PROGRAMME FOR THE QUANTITATIVE DISCUSSION OF ELECTRON ORBITS IN THE FIELD OF A MAGNETIC DIPOLE, WITH APPLICATION TO COSMIC RAYS AND KINDRED PHENOMENA

By CARL STØRMER, Oslo

In applied mathematics one is very often led to the problem of integrating a system of ordinary differential equations. For the application, a complete qualitative and quantitative study of all the integrals for values of the independent variable from $-\infty$ to $+\infty$ may be desirable. But, in general, this is such a vast problem that all the resources of contemporary mathematical methods are unable to solve it.

From the standpoint of a pure mathematician, only proofs of the existence and behaviour of the integrals in the neighbourhood of given values, together with some qualitative properties of the integrals, are generally all that is to be obtained.

But, for the applications, such results are generally of very little use. What is wanted is a quantitative study of the integrals for real values of the independent variable, and over as large an interval as possible.

It seems that both pure mathematicians and physicists generally neglect to pay enough attention to the fact that for an approximate quantitative study of the integrals, there exist methods sufficiently effective for the applications, and also of great importance in pure mathematics as heuristic means to suggest new facts.

Among the methods most important in this respect, I may first mention the use of the new integrating machine invented by V. Bush (1)* and by him called "differential analyzer". As far as I know, only a few of these machines have hitherto been built.

At the expense of the Rockefeller Foundation, an improved Bush machine is now being built here for the Institute of Theoretical Astrophysics by Gundersen and Løken under the supervision of Professor Rosseland.

We hope that this machine will be ready for use within a year, for its capacity will be so great that it may be able to integrate even a system of 12 simultaneous differential equations of the first order.

If such a machine, however, is not available, the work can nevertheless be done by methods of numerical integration. Of such methods, there

* The numbers refer to the bibliography at the end of the paper.

exist several, but those which use series of differences corresponding to equidistant values of the independent variable are to be preferred, also because errors in the calculations are then easy to discover by reason of the corresponding irregularities of the series of differences of the highest orders.

Such methods have been used by astronomers in the theory of planetary perturbations, and especially by Darwin (2) and Strömgren (3) and their assistants in the problem of the three bodies.

For a system of differential equations of the second order of the form: second derivatives equal to given functions of the variables, I elaborated in 1904 a very practical method which has since been used for about 18000 hours of work (4). At that time I did not know that a similar method for differential equations of the first order had already been given by Adams in 1883 (5).

By such numerical methods the integrals of the differential equations can be followed as far as one likes with an accuracy quite sufficient for the applications. One only needs enough time and sufficient money for paying the assistants.

In this lecture I shall mention a case where extended numerical integrations during a series of years have thrown much light upon all the integrals of a certain system of differential equations of the sixth order. This system is very interesting in itself, because we here meet types of orbits treated by Poincaré, such as periodic and asymptotic trajectories. But its chief importance lies in the applications, because two of the most interesting phenomena in nature, the polar aurora and the cosmic rays, both lead to this system.

The system in question is that of the equations of motion of an electron in the field of a magnetic dipole. In cartesian coordinates it has the form (6)

$$\frac{d^2x}{dt^2} = a\left[\frac{3yz}{r^5}\frac{dz}{dt} - \frac{3z^2-r^2}{r^5}\frac{dy}{dt}\right]$$

$$\frac{d^2y}{dt^2} = a\left[\frac{3z^2-r^2}{r^5}\frac{dx}{dt} - \frac{3xz}{r^5}\frac{dz}{dt}\right]$$

$$\frac{d^2z}{dt^2} = a\left[\frac{3xz}{r^5}\frac{dy}{dt} - \frac{3yz}{r^5}\frac{dx}{dt}\right]$$

where a is a constant, $r^2=x^2+y^2+z^2$, and where t is the time.

The general integral contains 6 arbitrary constants. As I have shown in 1904, one easily finds two "first integrals" by which the integration is reduced to a differential equation of the second order and two quadratures.

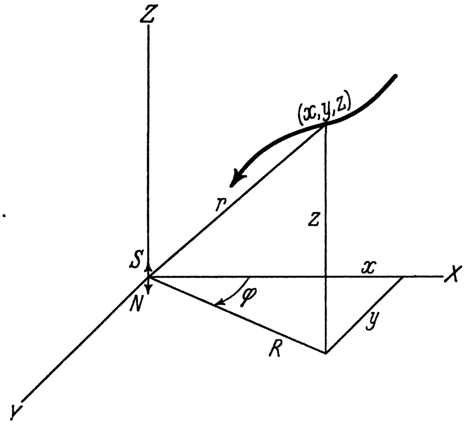

Fig. 1. System of coordinates.

The first of the two integrals is, that the velocity is constant. From this is seen that every trajectory can be obtained from a corresponding trajectory in the case where $a=1$ and $t=s=$ the arc of the trajectory, by enlarging all dimensions in the same ratio. Using this, the second integral can be written (Fig. 1):

$$R^2 \frac{d\varphi}{ds} = \frac{R^2}{r^3} + 2\gamma$$

where $x=R\cos\varphi$, $y=R\sin\varphi$ and where γ is a constant of integration. Further we find (7):

$$\frac{d^2 R}{ds^2} = \frac{1}{2}\frac{\partial Q}{\partial R}, \quad \frac{d^2 z}{ds^2} = \frac{1}{2}\frac{\partial Q}{\partial z}$$

$$\left(\frac{dR}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = Q$$

where

$$Q = 1 - \left(\frac{2\gamma}{R} + \frac{R}{r^3}\right)^2.$$

The motion of the particle can then be described in the following manner: We imagine a plane E through the z-axis which follows the motion of the particle in such a way that the particle is always situated in that plane. The motion can then be decomposed in the following two separate motions:

1. The motion in the plane E (R, z as functions of s).
2. The motion of the plane E (φ as a function of s).

The first motion is the motion of a particle moving under the action of a force depending on the force function Q, s being considered as the

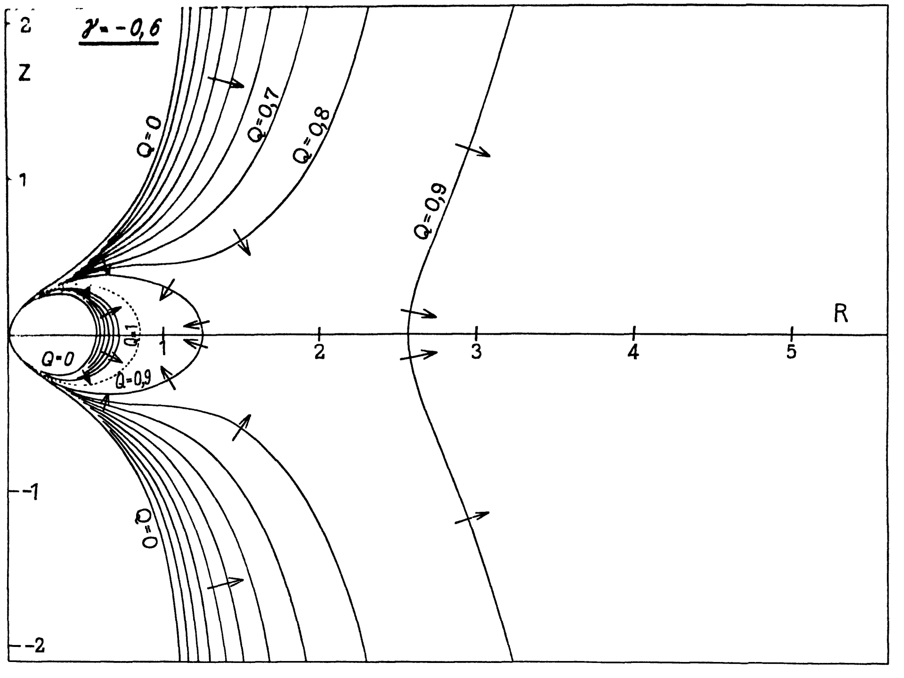

Fig. 2. Field of force $Q =$ constant, in the plane E.

time. If we draw the level lines $Q =$ constant, we get for each γ such a field of force as seen in figure 2, with arrows pointing in the direction of the force.

The field can be interpreted as a map of a landscape with the lines $Q =$ const. going through all points of equal height over the sea level. The orbits in this field resemble, then, the orbits of a little sphere rolling in this landscape.

The point cannot get out of the region limited by the branches of the level line $Q=0$.

Another important interpretation is obtained in the following manner (8): Suppose γ negative and $= -\gamma_1$, and put

$$x = \frac{1}{2\gamma_1} R_1 \cos\varphi, \quad z = \frac{1}{2\gamma_1} z_1$$

$$y = \frac{1}{2\gamma_1} R_1 \sin\varphi, \quad r = \frac{1}{2\gamma_1} r_1$$

$$ds = \left(\frac{1}{2\gamma_1}\right)^3 d\tau .$$

Then

$$\frac{d\varphi}{d\tau} = \frac{1}{r_1^3} - \frac{1}{R_1^2}$$

$$\frac{d^2R_1}{d\tau^2} = \frac{1}{2}\frac{\partial U}{\partial R_1}, \quad \frac{d^2z_1}{d\tau^2} = \frac{1}{2}\frac{\partial U}{\partial z_1}$$

$$\left(\frac{dR_1}{d\tau}\right)^2 + \left(\frac{dz_1}{d\tau}\right)^2 = U + w_0^2$$

where

$$w_0 = \left(\frac{1}{2\gamma_1}\right)^2$$

and the force function U does not contain the constant γ_1 any longer, because

$$U = -\left(\frac{1}{R_1} - \frac{R_1}{r_1^3}\right)^2.$$

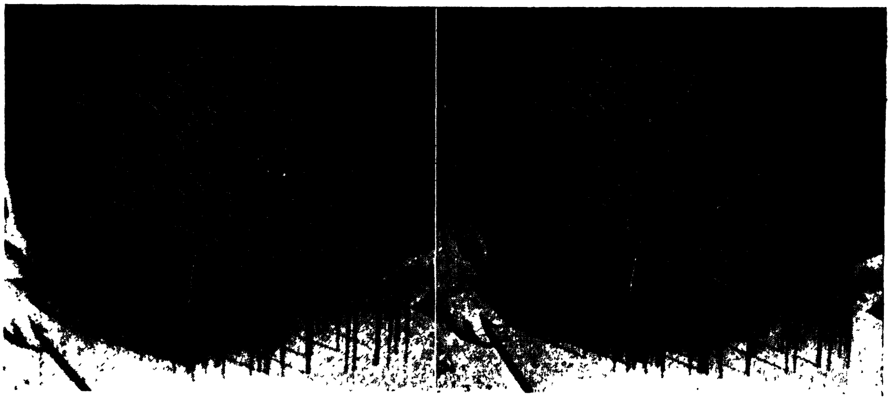

Fig. 3. Stereoscopic picture of a bundle of trajectories from infinity towards the dipole and with asymptotes parallel to the magnetic equatorial plane.

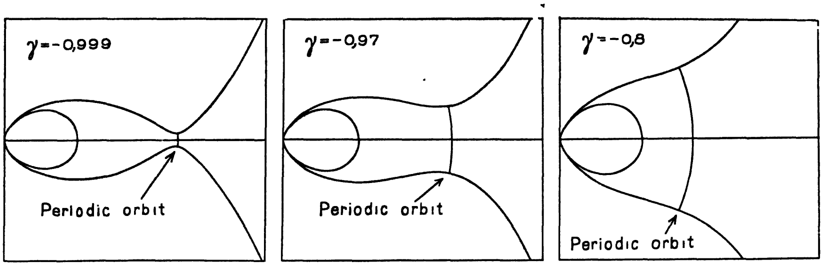

Fig. 4. Periodic orbits in the E plane for $\gamma = -0{,}999$, $\gamma = -0{,}97$ and $\gamma = -0{,}8$, calculated by numerical integration.

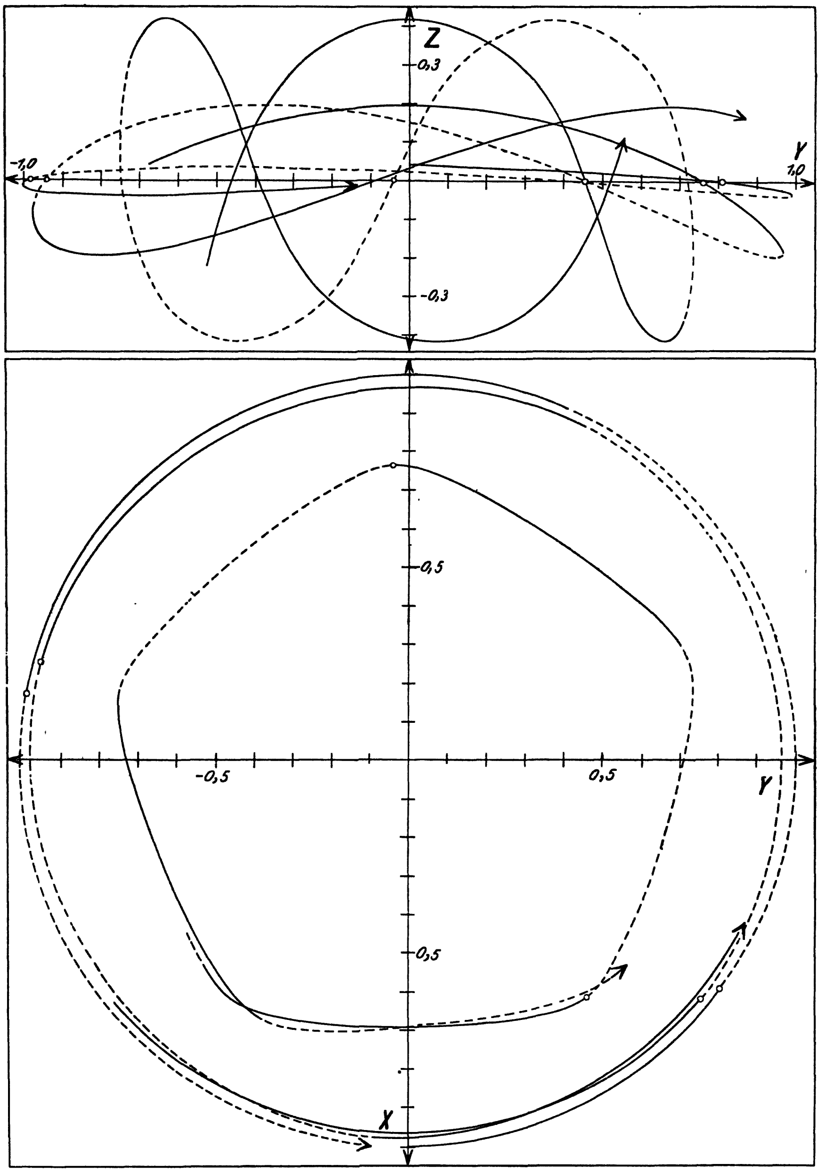

Fig. 5. Corresponding periodic orbits in space.

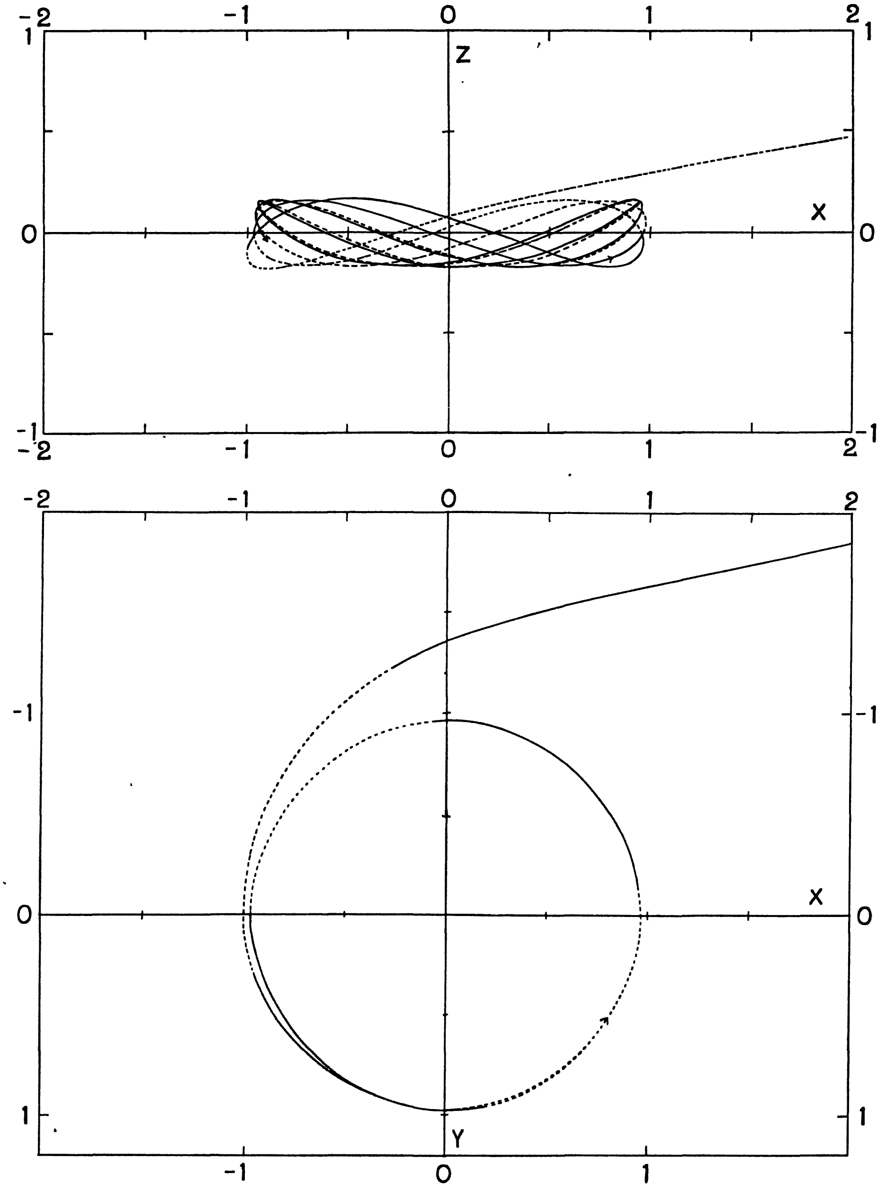

Fig. 6. An asymptotic trajectory in space seen from the side and from above (here some loops only are drawn).

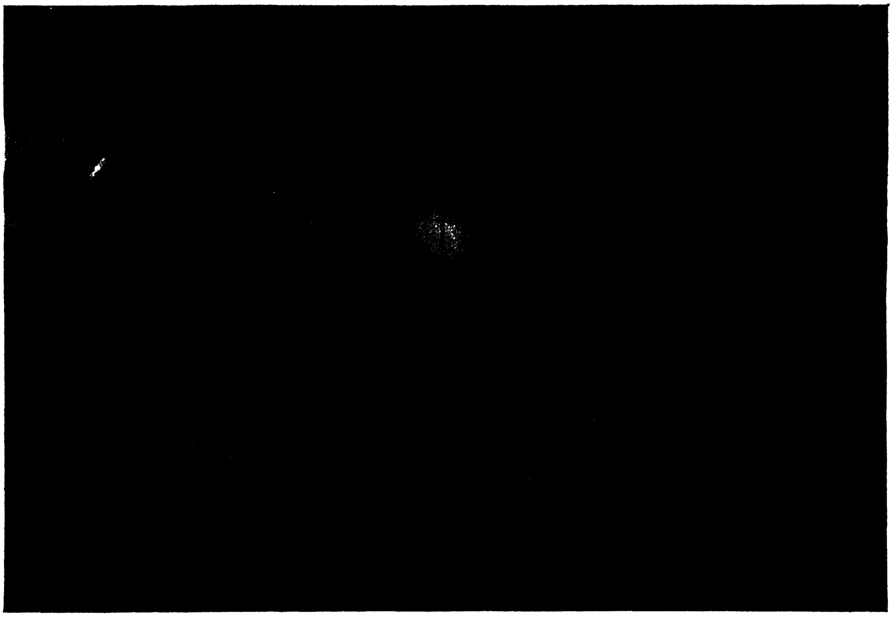

Fig. 7. Periodic orbit in space round a magnetic dipole (in the centre of the sphere). The circle (with radius unity) is also a periodic orbit.

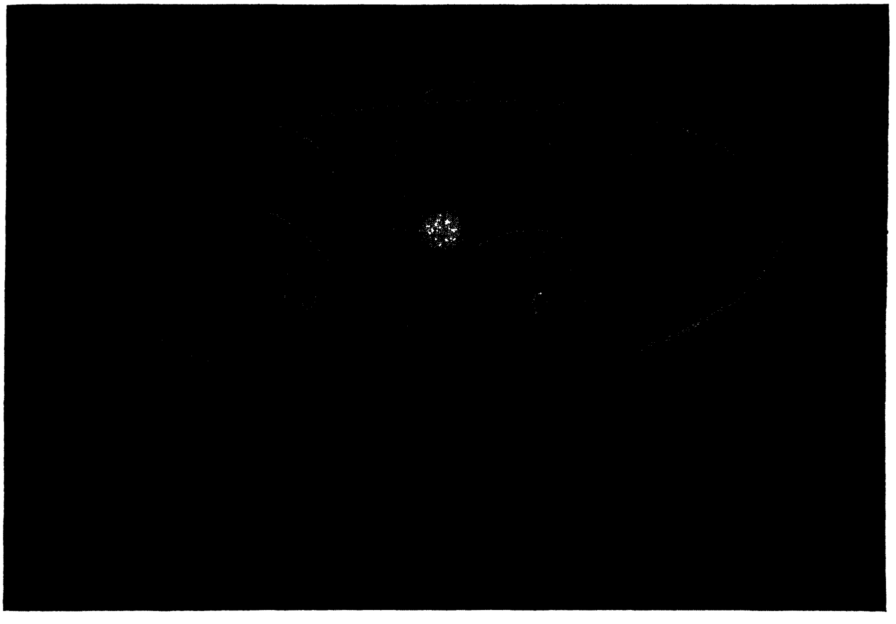

Fig. 8. Another periodic orbit.

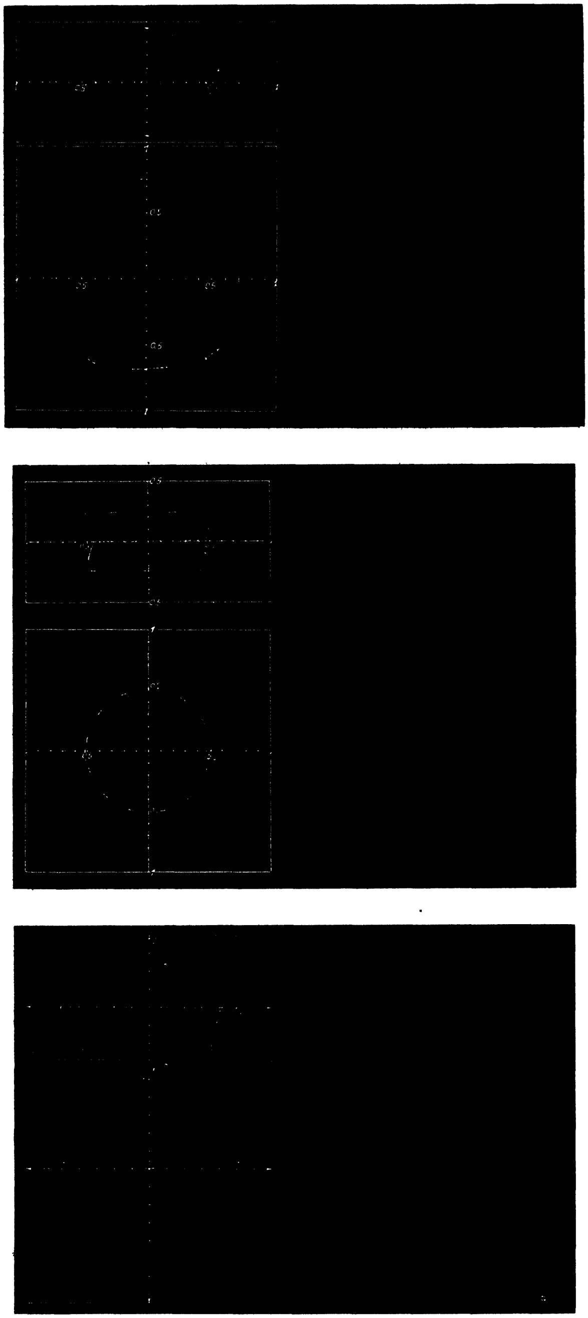

Fig. 9, 10, 11. Periodic orbits calculated by numerical integration and verified by the physical experiments of Brüche.

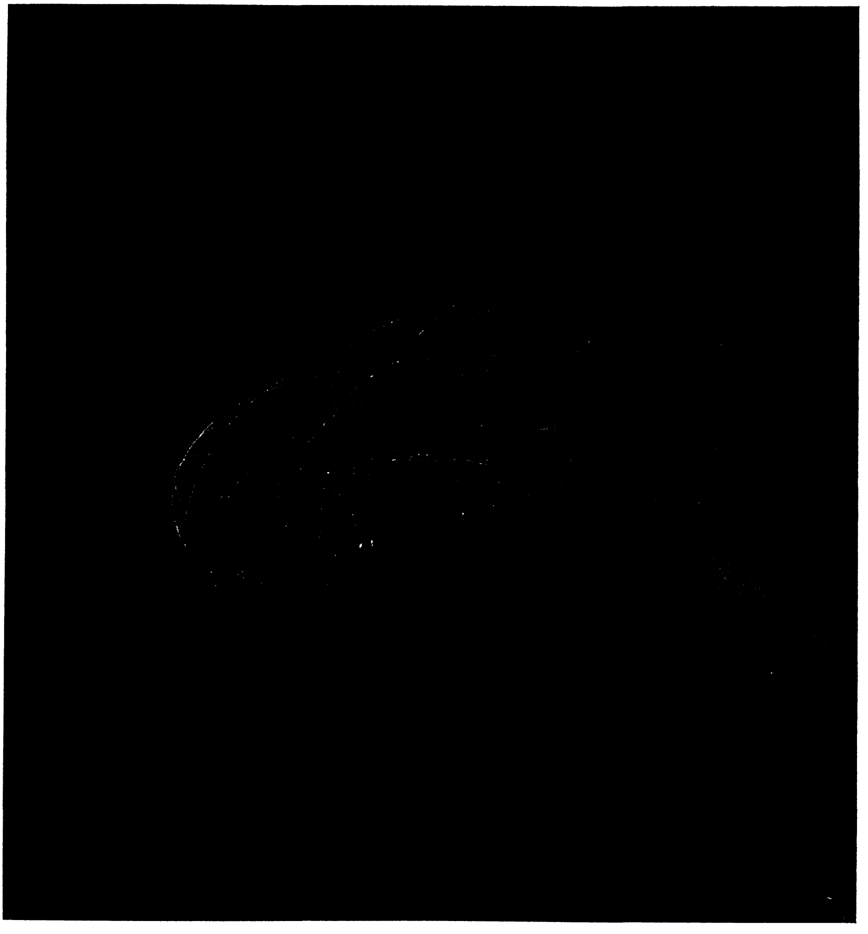

Fig. 12. Series of trajectories coming from infinity and going straight to the dipole, calculated in 1904—1907.

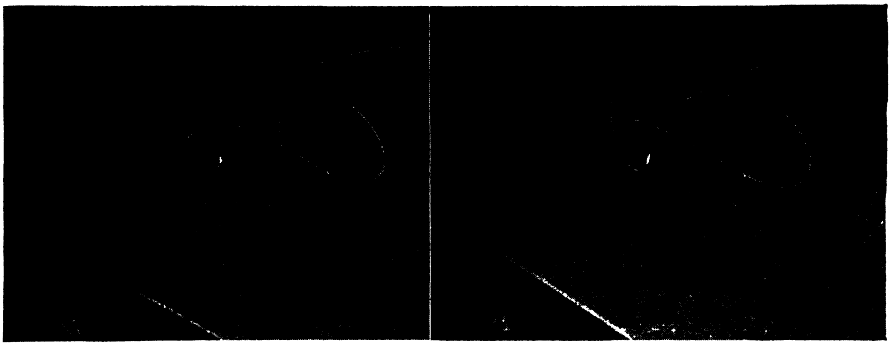

Fig. 13. Stereoscopic picture of a more complicated trajectory of the same kind as those in fig. 12.

Fig. 14. Another similar trajectory. Note the spiral towards the dipole and the big loops.

(Similar reductions are obtained for γ positive). Here the particle is moving in the field $U=$ const., and has the velocity w_0 each time it passes the level line $U=0$. The same field can thus be used for all ***negative values*** of γ and we have not one new field for each γ as in the former case.

We shall return to this field later. We now pass over to the programme for a complete quantitative study of the orbits in space.

It is, then, natural to begin with trajectories having infinite branches, and follow these trajectories from infinity towards the dipole. This is also very important for the application to cosmic radiation. We have here adopted the big programme (9) to calculate 10 bundles, each containing about 150 trajectories and with asymptotes making angles from 90° to 180° degrees with the z-axis. Two such bundles are calculated (10) and a wire-model of the first of them corresponding to 90° is seen in fig. 3.

For the application to cosmic radiation, a long series of further calculations have been made (10) giving the points of precipitation of these trajectories on the earth, the dipole being in the earth's centre.

For the discussion of the orbits in the inner part where $r < 1$, interesting families of periodic and asymptotic orbits are of fundamental importance.

In the field of force $Q =$ const., such periodic orbits are seen in fig. 4 and in space we have the corresponding ones in fig. 5. We will call these orbits in the plane E, *periodic orbits in the pass* (11). We have also in the plane E asymptotic orbits which approach to them asymptotically. They have been studied in a paper I published in 1911 (12), and more in detail, by Lemaitre and Vallarta, in a newly-published paper where several hundreds of them have been traced by the Bush-machine (13).

They are very similar to the asymptotic orbits seen in fig. 6, where the field $Q =$ const. has been replaced by a little simpler field chosen as a fairly good approximation. This picture is taken from a paper I published in 1934 (14).

The importance of these asymptotic orbits is that their envelopes separate regions from which trajectories can or cannot, penetrate the periodic trajectory in the pass.

There is also an infinity of other families of periodic trajectories (15). In figs. 7 and 8 are seen some wire-models of these orbits.

The calculation of these orbits has later been verified by some most striking experiments on cathode rays by the German physicist Brüche (16). As you see in figs. 9—11, the verification is most satisfactory.

Among the orbits coming from infinity there are series going straight to the dipole. These orbits are of outstanding importance for the theory of the polar aurora and several thousand hours have been sacrificed to calculate them (17). The corresponding orbits in the field $U =$ const. have a very simple meaning. They are the orbits of a point shot out from the dipole with velocity w_0 (18).

Some of the corresponding orbits in space are shown in figs. 12, 13 and 14. By means of these orbits it has been possible to explain a great many peculiarities of the aurora borealis (19).

Other interesting families of orbits are on the programme, for instance, orbits cutting a sphere with centre in the dipole at right angles.

From the theory of the cosmic radiation the following problem may be mentioned. At a given moment a cosmic particle with given energy comes down at a given place and from a given point of the sky. Find the regions among the stars from which the particle comes.

This problem can be solved by numerical or mechanical integration. On the film will be shown a case where such rays were coming down from zenith in Friedrichshafen and in Bergen, according to the experience by Ehmert and Trumpy (20).[1]

I hope you will agree with me, that the methods of numerical and mechanical integration of differential equations are very important not only in the applications but also in pure mathematics.

Bibliography.

1. V. Bush: "The differential analyzer. A new machine for solving differential equations" (Massachusetts Inst. of Techn. Cambridge U. S. A.). J. Franklin Inst. *212*, 447—448 (1931), and Zentralblatt für Mathematik und ihre Grenzgebiete, *3*, p. 65 (1932).

2. G. H. Darwin: "Periodic Orbits", Acta mathematica *21*, p. 99 – 242.

3. Elis Strömgren: "Connaissance actuelle des orbites dans le problème des trois corps", Bulletin Astronomique, Deuxième série, Tome IX, Fasc. 2.

4. Carl Størmer: "Sur les trajectoires des corpuscules électrisés dans l'espace sous l'action du magnétisme terrestre avec applications aux aurores boréales", Arch, d. Scien. phys. et nat. *24*, Genève 1907, — and Günther Schulz: "Fehlerabschätzung für das Størmersche Integrationsverfahren", Zeitschr. f. angew. Math. und Mech. B. 14, 1934.

5. F. Bashforth and J. C. Adams: "An attempt to test the theory of Capillary Action". Cambridge 1883, E. Y. Nyström: "Über die numerische Integration von Differentialgleichungen", Acta Soc. Scient. Fennicæ *L*, No. 13, 1925. — and Ernst Lindelöf: "Differentiali- ja Integralilasku", III, p. 315, Helsinki 1935.

6. Carl Størmer: "Sur le mouvement d'un point matériel portant une charge d'électricité sous l'action d'un aimant élémentaire", Vid.-Selsk. Skr., Math.-naturv. Kl. 1904, No. 3, Christiania.

7. Carl Störmer: "On the trajectories of electric corpuscles in space under the influence of terrestrial magnetism applied to aurora borealis and to magnetic disturbances", Arch. f. Math. og Naturv. *28*, No. 2, 1906, — and "Sur les trajectoires" etc., mentioned under (4).

8. Carl Størmer: "Sur une classe de trajectoires remarquables dans le mouvement d'un corpuscule électrique dans le champ d'un aimant élémentaire", Arch f. Math. og Naturv. *31*, No. 1911, — et "Sur le mouvement de corpuscules électriques dans le champ d'un aimant élémentaire et la forme de leur trajectoire à leur arrivé à l'aimant", Arch. des Sc. phys. et Natur., Genève 1913.

9. Carl Størmer: "On the trajectories of electric particles in the field of a magnetic dipole with applications to the theory of cosmic radiation, III", Astrophysica Norvegica I, No. 1, 1934.

10. Carl Størmer: The same, V, 1936, and "Résultats des calculs numériques des trajectoires de corpuscules électriques dans le champ d'un aimant élémentaire, IV et V", Skrifter, utgitt av Det Norske Videnskaps-Akademi i Oslo, 1936.

11. Carl Størmer: "Sur les trajectoires périodiques des corpuscules électriques dans l'espace sous l'influence du magnétisme terrestre avec application aux perturbations magnétiques", Comptes Rendus, Paris 1906, — and G. Lemaitre: "Trajectoires périodiques", Ann. de la Soc. Scient. de Bruxelles, Série A, t. LIV, p. 162 – 207.

[1] After the lecture a moving picture was shown of a series of wire models of interesting trajectories. Some of the pictures of this film have been combined to stereoscopic pictures in figs. 3, 13, and 14.

12. Carl Størmer: "Sur une classe etc.", mentioned under (8).

13. G. Lemaitre et M. S. Vallarta: "On the Geomagnetic Analysis of Cosmic Radiation", Physical Review *49*, p. 719 and On the Allowed Cone of Cosmic Radiation, ibid. *50*, p. 493.

14. Carl Størmer: "On the trajectories of electric particles in the field of a magnetic dipole with applications to the theory of cosmic radiation, II", University Observatory publication No. 12, Oslo 1934.

15. Carl Størmer: „Periodische Elektronenbahnen im Felde eines Elementarmagneten und ihre Anwendung auf Brüches Modellversuche und auf Eschenhagens Elementarwellen des Erdmagnetismus", Zeitschr. f. Astroph. I, — 1930, and "On pulsations of terrestrial magnetism and their possible explanation by periodic orbits of corpuscular rays", Terr. Magn. and Atmos. Elec., Washington 1931.

16. Ernst Brüche: "Experimente zu Størmers Polarlichttheorie", Phys, Zeitschr. 31, 1930, "Modellversuche mit sichtbaren Elektronenstrahlen zu Størmers Theorie etc.", Die Naturw. 18, p. 1085—1093, — and "Some new theoretical and experimental Results on the Aurora Polaris", Terr. Magn. and Atmos. Elec., Washington 1931.

17. Carl Størmer: "Résultats des calculs numériques des trajectoires des corpuscules électriques dans le champ d'un aimant élémentaire I". Vid.-Selsk. Skr., Math.-naturv. Kl. 1913, No. 4, Kristiania, — and "How the horse-shoe-formed auroral curtains can be explained by the corpuscular theory", Terr. Magn. and Atmos. Elec. Washington 1931.

18. Ibid.

19. Carl Størmer: "Über die Probleme des Polarlichtes", Ergebnisse der kosmischen Physik, I, Leipzig 1931.

20. Carl Størmer: On the trajectories of electric particles in the field of a magnetic dipole with applications to the theory of cosmic radiation, VI, Astrophysica Norvegica II, 1936.