NEW LINES IN HYDRODYNAMICS

By V. Bjerknes, Oslo.

My aim with this address is not so much to present solutions of hydrodynamic problems as to direct the attention of my colleagues mathematicians and physicists to an enormous field of ***unsolved problems***, problems belonging to a generalized hydrodynamics.

When the hydrodynamist of to-day writes his system of equations the concluding one is ***not the true equation of state***

$$q=f(p, \vartheta) \tag{1,1}$$

which expresses density q of the fluid as a function of two variables, pressure p and temperature ϑ, — *but a simplified form of this equation*, viz.

$$q=f(p) \tag{1,2}$$

where temperature has disappeared. This variable is allowed to enter only implicitely, in as much as it can be expressed as a function of one of the variables p or q.

The hydrodynamist has good reason for making this simplification. It gives him a closed system of equations, with an equal number of variables and equations. But the true equation of state brings in an additional variable, temperature, which makes him look for further equations. And the obvious further equations at disposal bring in still further variables: it is difficult to find new rational boundaries, — while the boundary set by the simplified equation of state (1,2) gives a harmonious science, — a model academic science, — characterized not least by the wonderful theorems of *Helmholtz* and *Kelvin* of the conservation of vortices and circulation.

But what do we loose?

We know that every *real* fluid can serve as working substance in a thermodynamical engine. But the unreal fluid defined by equation (1,2) can not. The science of thermodynamics can be built on no other equation than the true equation of state (1,1). The simplified equation of state keeps the two sciences of hydrodynamics and thermodynamics carefully separated.

Now we are living in a mainly fluid, i. e. mainly hydrodynamical world. With exception of the crusts of the planets and the meteoric stones, everything material in the universe seems to be in the fluid or the gaseous-fluid state; and even the mentioned exceptions, the meteoric stones and the crusts of the planets, are believed to have been fluid. And whenever or whereever deep going material changes are to take place, the fluid or gaseous state has to be passed, while at the same time thermodynamics has its share in determining the process. Keeping these two sciences separated, we loose the full scientific control.

Therefore we can not give up the creation of a generalized *physical hydrodynamics*, which is a synthesis of the two separate sciences. My address is a plea for cooperation for the creation of this combined science. The task is a great one. The synthesis can not be made in one step. It must be created by and by, by systematic work to be performed outside the barrier set by the simplified equation of state.

The ideal aim is of course to arrive at suitable *closed systems of equations*, of suitable generality for the different types of problems in the new field, and to solve well defined problems by exact integration.

But provisionally there are also other ways of performing pioneer work. A consequence of the fundamental hydrodynamic equations (equations of motion and equation of continuity) which has not yet been specialized by the simplified equation of state, will be in no contradiction with the principles of thermodynamics. Such a consequence is therefore open to discussion and to quantitative evaluation even from the point of view of thermodynamics.

I shall occupy myself mainly with *two hydrodynamic theorems* of this kind, twin theorems they may be called on account of their parallelism. Each of them will, in view of practical applications, be given in two forms, a "primary" or "primitive" form, and a "secondary" or "developed" form. The theorems are not new. I gave them for the first time in my lectures at the University of Stockholm in the year 1897. When I still call this address *new lines* in hydrodynamics, it is by two reasons. These theorems and their applications have not yet found their way to the textbooks, the authors of which almost without exception still barricade themselves behind the classical boundary set by the simplified equation of state. And to the great number of earlier applications of the theorems, a series of new applications have quite recently been added.

Summary.

(For the details see V. Bjerknes: Application of Line Integral Theorems to the Hydrodynamics of Terrestrial and Cosmic Vortices. Astrophysica Norvegica 1937.)

Passing to his main subject the lecturer started with two forms of the hydrodynamic equation of motion, one in which the forces are referred to unit of mass and one in which the forces are referred to unit of volume. Integration along a closed curve gives two integral equations, one which does not contain the potential and one which does not contain the pressure. These integral equations contain the circulation theorems in their primitive forms. The developed forms are obtained when the closed curve is sup-

posed to be a physical one, which participates in the motion, and under this supposition a total time derivative is separated from the inertia term of the equation.

The theorems were applied to the following problems.

Calculation of the periods of gravity oscillations of heterogeneous fluid masses.

Calculation of the periods of inertia oscillations of rotating fluid masses.

Calculation of the field of motion when the field of temperature is given and of the field of temperature when the field of motion is given in a circular vortex, with application to the fields of motion and of temperature in waterspouts, in sunspots and in the earth's atmosphere.

Applying these special results the lecturer gave finally a discussion of the general circulation of the earth's atmosphere, and made corresponding remarks on possible circulations in the sun.