = f(z), \quad \text{for } g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma,$$

we have

$$f(z+1) = f(x+1+iy) = f(z) = f(x+iy),$$

and therefore  $f(z)$  must have the Fourier expansion

$$f(z) = \sum_{n \in \mathbb{Z}} c_n(y) e^{2\pi i n x} = \sum_{n \in \mathbb{Z}} c_n(y) e(nx). \quad (3.2)$$

It remains to determine the coefficients  $c_n(y)$ .

By definition, we have  $\Delta f = \lambda f$ , and hence  $c_n(y)$  satisfies

$$-y^2 c_n'' + 4\pi^2 n^2 y^2 c_n = \lambda c_n. \quad (3.3)$$

If  $n \neq 0$ , this is a modified Bessel equation, which has the general solution

$$c_n(y) = a_n \sqrt{y} K_{i\nu}(2\pi|n|y) + b_n \sqrt{y} I_{i\nu}(2\pi|n|y). \quad (3.4)$$

Here  $\nu = \sqrt{\lambda - 1/4}$ ,  $K_{i\nu}$  is the modified Bessel function of the third kind which is rapidly decreasing when  $y \rightarrow \infty$ , and  $I_{i\nu}(y)$  is the modified Bessel function of the first kind which is rapidly increasing. More precisely, we have

$$I_{i\nu} = \frac{e^y}{\sqrt{2\pi y}} \left( 1 + O\left(\frac{1 + |\nu|^2}{y}\right) \right),$$

and

$$K_{i\nu} = e^{-y} \sqrt{\frac{\pi}{2y}} \left( 1 + O\left(\frac{1 + |\nu|^2}{y}\right) \right).$$

From (iii) in Definition 2.1, we see that  $b_n$  must equal zero, and hence (3.1) follows from (3.2) and (3.4).

If  $n = 0$ , the equation (3.3) has the solution  $c_0(y) = y^{1/2+i\nu}$ , where  $\nu = \pm\sqrt{\lambda - 1/4}$ . If  $\lambda \geq 1/4$ , then  $\nu$  is real, and consequently,

$$\int_{\Gamma \setminus \mathbb{H}} |c_0(y)|^2 dz \geq \int_{-1/2}^{1/2} dx \int_1^\infty \frac{|y^{1+2i\nu}|}{y^2} dy \geq \int_1^\infty \frac{dy}{y} = \infty.$$

For a square-integrable Maass form  $f$ , we then must have  $c_0(y) = 0$ . Therefore for  $\lambda \geq 1/4$ , a square-integrable Maass form must be a cusp form, and its Fourier expansion takes the form

$$f(z) = \sum_{n \neq 0} a_n \sqrt{y} K_{i\nu}(2\pi|n|y) e(nx).$$

This proves the theorem. □

Let  $\iota : \mathbb{H} \rightarrow \mathbb{H}$  be the antiholomorphic involution

$$\iota(x + iy) = -x + iy.$$

If  $f$  is an eigenfunction of  $\Delta$ , then  $f \circ \iota$  is an eigenfunction with the same eigenvalue. Since  $\iota^2 = 1$ , its eigenvalue are  $\pm 1$ . We may therefore diagonalize the Maass cusp forms with respect to  $\iota$ . If  $f \circ \iota = f$ , we call  $f$  even. In this case  $a_n = a_{-n}$ . If  $f \circ \iota = -f$ , then we call  $f$  odd. Then  $a_n = -a_{-n}$ .

**3.2. Analytic continuation of Eisenstein series.** We will establish the analytic continuation of  $E^*(z, s)$  via its Fourier expansion.

**Theorem 3.2.** *Let  $s \in \mathbb{C}$ .*

- *The Eisenstein series  $E^*(z, s)$ , originally defined for  $\sigma > 1$ , has meromorphic continuation to all  $s$ ; it is analytic except at  $s = 1$  and  $s = 0$ , where it has simple poles. The residue at  $s = 1$  is the constant function  $z = 1/2$ .*
- *The Eisenstein series satisfies the functional equation*

$$E^*(z, s) = E^*(z, 1 - s). \tag{3.5}$$

- *We have*

$$E^*(x + iy, s) \ll y^{\max(\sigma, 1-\sigma)}, \quad y \rightarrow \infty. \tag{3.6}$$

It follows that the Eisenstein series  $E^*(z, s)$  is a Maass form with Laplace eigenvalue  $s(1-s)$ .

*Proof.* We need some basic properties of Bessel function. The  $K$ -Bessel function is defined by

$$K_s(y) = \frac{1}{2} \int_0^\infty \exp\left(-\frac{y}{2}\left(t + \frac{1}{t}\right)\right) t^s \frac{dt}{t}. \quad (3.7)$$

If  $y > 0$ , the integrand above decays rapidly as  $t \rightarrow 0$  or  $\infty$ , so the integral is convergent for all values of  $s$ . We have the estimate

$$|K_s(y)| \leq e^{-y/2} K_\sigma(2), \quad y > 4. \quad (3.8)$$

To see this, note that if  $a, b > 2$  then  $ab > a + b$ , so

$$e^{-ab} < e^{-a} e^{-b}.$$

Applying this with  $a = y/2$  and  $b = t + t^{-1}$ , and integrating with respect to  $t$ , one gets (3.8). Also, the integrand in (3.7) is invariant under  $t \mapsto t^{-1}, s \mapsto -s$ , we see that

$$K_s(y) = K_{-s}(y). \quad (3.9)$$

We will show that if  $\sigma > 1/2$  and  $r$  is real, then

$$\begin{aligned} & \left(\frac{y}{\pi}\right)^s \Gamma(s) \int_{-\infty}^\infty \frac{e(rx)}{(x^2 + y^2)^s} dx \\ &= \begin{cases} \pi^{-s+1/2} \Gamma(s-1/2) y^{1-s}, & \text{if } r = 0, \\ 2|r|^{s-1/2} \sqrt{y} K_{s-1/2}(2\pi|r|y), & \text{if } r \neq 0. \end{cases} \end{aligned} \quad (3.10)$$

Substituting the definition of gamma function as an integral, we write the left-hand side of (3.10) as

$$\begin{aligned} & \int_{-\infty}^\infty \int_0^\infty e^{-t} \left(\frac{ty}{\pi(x^2 + y^2)}\right)^s e(rx) \frac{dt}{t} dx \\ &= \int_0^\infty \int_{-\infty}^\infty \exp\left(-\pi t \frac{x^2 + y^2}{y}\right) t^s e(rx) dx \frac{dt}{t}, \end{aligned}$$

where we have changed the order of integration, and made a change of variables  $t \mapsto \pi t(x^2 + y^2)/y$ . Now we have

$$\int_{-\infty}^\infty e^{-t\pi x^2/y} e(rx) dx = \begin{cases} \sqrt{y/t}, & \text{if } r = 0, \\ \sqrt{y/t} \exp(-y\pi r^2/t), & \text{if } r \neq 0. \end{cases}$$

We substitute this into the previous integral. If  $r = 0$ , then we get (3.10); if  $r \neq 0$ , the substitution  $t \mapsto |r|t$  also gives (3.10).

Now we compute the Fourier expansion of  $E^*(z, s)$ , and hence prove the theorem. By (2.4), we have

$$E^*(z, s) = E^*(z + 1, s).$$

Hence  $E^*(z, s)$  has a Fourier expansion

$$E^*(z, s) = \sum_{r=-\infty}^{\infty} a_r(y, s)e(rx). \quad (3.11)$$

we will compute the Fourier coefficients

$$a_r(y, s) = \int_0^1 E^*(x + iy, s)e(rx)dx.$$

Firstly, there is the contribution from the term with  $m = 0$ . Such a term is independent of  $x$ , and hence contributes only to  $a_0$ . Since  $n$  and  $-n$  contribute equally, this contribution equals

$$\pi^{-s}\Gamma(s)y^s \sum_{n=1}^{\infty} n^{-2s} = \pi^{-s}\Gamma(s)\zeta(2s)y^s. \quad (3.12)$$

This is part, but not all, of  $a_0$ , as we shall see shortly.

Next, we have the contributions with the terms with  $m \neq 0$ . Since  $(-m, -n)$  and  $(m, n)$  contribute equally, we may simply sum the terms with  $m > 0$ . The contribution to  $a_r$  equals

$$\begin{aligned} & \pi^{-s}\Gamma(s)y^s \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \int_0^1 \frac{e(rx)}{\{(mx+n)^2 + m^2y^2\}^s} dx \\ &= \pi^{-s}\Gamma(s)y^s \sum_{m=1}^{\infty} \sum_{n \bmod m} \int_{-\infty}^{\infty} \frac{e(rx)}{\{(mx+n)^2 + m^2y^2\}^s} dx. \end{aligned}$$

By making the substitution  $x \mapsto x - n/m$ , this is

$$\begin{aligned} & \pi^{-s}\Gamma(s)y^s \sum_{m=1}^{\infty} m^{-2s} \sum_{n \bmod m} e\left(\frac{rn}{m}\right) \int_{-\infty}^{\infty} \frac{e(rx)}{(x^2 + y^2)^s} dx \\ &= \pi^{-s}\Gamma(s)y^s \sum_{m|r} m^{1-2s} \int_{-\infty}^{\infty} \frac{e(rx)}{(x^2 + y^2)^s} dx. \end{aligned} \quad (3.13)$$



There are now two cases. Firstly, if  $r = 0$ , then the condition  $m|r$  is vacuous, so using (3.10), we see that (3.13) equals

$$\pi^{-s+1/2}\Gamma(s-1/2)\zeta(1-2s)y^{1-s}.$$

Combining this with (3.12) and invoking the functional equation of the zeta-function, we have computed the constant term

$$a_0(y, s) = \pi^{-s}\Gamma(s)\zeta(2s)y^s + \pi^{s-1}\Gamma(1-s)\zeta(2-2s)y^{1-s}. \quad (3.14)$$

If  $r \neq 0$ , then by (3.10) and (3.13),

$$a_r(y, s) = 2|r|^{s-1/2}\sigma_{1-2s}(|r|)\sqrt{y}K_{s-1/2}(2\pi|r|y), \quad (3.15)$$

where

$$\sigma_s = \sum_{m|r} m^s.$$

If  $\sigma > 1$ , then we have proved the Fourier expansion (3.11) with coefficients given by (3.14) and (3.15). Therefore the theorem now follows from an examination of this expansion. Each individual term of the series has analytic continuation to all  $s$ , except that  $a_0$  has simple poles at  $s = 0$  and  $s = 1$ . Each of the two terms in (3.14) has a pole at  $s = 1/2$ , but these cancel. The convergence of the infinite series follows from the rapid decay of the Bessel function. Thus we obtain the continuation. To get the functional equation (3.5), we observe that

$$a_r(y, s) = a_r(y, 1-s).$$

This is clearly if  $r = 0$ ; if  $r \neq 0$ , this follows from (3.9) and the identity

$$r^s\sigma_{-2s}(r) = \prod_{d_1 d_2 = r} d_1^s d_2^{-s} = r^{-s}\sigma_{2s}(r).$$

Hence we have (3.5). Regarding the statement about the residues, the only term in (3.11) with a pole at  $s = 1$  is  $a_0$ , and it is easy to check that the residue is the constant function  $z = 1/2$ . As for (3.6), it follows from (3.8) that the non-constant term (3.15) decay rapidly as  $y \rightarrow 0$ , so asymptotically the behavior of  $E^*(z, s)$  is the same as its constant term (3.14).

We conclude that  $E^*(z, s)$  is a Maass form.  $\square$

**Corollary 3.3.** *Let  $s \in \mathbb{C}$ . The Eisenstein series  $E^*(z, s)$  has the Fourier expansion*

$$E^*(z, s) = \sum_{r=-\infty}^{\infty} a_r(y, s)e(rx)$$

with

$$a_0(y, s) = \pi^{-s}\Gamma(s)\zeta(2s)y^s + \pi^{s-1}\Gamma(1-s)\zeta(2-2s)y^{1-s},$$

and, if  $r \neq 0$ ,

$$a_r(y, s) = 2|r|^{s-1/2}\sigma_{1-2s}(|r|)\sqrt{y}K_{s-1/2}(2\pi|r|y).$$

#### 4. SPECTRAL DECOMPOSITION OF NON-EUCLIDEAN LAPLACIAN $\Delta$

A function  $f : \mathbb{H} \rightarrow \mathbb{C}$  is said to be automorphic with respect to  $\Gamma$  if

$$f(gz) = f(z), \quad \text{for all } g \in \Gamma.$$

Therefore,  $f$  lives on  $\Gamma \backslash \mathbb{H}$ . We denote the space of such functions by  $\mathcal{A}(\Gamma \backslash \mathbb{H})$ . Our objective is to extend an automorphic function into automorphic forms subject to suitable growth condition. The main results holds in the Hilbert space

$$\mathcal{L}(\Gamma \backslash \mathbb{H}) = \{f \in \mathcal{A}(\Gamma \backslash \mathbb{H}) : \|f\| < \infty\}$$

with respect to the inner product

$$\langle f, g \rangle = \int_{\Gamma \backslash \mathbb{H}} f(z)\bar{g}(z)\frac{dx dy}{y^2}.$$

**Theorem 4.1.** *On  $\mathcal{L}(\Gamma \backslash \mathbb{H})$ , the non-Euclidean Laplace operator  $\Delta$  is positive semi-definite and self-adjoint.*

*Idea of Proof.* Define

$$\mathcal{D}(\Gamma \backslash \mathbb{H}) = \{f \in \mathcal{A}(\Gamma \backslash \mathbb{H}) : f, \Delta f \text{ smooth and bounded}\}.$$

- $\mathcal{D}(\Gamma \backslash \mathbb{H})$  is dense in  $\mathcal{L}(\Gamma \backslash \mathbb{H})$ .
- $\Delta$  is positive semi-definite and symmetric on  $\mathcal{D}(\Gamma \backslash \mathbb{H})$ . In fact, for  $f, g \in \mathcal{D}(\Gamma \backslash \mathbb{H})$ ,

$$\langle \Delta f, g \rangle = \int_{\Gamma \backslash \mathbb{H}} \nabla f \bar{\nabla} g dx dy,$$

and hence

$$\langle \Delta f, g \rangle = \langle f, \Delta g \rangle;$$

moreover,

$$\langle \Delta f, f \rangle = \int_{\Gamma \backslash \mathbb{H}} |\nabla f|^2 dx dy \geq 0,$$

so  $\Delta$  is non-negative.

- By Friedrichs' theorem in functional analysis,  $\Delta$  has a unique self-adjoint extension to  $\mathcal{L}(\Gamma \backslash \mathbb{H})$ .

The theorem is proved.  $\square$

It follows from the above argument that the eigenvalue  $\lambda = s(1 - s)$  of an eigenfunction  $f \in \mathcal{D}(\Gamma \backslash \mathbb{H})$  is real and non-negative. Therefore, either  $s = 1/2 + it$  with  $t \in \mathbb{R}$ , or  $0 < s < 1$ .

Let  $\psi(y)$  is smooth function with compact support on  $(0, +\infty)$ . An incomplete Eisenstein series is defined by

$$E(z|\psi) = \sum_{g \in \Gamma_\infty \backslash \Gamma} \psi(\Im gz).$$

Clearly,  $E(z|\psi)$  is invariant under  $\Gamma$ . Since  $\psi$  has compact support,  $E(z|\psi) \in \mathcal{L}(\Gamma \backslash \mathbb{H})$ . It is not an automorphic form, since it fails to be an eigenfunction of  $\Delta$ . However, by Mellin's inversion, one can represent the incomplete Eisenstein series as a contour integral of the Eisenstein series

$$E(z|\psi) = \frac{1}{2\pi i} \int_{(\sigma)} E(z, s) \hat{\psi}(s) ds \quad (4.1)$$

where  $\sigma > 1$  and

$$\hat{\psi}(s) = \int_0^\infty \psi(y) y^{-s} \frac{dy}{y}.$$

To pursue the analysis, we select two linear subspaces of  $\mathcal{L}(\Gamma \backslash \mathbb{H})$ :

- $\mathcal{B}(\Gamma \backslash \mathbb{H})$ , the space of smooth and bounded automorphic functions,
- $\mathcal{E}(\Gamma \backslash \mathbb{H})$ , the space of incomplete Eisenstein series.

We have the inclusions

$$\mathcal{E}(\Gamma \backslash \mathbb{H}) \subset \mathcal{B}(\Gamma \backslash \mathbb{H}) \subset \mathcal{L}(\Gamma \backslash \mathbb{H}) \subset \mathcal{A}(\Gamma \backslash \mathbb{H}). \quad (4.2)$$

The space  $\mathcal{B}(\Gamma \backslash \mathbb{H})$  is dense in  $\mathcal{L}(\Gamma \backslash \mathbb{H})$ , but  $\mathcal{E}(\Gamma \backslash \mathbb{H})$  need not be.

Let us examine the orthogonal complement of  $\mathcal{E}(\Gamma \backslash \mathbb{H})$  in  $\mathcal{B}(\Gamma \backslash \mathbb{H})$ .

Take  $f \in \mathcal{B}(\Gamma \backslash \mathbb{H})$ , and  $E(\cdot|\psi) \in \mathcal{E}(\Gamma \backslash \mathbb{H})$ . Then  $f$  is invariant under  $\Gamma$ , and therefore,

$$\langle E(\cdot|\psi), f \rangle = \int_{\Gamma \backslash \mathbb{H}} E(z|\psi) \bar{f}(z) d\frac{dx dy}{y^2} = \int_{\Gamma_\infty \backslash \mathbb{H}} E(y|\psi) \bar{f}(z) d\frac{dx dy}{y^2}.$$

Since

$$\Gamma_\infty \backslash \mathbb{H} = \{x + iy : 0 < x \leq 1, y > 0\},$$

we have

$$\langle E(\cdot|\psi), f \rangle = \int_0^\infty \psi(y) \left\{ \int_0^1 \bar{f}(z) dx \right\} \frac{dy}{y^2}.$$

Therefore,  $f \in \mathcal{B}(\Gamma \backslash \mathbb{H})$  is orthogonal to  $\mathcal{E}(\Gamma \backslash \mathbb{H})$  if and only if

$$\int_0^1 f(z) dx = 0 \quad \text{for all } y > 0,$$

i.e. the Fourier expansion of  $f$  has the form

$$f(z) = \sum_{n \neq 0} c_n(y) e(nx) \tag{4.3}$$

with  $c_0(y) = 0$ . Denote by  $\mathcal{C}(\Gamma \backslash \mathbb{H})$  the subspace of these  $f$ .

Applying  $\Delta$  to the Fourier expansion (4.3) of  $f$ , we see that the expansion of  $\Delta f$  will also have zero constant term, and therefore  $\Delta \mathcal{C}(\Gamma \backslash \mathbb{H}) \subset \mathcal{C}(\Gamma \backslash \mathbb{H})$ . Since  $\Delta$  is symmetric,  $\mathcal{C}(\Gamma \backslash \mathbb{H})$  is spanned by the eigenfunctions of  $\Delta$ , i.e. spanned by the Maass cusp forms. Hence,

$$\mathcal{B}(\Gamma \backslash \mathbb{H}) = \mathcal{C}(\Gamma \backslash \mathbb{H}) \oplus \mathcal{E}(\Gamma \backslash \mathbb{H}).$$

Noting that  $\mathcal{B}(\Gamma \backslash \mathbb{H})$  is dense in  $\mathcal{L}(\Gamma \backslash \mathbb{H})$ , we have

**Theorem 4.2.** *We have the orthogonal decomposition*

$$\mathcal{L}(\Gamma \backslash \mathbb{H}) = \tilde{\mathcal{C}}(\Gamma \backslash \mathbb{H}) \oplus \tilde{\mathcal{E}}(\Gamma \backslash \mathbb{H}),$$

where  $\mathcal{C}(\Gamma \backslash \mathbb{H})$  is the subspace spanned by all Maass cusp forms in  $\mathcal{L}(\Gamma \backslash \mathbb{H})$ , and  $\mathcal{E}(\Gamma \backslash \mathbb{H})$  the subspace spanned by all incomplete Eisenstein series in  $\mathcal{L}(\Gamma \backslash \mathbb{H})$ . Here tilde stands for closure in the Hilbert space  $\mathcal{L}(\Gamma \backslash \mathbb{H})$  with respect to the norm topology.

Clearly,

$$\Delta : \mathcal{C}(\Gamma \backslash \mathbb{H}) \rightarrow \mathcal{C}(\Gamma \backslash \mathbb{H}), \quad \Delta : \mathcal{E}(\Gamma \backslash \mathbb{H}) \rightarrow \mathcal{E}(\Gamma \backslash \mathbb{H}).$$

To describe the spectral decomposition on  $\mathcal{C}(\Gamma \backslash \mathbb{H})$  and  $\mathcal{E}(\Gamma \backslash \mathbb{H})$  in the following.

**Theorem 4.3.** *The automorphic Laplacian  $\Delta$  has a purely point spectral on  $\mathcal{C}(\Gamma \backslash \mathbb{H})$ , i.e. the space  $\mathcal{C}(\Gamma \backslash \mathbb{H})$  is spanned by cusp forms. The eigenvalues are*

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \rightarrow \infty$$

and the eigenspaces have finite dimension. For any complete orthonormal system of cusp forms  $\{u_j\}$ , every  $f \in \mathcal{C}(\Gamma \backslash \mathbb{H})$  has the expansion

$$f(z) = \sum_{j=0}^{\infty} \langle f, u_j \rangle u_j(z),$$

converging in the norm topology. If  $f \in \mathcal{C}(\Gamma \backslash \mathbb{H}) \cap \mathcal{D}(\Gamma \backslash \mathbb{H})$ , then the series converges absolutely and uniformly on compacta.

On the other hand, in the  $\mathcal{E}(\Gamma \backslash \mathbb{H})$  the spectrum turns out to be continuous. Here the analytic continuation of the Eisenstein series is the key issue. After this is established, the spectral resolution of  $\Delta$  in  $\mathcal{E}(\Gamma \backslash \mathbb{H})$  will evoke from (4.1) by contour integration. The eigenpacket of the continuous spectrum consists of the Eisenstein series  $E(z, s)$  on the line  $\sigma = 1/2$  (analytically continued).

**Theorem 4.4.** *The spectrum of  $\Delta$  on  $\mathcal{E}(\Gamma \backslash \mathbb{H})$  is absolutely continuous; it covers the segment  $[1/4, +\infty)$  uniformly with multiplicity 1. Every  $f \in \mathcal{E}(\Gamma \backslash \mathbb{H})$  has the expansion*

$$f(z) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \langle f, E(\cdot, 1/2 + it) \rangle E(z, 1/2 + it) dt. \quad (4.4)$$

which converges in the norm topology. If  $f \in \mathcal{E}(\Gamma \backslash \mathbb{H}) \cap \mathcal{D}(\Gamma \backslash \mathbb{H})$ , then the series converges pointwise absolutely and uniformly on compacta.

Combining Theorems 4.2-4.4, one gets the spectral decomposition of the whole space  $\mathcal{L}(\Gamma \backslash \mathbb{H})$ ,

$$f(z) = \sum_{j=0}^{\infty} \langle f, u_j \rangle u_j(z) + \frac{1}{4\pi} \int_{-\infty}^{\infty} \langle f, E(\cdot, 1/2 + it) \rangle E(z, 1/2 + it) dt. \quad (4.5)$$

**Theorem 4.5.** *The Laplace eigenvalue of any Maass cusp form for  $\Gamma$  in the discrete spectrum is  $> 3\pi^2/2$ .*

**Theorem 4.6.** (Weyl's law) *Let  $N_{\Gamma}(x)$  denote the number of Laplace eigenvalues up to  $x$ . Then*

$$N_{\Gamma}(x) = \frac{x}{12} + O(x^{1/2} \log x).$$

The proof of Theorem 4.6 depends on Selberg's trace formula which will not be presented in the notes. It follows from Theorem 4.6 that Maass cusp forms exist.

## 5. HECKE THEORY FOR MAASS FORMS

For  $n \geq 1$ , define the set

$$\Gamma_n = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = n \right\}. \quad (5.1)$$

In particular,  $\Gamma_1$  is the modular group. Naturally  $\Gamma_1$  acts on  $\Gamma_n$ . For  $n \geq 1$ , the Hecke operator

$$T_n : \mathcal{L}(\Gamma_1 \backslash \mathbb{H}) \rightarrow \mathcal{L}(\Gamma_1 \backslash \mathbb{H})$$

is defined by

$$(T_n f)(z) = \frac{1}{\sqrt{n}} \sum_{g \in \Gamma_1 \backslash \Gamma_n} f(gz). \quad (5.2)$$

Picking up specific representations of  $\Gamma_1 \backslash \Gamma_n$ , we can also write

$$(T_n f)(z) = \frac{1}{\sqrt{n}} \sum_{ad=n} \sum_{b \pmod{d}} f\left(\frac{az+b}{d}\right). \quad (5.3)$$

Clearly, this sum is finite, the number of terms is

$$[\Gamma_n : \Gamma_1] = \sum_{d|n} d = \sigma(n),$$

and therefore  $T_n$  is bounded on  $\mathcal{L}(\Gamma_1 \backslash \mathbb{H})$  by  $\sigma(n)n^{-1/2}$ .

We first examine the action of  $T_n$  on a Maass cusp form  $f$ .

**Theorem 5.1.** *Let  $f$  be a Maass cusp form with eigenvalue  $1/4 + \nu^2$ , and*

$$f(z) = \sum_{m \neq 0} a(m) \sqrt{|y|} K_{i\nu}(2\pi|m|y) e(mx).$$

*Then*

$$(T_n f)(z) = \sum_{m \neq 0} t_n(m) \sqrt{|y|} K_{i\nu}(2\pi|m|y) e(mx),$$

*with*

$$t_n(m) = \sum_{d|(m,n)} a(mn/d^2).$$

Now we are ready to establish some important facts for the Hecke operators  $T_n$ .

**Theorem 5.2.** *Let  $T_n$  be defined as above. Then*

- We have

$$T_m T_n = \sum_{d|(m,n)} T_{mn/d^2},$$

so that in particular  $T_m$  and  $T_n$  commute.

- Moreover the Hecke operators commute with the Laplace operator  $\Delta$ .
- $T_n$  is also self-adjoint in  $\mathcal{L}(\Gamma_1 \backslash \mathbb{H})$ , i.e.

$$\langle T_n f, g \rangle = \langle f, T_n g \rangle.$$

Therefore, in the space  $\mathcal{C}(\Gamma_1 \backslash \mathbb{H})$  of cusp forms, an orthonormal basis  $\{u_j(z)\}$  can be chosen which consists of simultaneous eigenfunctions for all  $T_n$ , i.e.

$$T_n u_j(z) = \lambda_j(n) u_j(z), \quad j \geq 1, \quad n \geq 1, \quad (5.4)$$

where  $\lambda_j(n)$  is the eigenvalue of  $T_n$  for  $u_j(z)$ .

- Up to a constant,  $\lambda_j(n)$  and the Fourier coefficient  $a_j(n)$  are equal. More precisely,

$$\lambda_j(n) a_j(1) = a_j(n), \quad \text{for all } n \geq 1, \quad j \geq 1. \quad (5.5)$$

- The Eisenstein series  $E(z, 1/2 + it)$  is an eigenfunction of all the Hecke operators  $T_n$  with eigenvalue  $\eta_t(n)$ , i.e.

$$T_n E(z, 1/2 + it) = \eta_t(n) E(z, 1/2 + it), \quad n \geq 1, \quad t \in \mathbb{R}, \quad (5.6)$$

where

$$\eta_t(n) = \sum_{ad=n} (a/d)^{it}. \quad (5.7)$$

*Proof.* We prove only (5.5). Erase the  $j$  for simplicity, and let  $u(z)$  denote  $u_j(z)$ . Suppose

$$u(z) = \sum_{m \neq 0} a(m) \sqrt{y} K_{i\nu}(2\pi|m|y) e(mx).$$

Then by (5.4),

$$(T_n u)(z) = \sum_{m \neq 0} \lambda(n) a(m) \sqrt{y} K_{i\nu}(2\pi|m|y) e(mx).$$

On the other hand, it follows from Theorem 5.1 that

$$(T_n u)(z) = \sum_{m \neq 0} t_n(m) \sqrt{y} K_{i\nu}(2\pi|m|y) e(mx),$$

with

$$t_n(m) = \sum_{d|(m,n)} a(mn/d^2).$$

Hence,

$$\lambda(n)a(m) = \sum_{d|(m,n)} a(mn/d^2),$$

and the desired result (5.5) follows on letting  $m = 1$ .  $\square$

The Generalized Ramanujan Conjecture (GRC) states that

$$|\lambda_j(n)| \leq \tau(n). \quad (5.8)$$

By (5.7), the conjecture is obvious true for in the space of continuous spectrum. In the cuspidal space, the best known bound towards the conjecture is due to Kim and Sarnak [?]:

$$|\lambda_j(n)| \leq \tau(n)n^\theta \quad (5.9)$$

with  $\theta = 7/64$ . The trivial bound is

$$\theta = 1/2. \quad (5.10)$$

## 6. THE KUZNETSOV TRACE FORMULA

We state without proof the following Kuznetsov formula. Let  $\{u_j\}$  be an orthonormal basis of the space of Maass forms for  $\Gamma$ . Denote by  $1/4 + \nu_j^2$  the Laplace eigenvalue for  $u_j$ . Let

$$u_j(z) = (y \cosh \pi \nu_j)^{1/2} \sum_{n \neq 0} \lambda_j(n) K_{i\nu_j}(2\pi|n|y) e(nx)$$

be the Fourier expansion of  $u_j$ .

**Theorem 5.1.** (Kuznetsov [8]). *Let  $h(r)$  be an even function of complex variable, which is analytic in  $-\Delta \leq \Im r \leq \Delta$  for some  $\Delta \geq 1/4$ . Assume in this region that  $h(r) \ll r^{-2-\delta}$  for some  $\delta > 0$  as  $r \rightarrow \infty$ . Then for any  $n, m \geq 1$ ,*

$$\begin{aligned} & \sum_{u_j} h(\nu_j) \lambda_j(m) \overline{\lambda_j(n)} + \frac{1}{\pi} \int_{\mathbb{R}} \tau_{ir}(n) \tau_{ir}(m) h(r) \frac{1}{|\zeta(1+2ir)|^2} dr \\ &= \frac{\delta_{n,m}}{\pi^2} \int_{\mathbb{R}} \tanh(\pi r) h(r) dr + \frac{2i}{\pi} \sum_{c \geq 1} \frac{S(n, m; c)}{c} \int_{\mathbb{R}} J_{2ir} \left( \frac{4\pi\sqrt{mn}}{c} \right) \frac{h(r)r}{\cosh(\pi r)} dr. \end{aligned} \quad (6.1)$$



Here

$$\tau_v(n) = \sum_{ab=|n|} (a/b)^v$$

and  $S(m, n; c)$  is the classical Kloosterman sum defined as

$$S(m, n; c) = \sum_{a \bmod c}^* e\left(\frac{am + \bar{a}n}{c}\right).$$

The left-hand side of (6.1) is the spectral side, and the right the geometric side of the trace formula. The integral on the spectral side represents the continuous spectrum of the Laplace operator. Indeed, the divisor function  $\tau_{ir}(n)$  is the Fourier coefficients of Eisenstein series.

The integral on the left-hand side converges absolutely; this can be seen from

$$|\zeta(1 + 2ir)| \gg \frac{1}{\log(2 + |r|)},$$

a bound of de la Vallée Poussin. The absolute convergence is then from the fact that  $\tau_{ir}(n) \ll_n 1$  and  $h(r) \ll r^{-2-\delta}$ . The same bound for  $h(r)$  also give us convergence of the first integral on the right-hand side of (6.1).

The sum of Fourier coefficients on the left side of (6.1) is an infinite sum. For a proof of the Kuznetsov trace formula, as well as a discussion of convergence, see Kuznetsov [8] or Iwaniec [4].

The normalization for  $u_j$  is crucial here. Without suitable normalization of the Maass forms  $u_j$ , there would be no reason to have this sum convergent. In fact, the Kuznetsov trace formula (6.1) is only valid for Maass forms  $u_j$  which form an orthonormal basis of the space of the Maass cusp forms.

## 7. AUTOMORPHIC $L$ -FUNCTIONS

**7.1. Classical case.** First let us recall the Riemann zeta-function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \quad (7.1)$$

for  $\sigma > 1$ . The zeta-function has a functional equation and analytic continuation to the whole complex plane  $\mathbb{C}$ , with only a simple pole at  $s = 1$ . It has simple zeros at negative even integers, which are called trivial zeros. Other zeros of  $\zeta(s)$  lie in the critical strip  $0 \leq \sigma \leq 1$ ,

and are called nontrivial zeros. The Riemann Hypothesis predicts that all nontrivial zeros of  $\zeta(s)$  are on the line  $\sigma = 1/2$ .

Note that

$$\zeta(s) \left(1 - \frac{2}{2^s}\right) = \sum_{n \geq 1} \frac{1}{n^s} - 2 \sum_{n \geq 1} \frac{1}{(2n)^s} = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n^s}.$$

Since the series on the right side converges for real  $s > 0$ , we get an expression of  $\zeta(s)$  for real  $s > 0$

$$\zeta(s) = \left(1 - \frac{2}{2^s}\right)^{-1} \sum_{n \geq 1} \frac{(-1)^{n-1}}{n^s}. \quad (7.2)$$

By a general argument on the convergence of Dirichlet series, the right side of (7.2) also converges for all values of  $s$  with  $\sigma > 0$  (cf. Titchmarsh [24], pp.16-17). The function  $\zeta(s)$  has a functional equation and analytic continuation to  $\mathbb{C}$ . As a meromorphic function of  $s$ , it has a simple pole at  $s = 1$ , and has infinitely many trivial zeros and infinitely many nontrivial zeros. The Riemann Hypothesis (RH) claims that all nontrivial zeros have real part equal to  $1/2$ .

**7.2. Automorphic  $L$ -functions attached to Maass forms.** Now we turn to  $L$ -functions attached to Maass cusp forms for  $SL_2(\mathbb{Z})$ . Let  $f$  be a Maass cusp form with Laplace eigenvalue  $1/4 + \nu^2$  and Fourier expansion

$$f(z) = \sum_{n \neq 0} \sqrt{|y|} \lambda_f(n) K_{i\nu}(2\pi|n|y) e(nx). \quad (7.3)$$

Here we have normalized  $f$  so that  $\lambda_f(1) = 1$ . Then the  $L$ -function attached to  $f$  is defined as

$$L(s, f) = \sum_{n \geq 1} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{1}{p^{2s}}\right)^{-1}. \quad (7.4)$$

Note that we have normalized  $f$  by  $\lambda_f(1) = 1$  so that its  $L$ -function has the first term equal to 1.

By a trivial bound for  $\lambda_f(n)$ ,

$$\lambda_f(n) \ll n^{1/2},$$

the series in (7.4) and product in (7.4) converge absolutely for  $\sigma > 3/2$ . Using the Rankin-Selberg method, one can further prove that

$$\sum_{n \leq x} |\lambda_f(n)|^2 \ll x,$$

and hence (7.4) indeed converge absolutely for  $\sigma > 1$ .

**Theorem 7.1.** *The series in (7.4) converge absolutely for  $\sigma > 1$ .*

We define  $\alpha_f(p)$  and  $\beta_f(p)$  by

$$\alpha_f(p) + \beta_f(p) = \lambda_f(p), \quad \alpha_f(p)\beta_f(p) = 1.$$

Then the  $L$ -function can be written as

$$L(s, f) = \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_f(p)}{p^s}\right)^{-1}.$$

GRC actually predicts that

$$|\alpha(p)| = |\beta(p)| = 1.$$

**Theorem 7.2.** *Let  $f$  be a Maass for with eigenvalue  $1/4 + \nu^2$ . Let  $\varepsilon = 0$  or  $1$  according as  $f$  is even or odd. Let*

$$\Lambda(s, f) = \pi^{-s} \Gamma\left(\frac{s + \varepsilon + i\nu}{2}\right) \Gamma\left(\frac{s + \varepsilon - i\nu}{2}\right) L(s, f).$$

Then

- $\Lambda(s, f)$  has analytic continuation to all  $s$  and satisfies the functional equation

$$\Lambda(s, f) = (-1)^\varepsilon \Lambda(1 - s, f). \tag{7.5}$$

- It is indeed an entire function.

*Proof.* First consider the case where  $f$  is even. We start from the integral

$$\int_0^\infty f(iy) y^{s-1/2} \frac{dy}{y}. \tag{7.6}$$

As  $y \rightarrow \infty$ , the integrand is vanishingly small because of the rapid decay of the Bessel function; and as  $y \rightarrow 0$ , one uses the identity

$$f(iy) = f(i/y) \tag{7.7}$$

to see that the integrand is small. Hence (7.6) is convergent for all  $s \in \mathbb{C}$ . If  $\sigma$  is large, one substitutes the Fourier expansion of  $f$  and uses

$$\int_0^\infty K_{i\nu} y^s \frac{dy}{y} = 2^{s-2} \Gamma\left(\frac{s+i\nu}{2}\right) \Gamma\left(\frac{s-i\nu}{2}\right),$$

to see that (7.6) equals  $\frac{1}{2}\Lambda(s, f)$ . The functional equation now follows from (7.7).

If  $f$  is odd, then we consider instead the integral

$$\int_0^\infty g(iy) y^{s+1/2} \frac{dy}{y},$$

where

$$g(z) = \frac{1}{4\pi i} \frac{\partial f}{\partial x}(z) = \sum_{n \geq 1} a(n) n \sqrt{y} K_{i\nu}(2\pi i |n| y) \cos(2\pi n x).$$

We have

$$g(iy) = -\frac{1}{y^2} g\left(\frac{i}{y}\right),$$

and the functional equation follows.  $\square$

It follows that the function  $L(s, f)$  has a functional equation and analytic continuation. It is indeed an entire function. We also know that it is nonzero on  $\sigma = 1$ . The Generalized Riemann Hypothesis (GRH) predicts that all the nontrivial zeros of  $L(s, f)$  are on the line  $\sigma = 1/2$ .

**7.3. Rankin-Selberg automorphic  $L$ -functions.** The next  $L$ -function we will consider is the Rankin-Selberg  $L$ -function. Let  $f$  and  $g$  be two Maass cusp forms with Laplacian eigenvalue  $1/4 + k^2$  and  $1/4 + l^2$ , respectively, and have Fourier expansion like in (7.3). Their Rankin-Selberg  $L$ -function is defined by

$$\begin{aligned} L(s, f \times g) &= \prod_p \left(1 - \frac{\alpha_f(p)\alpha_g(p)}{p^s}\right)^{-1} \left(1 - \frac{\alpha_f(p)\beta_g(p)}{p^s}\right)^{-1} \\ &\quad \times \left(1 - \frac{\beta_f(p)\alpha_g(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_f(p)\beta_g(p)}{p^s}\right)^{-1} \\ &= \zeta(2s) \sum_{n \geq 1} \frac{\lambda_f(n)\lambda_g(n)}{n^s}. \end{aligned} \tag{7.8}$$

**Theorem 7.3.** *Let  $f$  and  $g$  be two Maass cusp forms with Laplacian eigenvalue  $1/4 + k^2$  and  $1/4 + l^2$ , respectively, and have Fourier expansion like in (7.3). Then*

- *The product and the series in (7.8) are absolutely convergent for  $\sigma > 1$ . Because of our normalization of  $f$  and  $g$ , the first term in  $L(s, f \times g)$  is again equal to 1.*
- *The function  $L(s, f \times g)$  has a functional equation and analytic continuation. When  $f = g$ , it has a simple pole at  $s = 1$ .*

The functional equation is actually of the form

$$L(s, f \times g) = \gamma(s)L(1 - s, f \times g),$$

where

$$\gamma(s) = \text{quotient of } \Gamma\text{-factors.}$$

We will also need an approximate functional equation when  $l$  is fixed but  $k \rightarrow \infty$ . Applying Stirling's formula,

$$\gamma(s) = \left( \frac{16\pi^2}{(k+l-2)(k-l+2)} \right)^{2s-1} (1 + \eta_k(s))$$

where  $\eta_k(s) \rightarrow 0$  when  $k \rightarrow \infty$ . In fact, one can prove that

$$\eta_k(s) \ll (1 + |s|)^3/k^2.$$

Then the approximate functional equation is

$$L(s, f \times g) \sim \left( \frac{16\pi^2}{(k+l-2)(k-l+2)} \right)^{2s-1} L(1 - s, f \times g).$$

We can use this approximate functional equation to get an approximation to the central value of  $L(s, f \times g)$  at  $s = 1/2$ :

$$L\left(\frac{1}{2}, f \times g\right) = 2 \sum_{b \geq 1} \frac{1}{b} \sum_{a \geq 1} \frac{\lambda_f(a)\lambda_g(a)}{\sqrt{a}} V\left(\frac{ab^2}{X}\right) + O(k^\epsilon) \quad (7.9)$$

where

$$X = (k+l-2)(k-l+2)/(16\pi^2) \asymp k^2,$$

and

$$V(y) = \frac{1}{2\pi i} \int_{\sigma=1} G(s)y^{-s} \frac{ds}{s} \quad (7.10)$$

for a good function  $G(s)$ .

## 8. NUMBER THEORETICAL BACKGROUND

Gauss and Legendre conjectured that

$$\sum_{p \leq x} 1 \sim \frac{x}{\log x}, \quad x \rightarrow \infty,$$

which is called the prime number theorem. Riemann established a deep connection between the distribution of primes and distribution of zeros of the zeta-function. The following are some basic properties of the zeta-function.

The zeta-function satisfies the functional equation

$$\zeta(s) = \pi^{s-1/2} \frac{\Gamma((1-s)/2)}{\Gamma(s/2)} \zeta(1-s).$$

This gives the analytic continuation of  $\zeta(s)$  to the whole plane.  $\{-2n\}_{n=1}^{\infty}$  comprises all the trivial zeros of  $\zeta(s)$ . There are infinitely many non-trivial zeros, all of which lie in the critical strip  $0 \leq \sigma \leq 1$ .

Let  $\Lambda(n)$  be the von Mangoldt function, i.e.  $\Lambda(n) = \log p$  for  $n = p^a$  with  $a \geq 1$ , and zero otherwise. Then the prime number theorem is equivalent to

$$\sum_{n \leq x} \Lambda(n) \sim x. \tag{8.1}$$

The connection between the Riemann zeta-function and primes is given by the so-called explicit formula, which states that, for  $2 \leq T \leq x$ ,

$$\sum_{n \leq x} \Lambda(n) = x - \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} + O\left(\frac{x \log^2 x}{T}\right), \tag{8.2}$$

where the sum is taken over nontrivial zeros  $\rho = \beta + i\gamma$  of  $\zeta(s)$  with  $|\gamma| \leq T$ .

If there is a constant  $B < 1$  such that all the non-trivial zeros satisfy  $\beta \leq B$ , then the sum over  $\rho$  in (8.2) is

$$\ll \sum_{|\gamma| \leq T} \frac{x^\beta}{|\rho|} \leq x^B \sum_{|\gamma| \leq T} \frac{1}{|\rho|} \ll x^B \log^2 x,$$

and this gives the prime number theorem in the form

$$\sum_{n \leq x} \Lambda(n) = x + O(x^B \log^2 x). \tag{8.3}$$

However, we cannot establish such zero-free region as above for  $\beta \leq B < 1$ . In 1896, Hadamard and de la Vallée Poussin proved that  $\zeta(1 + it) \neq 0$ . This weak information on the zeros of  $\zeta(s)$  is sufficient to prove the prime number theorem (8.1). De la Vallée Poussin proved that  $\zeta(s)$  is zero-free in the region

$$\sigma \geq 1 - \frac{c}{\log(|t| + 2)}.$$

The Riemann Hypothesis says that all non-trivial zeros of  $\zeta(s)$  lie on the vertical line  $\sigma = 1/2$ . This is equivalent to

$$\sum_{n \leq x} \Lambda(n) = x + O(x^{1/2} \log^2 x).$$

By the Phragmén-Lindelöf method, we have

$$\zeta\left(\frac{1}{2} + it\right) \ll |t|^{1/4+\varepsilon}.$$

This is called the convexity bound. In 1921, Weyl improved the bound to  $|t|^{1/6+\varepsilon}$ . Over the years, there have been many improvements. Under the Riemann Hypothesis, one can get

$$\zeta\left(\frac{1}{2} + it\right) \ll |t|^\varepsilon,$$

which is known as the Lindelöf Hypothesis.

## 9. SUBCONVEXITY BOUNDS FOR RANKIN-SELBERG $L$ -FUNCTIONS

Let  $f$  and  $g$  be two Maass cusp forms with Laplacian eigenvalue  $1/4 + k^2$  and  $1/4 + l^2$ , respectively. Recall their Fourier expansions (7.3). The convexity bound is, for fixed  $g$  and  $t \in \mathbb{R}$ ,

$$L\left(\frac{1}{2} + it, f \times g\right) \ll_{\varepsilon, t, g} k^{1+\varepsilon}.$$

The first subconvexity in the  $k$ -aspect is due to Sarnak.

**Theorem 9.1.** (Sarnak [16]). *Let  $f$  be a holomorphic cusp form with weight  $k$ . Then*

$$L\left(\frac{1}{2} + it, f \times g\right) \ll_{\varepsilon, t, g} k^{18/(19-2\theta)+\varepsilon},$$

where  $\theta$  is given by (5.9) and we can take  $\theta = 7/64$ .

The first result for Maass forms  $f$  and  $g$  is the following

**Theorem 9.2.** (Liu-Ye [11]). *Let  $g$  be a fixed Maass cusp form for  $\Gamma$ . Let  $f$  be a Maass cusp form for  $\Gamma$  with Laplace eigenvalue  $1/4 + k^2$ . Then we have for any  $\varepsilon > 0$  and  $t \in \mathbb{R}$  that*

$$L\left(\frac{1}{2} + it, f \times g\right) \ll_{\varepsilon, t, g} k^{(11+2\theta)/12+\varepsilon}.$$

Note that  $L(s, f \times g)$  is indeed an  $L$ -function attached to a group representation of  $GL_4$  over  $\mathbb{Q}$ . A proof of Theorems 9.1 and 9.2 need  $GL_4$  techniques. The arguments of [16] and [12] use

- the Petersson or Kuznetsov trace formula once, which is a  $GL_2$  technique;
- an estimate of a shifted sum  $\sum_n \lambda_g(n) \bar{\lambda}_g(n+h)$  of Fourier coefficients of cusp form  $g$  for  $h \neq 0$ .

This latter estimate is again a  $GL_2$  technique, proved by Sarnak [16], Appendix. More precisely, for  $\nu_1, \nu_2 > 0$  let us define

$$D_g(s, \nu_1, \nu_2, h) = \sum_{\substack{m, n \neq 0 \\ \nu_1 m - \nu_2 n = h}} \lambda_g(n) \overline{\lambda}_g(m) \left( \frac{\sqrt{\nu_1 \nu_2 |mn|}}{\nu_1 |m| + \nu_2 |n|} \right)^{2il} (\nu_1 |m| + \nu_2 |n|)^{-s},$$

if  $g$  is a Maass cusp form with Laplace eigenvalue  $1/4 + l^2$ .

**Theorem 9.2.** (Sarnak [16]). *Let  $g$  be a Maass cusp form of Laplace eigenvalue  $1/4 + l^2$ . For  $\sigma > 1$ ,  $\nu_1, \nu_2 > 0$ , and  $h \in \mathbb{Z}$  define  $D_g(s, \nu_1, \nu_2, h)$  as above. Then  $D_g(s)$  extends to a holomorphic function for  $\sigma \geq 1/2 + \theta + \varepsilon$ , for any  $\varepsilon > 0$ . Moreover, in this region it satisfies*

$$D_g(s, \nu_1, \nu_2, h) \ll_{g, \varepsilon} (\nu_1 \nu_2)^{1/2+\varepsilon} |h|^{1/2+\theta+\varepsilon-\sigma} (1 + |t|)^3.$$

Theorem 9.2 has been improved.

$\beta$	author(s)	the shifted convolution sum is treated by
$\frac{15+2\theta}{16}$	Liu and Ye [11]	spectral method
$\frac{6-2\theta}{7-4\theta}$	Blomer [1]	circle method
$1 - \frac{1}{8+4\theta}$	Lau, Liu, and Ye [9]	spectral method
$\frac{2}{3}$	Lau, Liu, and Ye [10]	spectral method
$\frac{2}{3}$	Jutila and Motohashi [6]	spectral method



The best known result is the following Weyl's bound for general congruence subgroup  $\Gamma_0(N)$  due to Lau, Liu, and Ye [10]. For  $N = 1$ , it is also proved in Jutila and Motohashi [6].

**Theorem 9.4.** (Lau-Liu-Ye [10], Jutila-Motohashi [6]). *Let  $f$  and  $g$  be as in Theorem 9.1. Then we have for any  $\varepsilon > 0$  and  $t \in \mathbb{R}$  that*

$$L\left(\frac{1}{2} + it, f \times g\right) \ll_{\varepsilon, t, g} k^{2/3+\varepsilon}. \quad (9.1)$$

**Theorem 9.5.** *Let  $f$  and  $g$  be as in Theorem 9.2. Then we have for any  $\varepsilon > 0$  and  $t \in \mathbb{R}$  that*

$$\sum_{K-L \leq k \leq K+L} \left| L\left(\frac{1}{2} + it, f \times g\right) \right|^2 \ll_{\varepsilon, t, g} (KL)^{1+\varepsilon}, \quad (9.2)$$

for

$$K^{1/3+\varepsilon} \leq L \leq K^{1-\varepsilon}. \quad (9.3)$$

By Weyl's law, which states that  $\#\{k : 1/4 + k^2 \leq T\} \sim cT$ , we have

$$\begin{aligned} & \#\{k : K - L \leq k \leq K + L\} \\ &= \#\{k : 1/4 + (K - L)^2 \leq 1/4 + k^2 \leq 1/4 + (K + L)^2\} \\ &\sim c((1/4 + (K + L)^2) - (1/4 + (K - L)^2)) \\ &= 4cKL. \end{aligned}$$

Thus from Theorem 9.5, the generalized Lindelöf hypothesis

$$L(1/2 + it, f \times g) \ll k^\varepsilon$$

is true on average for  $K - L \leq k \leq K + L$  with  $L$  in the range (9.3).

*Proof of Theorem 9.4.* Now we derive Theorem 9.4 from Theorem 9.5. We take only one term from the left side of (9.2) and get

$$\left| L\left(\frac{1}{2} + it, f \times g\right) \right|^2 \ll (KL)^{1+\varepsilon}$$

for  $K - L \leq k \leq K + L$  and  $K^{1/3+\varepsilon} \leq L \leq K^{1-\varepsilon}$ . Taking  $K = k$  and  $L = K^{1/3+\varepsilon}$ , we prove Theorem 9.4.  $\square$

## 10. AN APPLICATION OF SUBCONVEXITY BOUNDS

Now let us turn to an application of our subconvexity bounds in Theorem 9.2 or 9.4. Let  $f$  be a Maass Hecke eigenform for  $\Gamma$  with Laplace eigenvalue  $\lambda$ . Normalize  $f$  so that

$$\mu_f = |f(z)|^2 \frac{dx dy}{y^2}$$

is a probability measure on  $\Gamma \backslash \mathbb{H}$ , i.e.,

$$\mu_f(\Gamma \backslash \mathbb{H}) = 1.$$

The equi-distribution conjecture predicts that

**Conjecture 10.1.** (Rudnick and Sarnak [14]). *As  $\lambda \rightarrow \infty$ ,*

$$\mu_f \longrightarrow \frac{1}{\text{vol}(\Gamma \backslash \mathbb{H})} \frac{dx dy}{y^2}.$$

According to Sarnak [15] and Watson [19], this conjecture would follow from a subconvexity bound for

$$L(1/2, f \times f \times g),$$

where  $f$  is as above and  $g$  is a fixed Maass Hecke eigenform. If  $f$  is a CM form corresponding to a representation of the Weil group  $W_{\mathbb{Q}}$ , i.e., if  $L(s, f) = L(s, \eta)$  for some grossencharacter  $\eta$  on a quadratic number field, then the triple Rankin-Selberg  $L$ -function can be factored as

$$L(s, F \times g)L(s, g)^2$$

Here  $F$  is a Maass cusp form with Laplace eigenvalue  $1/4 + (2k)^2$ , if  $\lambda = 1/4 + k^2$ . This way the equidistribution conjecture for CM Maass forms is reduced to a subconvexity estimate of  $L(s, F \times g)$ . Our Theorem 9.2 or 9.4 therefore implies the following theorem.

**Theorem 10.2.** *The equidistribution conjecture is true for CM Maass forms.*

## 11. A PROOF OF SUBCONVEXITY BOUNDS

**11.1. Approximate functional equation.** We have from (7.9) that

$$\begin{aligned} L\left(\frac{1}{2}, f \times g\right) &= 2 \sum_{1 \leq b \leq X^{1/2+\varepsilon}} \frac{1}{b} \sum_{a \geq 1} \frac{\lambda_f(a) \lambda_g(a)}{\sqrt{a}} V\left(\frac{ab^2}{X}\right) + O(k^\varepsilon) \\ &= \frac{2}{\sqrt{X}} \sum_{1 \leq b \leq X^{1/2+\varepsilon}} \sum_{a \geq 1} \lambda_f(a) \lambda_g(a) V\left(\frac{a}{X/b^2}\right) \sqrt{\frac{X/b^2}{a}} + O(k^\varepsilon), \end{aligned} \quad (11.1)$$

where

$$X \asymp k^2$$

and  $V$  is the same as in (7.10). Here the outer sum is written as a finite sum because terms with  $b > X^{1/2+\varepsilon}$  are negligible. The estimation of (11.1) is thus reduced to that of

$$S_Y(f) = \sum_{a \geq 1} \lambda_f(a) \lambda_g(a) H\left(\frac{a}{Y}\right),$$

where  $Y = X/b^2$  and

$$H\left(\frac{a}{Y}\right) = V\left(\frac{a}{X/b^2}\right) \sqrt{\frac{X/b^2}{a}}.$$

By an argument of smooth dyadic subdivision we can assume that  $H$  is a smooth function of compact support in  $[1, 2]$ .

**11.2. Application of the Kuznetsov trace formula.** No one can get a nontrivial estimate for individual  $S_Y(f)$  directly. In order to use the Kuznetsov trace formula, let us consider a smoothly weighted average

$$\begin{aligned} &\sum_{u_j} |S_Y(u_j)|^2 \left( h\left(\frac{K - \nu_j}{L}\right) + h\left(\frac{K + \nu_j}{L}\right) \right) \\ &= \sum_{m, n} \lambda_g(n) \bar{\lambda}_g(m) H\left(\frac{n}{Y}\right) \overline{H\left(\frac{m}{Y}\right)} \\ &\quad \times \sum_{u_j} \left( h\left(\frac{K - \nu_j}{L}\right) + h\left(\frac{K + \nu_j}{L}\right) \right) \lambda_j(n) \bar{\lambda}_j(m). \end{aligned} \quad (11.2)$$

Now the Kuznetsov trace formula (6.1) can be applied to the inner sum on the right side above. This will give us

$$\sum_{u_j} |S_Y(u_j)|^2 \left( h\left(\frac{K-\nu_j}{L}\right) + h\left(\frac{K+\nu_j}{L}\right) \right) = \Sigma_1 + \Sigma_2 + \Sigma_3, \quad (11.3)$$

where

$$\begin{aligned} \Sigma_1 &= - \sum_{m,n} \lambda_g(n) \bar{\lambda}_g(m) H\left(\frac{n}{Y}\right) \bar{H}\left(\frac{m}{Y}\right) \\ &\quad \times \int_{\mathbb{R}} \left( h\left(\frac{K-r}{L}\right) + h\left(\frac{K+r}{L}\right) \right) \tau_{ir}(n) \tau_{ir}(m) \frac{1}{|\zeta(1+2ir)|^2} dr, \end{aligned} \quad (11.4)$$

$$\begin{aligned} \Sigma_2 &= \sum_{m,n} \lambda_g(n) \bar{\lambda}_g(m) H\left(\frac{n}{Y}\right) \bar{H}\left(\frac{m}{Y}\right) \\ &\quad \times \frac{\delta_{n,m}}{\pi} \int_{\mathbb{R}} \tanh(\pi r) \left( h\left(\frac{K-r}{L}\right) + h\left(\frac{K+r}{L}\right) \right) r dr, \end{aligned} \quad (11.5)$$

and

$$\begin{aligned} \Sigma_3 &= 2i \sum_{m,n} \lambda_g(n) \bar{\lambda}_g(m) H\left(\frac{n}{Y}\right) \bar{H}\left(\frac{m}{Y}\right) \sum_{c \geq 1} \frac{S(n, m; c)}{c} \\ &\quad \times \int_{\mathbb{R}} J_{2ir} \left( \frac{4\pi\sqrt{nm}}{c} \right) \left( h\left(\frac{K-r}{L}\right) + h\left(\frac{K+r}{L}\right) \right) \frac{r dr}{\cosh(\pi r)}. \end{aligned} \quad (11.6)$$

We note that (11.3) is positive, while the expression in (11.4) without the leading negative sign is also positive. In fact, by change of variables, (11.4) is

$$\begin{aligned} \Sigma_1 &= - \frac{2L}{\pi} \sum_{n,m} \lambda_g(n) \bar{\lambda}_g(m) H\left(\frac{n}{Y}\right) \bar{H}\left(\frac{m}{Y}\right) \\ &\quad \times \int_{\mathbb{R}} \tau_{i(uL+K)}(n) \tau_{i(uL+K)}(m) \frac{h(u) du}{|\zeta(1+2i(uL+K))|^2} \\ &= - \frac{2L}{\pi} \int_{\mathbb{R}} \left| \sum_n \lambda_g(n) H\left(\frac{n}{Y}\right) \tau_{i(uL+K)}(n) \right|^2 \frac{h(u) du}{|\zeta(1+2i(uL+K))|^2} \\ &\leq 0. \end{aligned}$$

Therefore estimation of (11.3) is reduced to estimation of (11.5) and (11.6).

The expression in (11.5) is bounded by

$$\Sigma_2 \ll \sum_n |\lambda_g(n)|^2 \left| H\left(\frac{n}{Y}\right) \right|^2 \int_{\mathbb{R}} \left| h\left(\frac{K-r}{L}\right) \right| |r| dr.$$

By the Rankin-Selberg method, we can get

$$\sum_n |\lambda_g(n)|^2 \left| H\left(\frac{n}{Y}\right) \right|^2 \ll Y^{1+\varepsilon}.$$

On the other hand,

$$\int_{\mathbb{R}} \left| h\left(\frac{K-r}{L}\right) \right| |r| dr \ll \int_{K-cL}^{K+cL} r dr \ll KL.$$

Therefore  $\Sigma_2$  in (11.5) contributes at most  $O(LKY^{1+\varepsilon})$ .

**11.3. The main argument.** Estimation of  $\Sigma_3$  in (11.6) is difficult and complicated ( $\sim 78$  pages, see [10]). The techniques include

- analytic continuation of the shift convolution sum  $D_g$ , instead of using Theorem 9.3 of Sarnak;
- Voronoi summation formula;
- spectral large sieve of Deshouillers and Iwaniec;
- Good's estimate;
- delicate analysis like stationary phase, etc.

All of these yield the bound

$$\Sigma_3 \ll LKY^{1+\varepsilon}$$

for (11.6). This proves Theorem 9.5. □

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