NON-EUCLIDEAN GEOMETRY FROM NON-PROJECTIVE STANDPOINT

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1. There are three ways of developing the elementary parts of non-euclidean geometry:

(*1*) The synthetic method of the founders Lobatschewsky and Bolyai, which in more recent days has been refined and extended by Hilbert and many others.

(*2*) The method of projective geometry in which the fundamental notions of distance and angle are defined as cross ratios relative to a conic or quadric surface. Klein has been the great protagonist of this school.

(*3*) The method of differential geometry inaugurated by Riemann and Beltrami.

So far as the writer knows this third method has been employed only in a fragmentary way to develop those subjects which in Euclidean geometry are found in treatises on analytic geometry. In a paper* which I gave last year at the annual meeting of the Mathematical Association of America an attempt was made to show how readily this method of approach lent itself to an elementary treatment of what I may call non-euclidean analytic geometry. In that paper I confined myself to elliptic geometry; here I will treat them together but more particularly the hyperbolic geometry.

2. I begin by assuming that *e*-geometry† has been established with all requisite rigour, and on this we shall build the elliptic and hyperbolic geometries. The method is analogous to that employed in general arithmetic. Everyone knows the difficulties which beset our efforts to establish in a rigorous manner the real number system. But this once effected, other number systems as quaternions may be established with comparative ease.

Let x_1, x_2, x_3 be the rectangular coordinates of ordinary analytic geometry. Then in *e*-geometry the element of arc is defined by

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 \tag{1}$$

and the angle θ between two curves meeting at the point x by

$$\cos\theta = \sum_a \frac{dx_a}{ds}\frac{\delta x_a}{\delta s}, \qquad (a=1,\ 2,\ 3.) \tag{2}$$

*The Amer. Math. Monthly, vol. XXX (1923) and vol. XXXI (1924).

†*e*-geometry, *E*-geometry, *H*-geometry are abbreviations for euclidean, elliptic and hyperbolic geometry.

Straight lines or e-straights are such that

$$\delta \int ds = 0. \tag{3}$$

To get a non-euclidean geometry we keep $x_1x_2x_3$ the same, but define ds by

$$ds^2 = \Sigma a_{ij} dx_i dx_j, \qquad (i, j = 1, 2, 3), \tag{4}$$

where $a_{ij} = a_{ji}$ are functions of the x. The angle θ we define by

$$\cos \theta = \sum a_{ij} \frac{dx_i}{ds} \frac{\delta x_j}{\delta s}. \tag{5}$$

Straight lines or "straights" in this geometry are such that (3) holds, ds having the value (4).

How many geometries are defined by (4)? Obviously there are an infinite number of differential forms (4) which define the same geometry, at least in regions of not too great extent. For if we change the variables x to x' in a 1 to 1 manner, curves in x space go over into curves in x' space so that corresponding arcs have the same length, straights go over into straights and angles are unaltered. Hence figures are unaltered in size and shape in their respective metrics. Riemann showed that by a proper change of variables (4) can be reduced to

$$ds = \frac{d\sigma}{1 \pm \dfrac{r^2}{4R^2}} \;;\quad r^2 = x_1^2 + x_2^2 + x_3^2, \; d\sigma^2 = dx_1^2 + dx_2^2 + dx_3^2. \tag{6}$$

If we set

$$\lambda = 4R^2 - r^2, \qquad \mu = 4R^2 + r^2, \tag{7}$$

formula (6) gives

$$ds = \frac{4R^2}{\mu}\, d\sigma \qquad \text{(elliptic geometry)}, \tag{8}$$

or

$$ds = \frac{4R^2}{\lambda}\, d\sigma \qquad \text{(hyperbolic geometry)}, \tag{9}$$

according as we take the $+$ or $-$ sign in (6).

The two geometries defined by (8), (9) have much in common and much that is different. In the first place (8) and (9) show at once in connection with (5) that the measure of the angle under which two curves cut is the same in these two geometries as in e-geometry. Secondly we observe that (8) and (9) interchange on replacing R by iR; we must expect this duality in our analytical formulae. Thirdly we can show at once, using (3), that e-straights through the origin O are also straights in E and H-geometry.

On the other hand (8) and (9) reveal a great difference in these geometries. In fact (8) holds for all points of space while (9) breaks down on the sphere $\lambda = 0$ or

$$G = x_1^2 + x_2^2 + x_3^2 - 4R^2 = 0, \tag{10}$$

and ds becomes even negative for points without. For this reason only points within G are regarded as real in H-geometry. This G-sphere plays a dominant role in both geometries; we call it the *fundamental sphere.* Associated with G is the imaginary sphere $\mu = 0$ or

$$G' = x_1^2 + x_2^2 + x_3^2 + 4R^2 = 0, \tag{11}$$

which is also important. We observe that they interchange on replacing R with iR.

Another difference is the following. Let ρ be the length of the segment OP of a straight through O. In E-geometry

$$\rho_E = \int_o^r \frac{dr}{1 + \dfrac{r^2}{4R^2}} = 2R \tan^{-1} \frac{r}{2R}, \tag{12}$$

while in H-geometry

$$\rho_H = 2R \tanh^{-1} \frac{r}{2R}. \tag{13}$$

Thus the length of the entire straight through O in E-geometry is $2\pi R$. On the other hand in H-geometry, when P approaches a point of G, $\rho_H \doteq \infty$. Hence all points in H-geometry are at an infinite distance from any point of the fundamental sphere. In e-geometry the introduction of imaginary points has proved indispensable, *e.g.*, the circular points at infinity. In non-euclidean geometry the same is true. As an example a straight in H-geometry through O cuts the sphere G' in two imaginary points for which $r^2 = -4R^2$. For these points (13) gives

$$\rho = \pm \frac{\pi i R}{2}. \tag{14}$$

Thus on each of these straights there are points whose distance from O in H-measure is given by (14). This is a particular case of a very important property.

3. Let us consider briefly some of the simple facts of plane E-geometry, which unroll with hardly any effort from the e-geometry on a sphere. As we deal here only with points x in the plane we drop x_3 from the foregoing formulae. If we introduce the variables

$$z_1 = \frac{4R^2 x_1}{\mu}, \qquad z_2 = \frac{4R^2 x_2}{\mu}, \qquad z_3 = R\frac{\lambda}{\mu}, \tag{15}$$

we find x is the stereographic projection of the point z on the sphere

$$z_1^2 + z_2^2 + z_3^2 = R, \tag{16}$$

while

$$ds^2 = dz_1^2 + dz_2^2 + dz_3^2.$$

This shows that when x describes a curve of length s in E-measure, z describes a curve on the sphere of equal length in e-measure. Thus to E-straights in the x plane correspond geodesics on the sphere by virtue of (3), that is, great circles. Any such great circle is determined by a plane through the centre of (16) whose equation is, say

$$A_1z_1+A_2z_2+A_3z_3=0.$$

Replacing the z's by (15) we obtain

$$A_3(x_1^2+x_2^2)-4R(A_1x_1+A_2x_2)=4A_3R^2.$$

Thus E-straights are from the standpoint of e-geometry, circles cutting the fundamental G circle in diametral points, and the imaginary G' circle orthogonally. To any two curves in the x plane meeting under the angle θ in E-measure correspond two curves on the sphere (16) meeting under the same angle θ in e-measure. Thus to a triangle in the E-plane corresponds a triangle on the sphere having respectively equal sides and angles. The trigonometry of the E-plane is thus identical with ordinary trigonometry on a sphere of radius R. In particular the sum of the angles of an E triangle is $>180°$. Finally since figures may be moved about freely without distortion on the sphere the same holds in plane E-geometry. Hence the length of all E straights is $2\pi R$ since this is the length in E measure of a straight through O. On the sphere two points determine uniquely a great circle unless they lie on a diameter, hence in E-geometry two points determine but one straight unless they lie on a diameter of G. Similarly two E-straights do not cut in one point, but in two.

To avoid this anomaly one has defined a *restricted E-geometry* by imposing the conditions that to diametral points on the sphere (16) shall correspond but one point in the x plane, viz., that one of the two points which lies within the fundamental G circle. The two end points of a diameter of G are regarded as identical points. There are no points outside G. In this geometry which we may denote by E^* two points determine uniquely a straight; two straights intersect in a single point. The length of any E^* straight is πR instead of $2\pi R$ as in E-geometry.

A peculiarity of E^*-geometry is illustrated by the accompanying figures: 1, 2, 3. The circle $ABCD$ in Fig. 1 is moved toward the right. When it meets G in Fig. 2, the two points L, M are identical with their diametral points $L'M'$ and when the circle reaches O again the figure has been turned through 180° as indicated in Fig. 3. This is analogous to the twisted band of Möbius.

We mention one other feature which is of great importance in the following. On the sphere (16) the locus of the points whose distance from a point A is $\pi R/2$ is a great circle, the polar of A. Thus in the E-plane the locus of all points at a distance $\pi R/2$ from A is an E-straight a, the polar of A, also A is the pole of a. In E-geometry a straight has two poles, in E^*-geometry only one. The three E-straights $z_1=0$, $z_2=0$, $z_3=0$ form a right polar triangle; each side is the polar of the opposite vertex. We may use this as a triangle of reference. From a point z let us drop E-perpendiculars on the three sides of this triangle. If $\delta_1, \delta_2, \delta_3$ are the lengths in E-measure of these perpendiculars we find

$$z_k=R.\sin\ \delta_k/R, \qquad (k=1,\ 2,\ 3),$$

which gives us another geometric interpretation of the z's, analogous to the homogeneous coordinates of projective geometry.

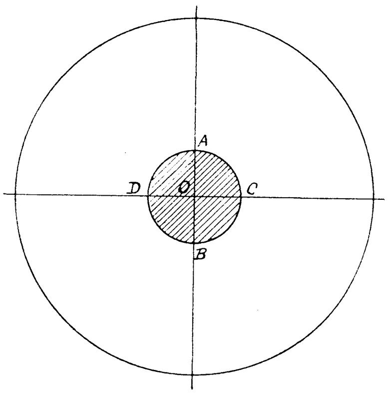


Fig. 1

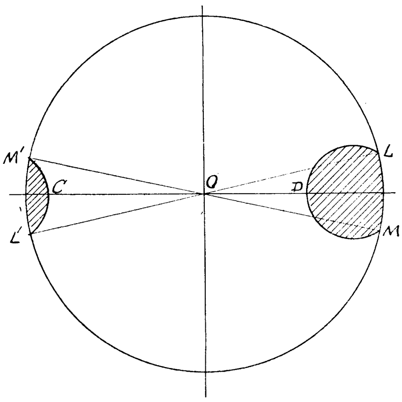


Fig. 2

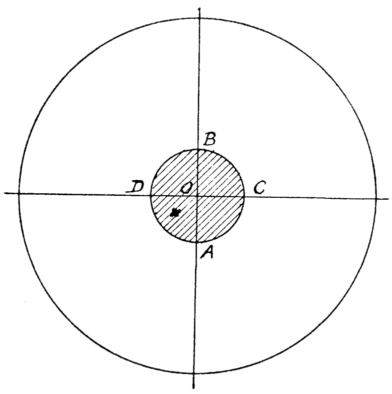


Fig. 3

4. We now leave these very instructive particulars and turn to hyperbolic geometry. First a word about plane H-geometry. If we replace R by iR the sphere (16) used for stereographic projection becomes imaginary. Perhaps for this reason one has introduced new variables so that (9) becomes

$$ds^2 = \frac{R^2(dy_1^2 + dy_2^2)}{y_1^2},$$

which defines the element of arc on a pseudosphere. This was first done by Beltrami and later by v. Escherich with considerable success, but it introduces unnecessary difficulties and one loses the close relations which exist between E and H-geometry. We shall therefore introduce variables analogous to those of (15), and as we propose to treat space and not the plane we need four, viz.:

$$\zeta_\alpha = \frac{4R^2 x_\alpha}{\lambda}, \quad (\alpha = 1, 2, 3), \quad \zeta_4 = \frac{R\mu}{\lambda} > 0. \tag{17}$$

The metric is that of (9). We shall find it convenient to use the abbreviations

$$[ab]=a_1b_1+\dots\dots+a_4b_4,\quad [a^2]=a_1^2+\dots\dots+a_4^2,$$
$$\{ab\}=a_1b_1+a_2b_2+a_3b_3-a_4b_4,\quad \{a^2\}=a_1^2+a_2^2+a_3^2-a_4^2.$$

We now find for all points x for which $\lambda\neq 0$, *i.e.*, points not on the fundamental G-sphere (10), that

$$\{\zeta^2\}=-R^2, \tag{18}$$

which, if we like, may be regarded as a hyperboloid in 4-way e-space, but we shall not urge this interpretation. We find also that the element of arc (9) satisfies

$$ds^2=\{d\zeta^2\}. \tag{19}$$

Let the straight joining O with the point $P(x_1,x_2,x_3)$ have the direction cosines l_1,l_2,l_3. The length ρ of the segment OP is given by (13). We find that

$$\zeta_a=Rl_a\sinh\rho/R,\quad (a=1,2,3),\quad \zeta_4=R\cosh\rho/R. \tag{20}$$

Let us first consider the locus defined by

$$[Az]=0, \tag{21}$$

which we call an *H-plane.* Setting (17) in this equation we get

$$A_4r^4+4R(A_1x_1+A_2x_2+A_3x_3)+4A_4R^2=0; \tag{22}$$

or

$$\sum_a\left(x_a+\frac{4RA_a}{A_4}\right)^2=\frac{4R^2}{A_4^2}\{A^2\},\quad A_4\neq 0. \tag{23}$$

Thus the H-plane (21) is in e-geometry a sphere cutting the fundamental G-sphere orthogonally. Since only the ratios of the A's in (21) are important we may suppose

$$\{A^2\}=R^2. \tag{24}$$

In this case we say (21) is in *normal form.* In case $A_4=0$, (21) reduces to $A_1z_1+A_2z_2+A_3z_3=0$ or in x coordinates $A_1x_1+A_2x_2+A_3x_3=0$. Hence H-planes through O are also e-planes. To the four H-planes $\zeta_1=0,\ \dots,\ \zeta_4=0$ correspond the three coordinate planes $x_1=0$, $x_2=0$, $x_3=0$ and the imaginary G' sphere. They form a tetrahedron which we shall always denote by τ. Let us now see what H-straights are. If we perform the variations indicated in (3), using the ds of (19), we get four differential equations

$$\frac{d^2\zeta_k}{ds^2}=\frac{\zeta_k}{R},\quad (k=1,\dots,4), \tag{25}$$

whose integrals are

$$\zeta_k=a_k\cosh\frac{s}{R}+b_k\sinh\frac{s}{R}. \tag{26}$$

The 8 constants of integration must be so chosen that ζ satisfies (18). When

$s=0$, $\zeta_k=a_k$, hence $\{a^2\}=-R^2$. To satisfy (18) we impose the further conditions

$$\{b^2\}=R^2, \quad \{ab\}=0. \tag{27}$$

With these conditions (26) are the parameter equations of an H-straight. We easily find that every H-straight is the intersection of two H-planes of the type (21) and conversely, as in e-geometry. This is our reason for calling (21) a plane. We have thus the important result: H-straights are e-circles cutting the fundamental G-sphere orthogonally and the imaginary G' sphere in diametral points. Just the opposite is true in E-geometry as we should expect, since these spheres like the two metrics (8), (9) interchange on replacing R by iR. A peculiarity of plane H-geometry is illustrated in the accompanying Fig. 4 which we suppose, to fix our ideas, lies in the x_1x_2 plane. G is the fundamental circle; e-circles cutting G orthogonally are H-straights. Straights as a, b which meet on G, *i.e.*, at infinity, are said to be *parallel*. Through a point A we can draw two parallels b, c to a given straight a. On the other hand there are an infinity of

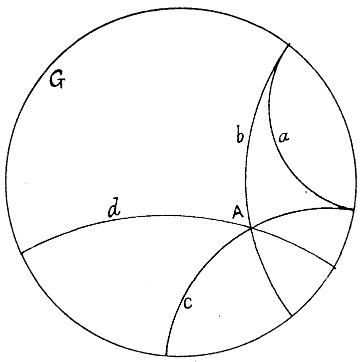


Fig. 4

straights through A which do not meet a given line, as for example a. Such a line is d, we observe that the parallels b, c are the limiting positions of lines through A which do not meet a.

Returning to our main theme, let us multiply (26) by $a_1, \ldots, a_4$; we get

$$\{a\zeta\}=\{a^2\}\cosh\frac{s}{R}+\{ab\}\sinh\frac{s}{R},$$

or

$$\cosh\frac{s}{R}=\frac{\{a\zeta\}}{-R^2}. \tag{28}$$

Similarly, multiplying (26) by $b_1, \ldots, b_4$ gives

$$\sinh\frac{s}{R}=\frac{\{b\zeta\}}{-R^2}. \tag{29}$$

The relation (28) is of utmost importance.

We have derived (25) and (26) on the supposition that the quantities involved are real. If we allow imaginary values we may regard (26) as defining an imaginary straight and (28) as defining the distance between the (imaginary) points a, ζ. The ambiguity of this definition is not disturbing as s is usually real or purely imaginary. Points whose coordinates are of the form $b_k = i\beta_k$, the β's real, are of extreme importance in H-geometry. Such points are, *e.g.*, the intersections of a straight through O with the imaginary G' sphere, *i.e.*, the H-plane $\zeta_4 = 0$. We saw in (14) that the distance of any point of this plane from O is $s = \pi i R/2$, agreeing with (28).

Let us generalize and say two points a, ζ are *associate* when

$$\{a\zeta\} = 0, \tag{30}$$

or, by (28), when their distance apart is $\pi i R/2$. The locus of all points ζ associated with a is thus an H-plane. We call this plane the *polar* of a and a the pole of the plane (30). If a is real a_4 is real and (23) shows that the polar of a is imaginary, since $\{a^2\} = -R^2$; on the other hand if a is imaginary, *i.e.* if its coordinates have the form $i\alpha_k$, the polar of a is real.

With this in mind let us return to (25). If we set $s = \pi i R/2$ we get $\zeta_k = ib_k$. The relations (27) therefore mean that the H-straight passes through a and the associate point whose coordinates are ib_k. The tetrahedron τ mentioned above is a polar tetrahedron; each of its faces is the polar of the opposite vertex.

Let us now consider the angle θ between two H-planes

$$[A\zeta] = 0, \quad [B\zeta] = 0, \tag{31}$$

which we will suppose are in normal form. To these planes correspond e-spheres whose equations are of the type (22). The angle under which these spheres cut is also θ in e-measure as we saw in § 2. Then from analytic geometry

$$\cos\theta = \frac{\{AB\}}{R^2}. \tag{32}$$

The two planes (31) are orthogonal when

$$\{AB\} = 0 \tag{33}$$

and in this relation we may note it is not necessary that the H-planes (31) should be in normal form.

Let b, c be two points in the plane $\alpha = \{a\zeta\} = 0$. The plane ω through a, b, c has the equation

$$\omega = \begin{vmatrix} \zeta_1 \ldots\ldots \zeta_4 \\ a_1 \ldots\ldots a_4 \\ b_1 \ldots\ldots b_4 \\ c_1 \ldots\ldots c_4 \end{vmatrix} = A_1\zeta_1 + \ldots + A_4\zeta_4 = 0.$$

This plane is perpendicular to α if $\{aA\} = 0$, by (33). But $\{aA\} = a_1A_1 + \ldots + a_4A_4$ is the development of the above determinant when we replace the ζ's by the a's; hence $\{aA\} = 0$ and ω is orthogonal to α. If we keep a, b fixed and let c vary in the plane α, we see that all planes through the join of a, b cut the α plane orthogonally. Hence the important theorem: The H-straight joining a point with any point of its polar is perpendicular to this plane, or all straights perpendicular to an H-plane meet in the pole of this plane. This is illustrated by the τ tetrahedron; we see now all its faces cut orthogonally.

Let $\alpha=[A\zeta]=0$ be a plane in normal form so that $\{A^2\}=R^2$. If we set $A_1=ia_1$ $A_2=ia_2$, $A_3=ia_3$, $A_4=-ia_4$ we have $\{a^2\}=-R^2$ and hence a_1, a_2, a_3, a_4 are the coordinates of a point a. Then $[A\zeta]=0$ becomes $\{a\zeta\}=0$, which shows that a is the pole of α. We can now find the length δ of the H-perpendicular p dropped from a point ζ on the plane α. For let p cut α in the point c, then a is the associate of c on p and (29) gives

$$\sinh\frac{\delta}{R}=\frac{\{a\zeta\}}{iR^2}=\frac{[A\zeta]}{R^2}. \tag{34}$$

Hence if δ_k is the length of the H-perpendicular let fall from the point ζ on the face $\zeta_k=0$ of the τ tetrahedron we get at once

$$\zeta_j=R\sinh\frac{\delta_j}{R},\quad j=1,2,3;\quad \zeta_4=\frac{R}{i}\sinh\frac{\delta_4}{R}. \tag{35}$$

5. As an application of these formulae let us show how easily the formulae of H-trigonometry may be deduced.

Let ABC be a triangle in the x_1x_2 plane whose opposite sides have the lengths a, b, c in H-measure. Let A coincide with the origin O and OC with the $+x$ axis as in Fig. 5. We shall show later that any triangle may be moved into this position without altering any of its dimensions.

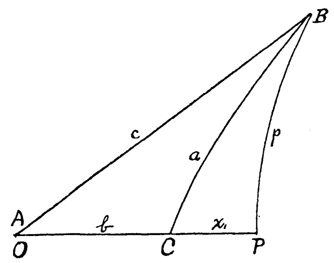


Fig. 5

Let the ζ coordinates of the vertex B be b_1, b_2, b_3 and c_1, c_2, c_3 those of C. Then by (20)

$$b_1=R\sinh\frac{c}{R}\cos A,\qquad b_2=R\sinh\frac{c}{R}\sin A,\qquad b_3=R\cosh\frac{c}{R},$$

$$c_1=R\sinh\frac{b}{R},\qquad c_2=0,\qquad c_3=R\cosh\frac{b}{R}.$$

These in combination with (28) give

$$\cosh\frac{a}{R}=\frac{\{bc\}}{-R^2}=\cosh\frac{b}{R}\cosh\frac{c}{R}-\sinh\frac{b}{R}\sinh\frac{c}{R}\cos A. \tag{36}$$

By cyclic permutation we obtain the corresponding formulae for $\cosh b/R$, $\cosh c/R$.

To get the sine formula we drop an H-perpendicular of length p on the side OC, or the line $\zeta_2=0$, or in normal form $R\zeta_1=0$. Then, by (34),

$$\sinh\frac{p}{R}=\frac{Rb_2}{R^2}=\sinh\frac{c}{R}\sin A.$$

This being true for any right triangle is true for CBP. Hence $\sinh\frac{p}{R}=\sinh\frac{a}{R}\sin C$.

Hence

$$\sinh\frac{a}{R}\sin C=\sinh\frac{c}{R}\sin A.$$

As these are entirely general we get as usual

$$\sinh\frac{a}{R}:\sinh\frac{b}{R}:\sinh\frac{c}{R}=\sin A:\sin B:\sin C. \tag{37}$$

The third fundamental formula is

$$\sinh\frac{a}{R}\cos B=\sinh\frac{c}{R}\cosh\frac{b}{R}-\cosh\frac{c}{R}\sinh\frac{b}{R}\cos A. \tag{38}$$

This is obtained from (36) in precisely the same manner as the corresponding formulae in ordinary spherical trigonometry. From (36), (37), (38) all the formulae of H-trigonometry may be obtained by following the steps employed in any elementary treatise on spherical trigonometry. We observe that we have merely to replace the sides a, b, c in such formulae by a/iR, b/iR, c/iR, to convert a formula of e-spherical trigonometry into one of H-trigonometry.

6. In e-geometry we assume the existence of rigid geometric figures, *i.e.*, figures that can be moved about freely without altering their size or shape. This fact is characterized by the existence of continuous point transformations which leave ds unaltered. We extend this to H-geometry. To simplify our analysis it will be convenient to set

$$z_j=\zeta_j,\ j=1,2,3;\quad z_4=i\zeta_4. \tag{39}$$

Then the z coordinates satisfy the relation

$$[z^2]=-R^2, \tag{40}$$

while ds^2 becomes

$$ds^2=[dz^2]. \tag{41}$$

Let us effect a linear transformation of the z's

$$z_k'=a_{k1}z_1+\ldots+a_{k4}z_4=\sum_a a_{ka}z_a,\quad (a,\ k=1,\ldots,4). \tag{42}$$

If the determinant A of the $a_{k\alpha}$ is $\neq 0$, (42) defines a one to one transformation of the point z to z' and conversely, provided the relation (40) is valid for z'. Now

$$[z'^2] = \sum_k z_k'^2 = \sum_k \sum_\alpha a_{k\alpha} z_\alpha \sum_\beta a_{k\beta} z_\beta = \sum_{\alpha,\beta} z_\alpha z_\beta \sum_k a_{k\alpha} a_{k\beta}.$$

Hence if the a's satisfy the so-called orthogonal relations

$$(43) \qquad \begin{cases} \sum_k a_{k\alpha} a_{k\beta} = 1 \ (\alpha = \beta), \\ \qquad\qquad\ = 0 \ (\alpha \neq \beta), \end{cases}$$

the condition (40) is satisfied for the z's, that is $z'_1, \ldots, z'_4$ are indeed the coordinates of a point z'.

By virtue of (43) we find $A = \pm 1$; if we take $A = +1$ we find $a_{k\alpha} = A_{k\alpha}$ the minor of $a_{k\alpha}$; also

$$(44) \qquad \begin{cases} \sum_k a_{\alpha k} a_{\beta k} = 1 \ (\alpha = \beta), \\ \qquad\qquad\ = 0 \ (\alpha \neq \beta). \end{cases}$$

We must subject the a's to another condition. When the point ζ is real, z_1, z_2, z_3 are real and z_4 is imaginary. Thus, as we wish the transformation (42) to convert a real point ζ into a real point ζ', we will take the a's according to the scheme

(45)

	$z_1 = \zeta_1$	$z_2 = \zeta_2$	$z_3 = \zeta_3$	$z_4 = i\zeta_4$
$z_1' = \zeta_1'$	C_{11}	C_{12}	C_{13}	iC_{14}
$z_2' = \zeta_2'$	C_{21}	C_{22}	C_{23}	iC_{24}
$z_3' = \zeta_3'$	C_{31}	C_{32}	C_{33}	iC_{34}
$z_4' = i\zeta_4'$	$-iC_{41}$	$-iC_{42}$	$-iC_{43}$	C_{44}.

Here the C's are real and the elements of this table satisfy the orthogonal relations by rows and by columns. We take $C_{44} > 0$.

Let (45) transform the two near-by points z, $z + dz$ whose distance apart is ds into the points z', $z' + dz'$ whose distance apart is ds'. Since the coefficients a in (42) are constants we see that the dz's transform the same as the z's. Since the orthogonal relations (43) now hold we see that $ds'^2 = ds^2$. Thus the linear orthogonal transformation (45) leaves all distances unaltered and hence also all angles. This is further confirmed by applying (45) to (28); we find at once that

$$\{a'\zeta'\} = \{a\zeta\}.$$

Obviously the transformation (45) does not transform a point ζ within the fundamental G sphere to one without it or on it. It is now not difficult to show that the tetrahedron τ can be made to coincide with any other tetrahedron τ' of the same character. In particular any triangle can be brought into the special position employed in § 5.

Let us briefly mention a few special cases of (45).

Example 1. $C_{44} = 1$, the other C's in the last row and column $= 0$. This defines a rotation about O in the e-sense.

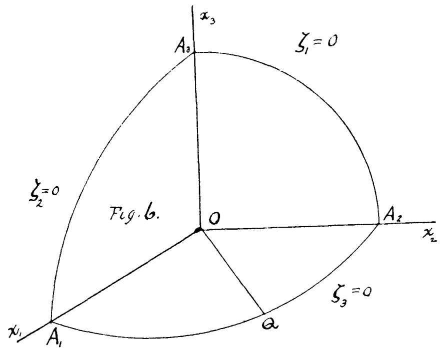
Example 2.

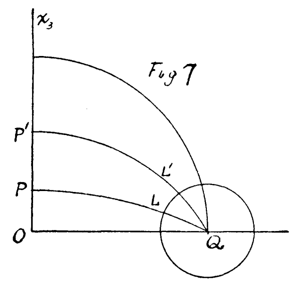
(46)
$$\begin{array}{c|cccc} & \zeta_1 & \zeta_2 & \zeta_3 & i\zeta_4 \\ \hline \zeta_1' & 1 & 0 & 0 & 0 \\ \zeta_2' & 0 & 1 & 0 & 0 \\ \zeta_3' & 0 & 0 & \cos i\theta & \sin i\theta \\ i\zeta_4' & 0 & 0 & -\sin i\theta & \cos i\theta, \end{array}$$

that is:

$$\zeta_1' = \zeta_1,\ \zeta_2' = \zeta_2,\ \zeta_3' = \zeta_3 \cosh\theta - \zeta_4 \sinh\theta,\ \zeta_4' = -\zeta_3 \sinh\theta + \zeta_4 \cosh\theta.$$

The Fig. 6 represents the tetrahedron τ. A plane $\alpha = \zeta_1 - g\zeta_2 = 0$ through the x_3 axis is unaltered. A plane $\beta = \zeta_3 - h\zeta_4 = 0$ through the edge A_1A_2 is rotated about it as an axis. Thus points on the x_3 axis are shifted along it, in such a manner, however, that points within G remain in it. The intersection of $\alpha = 0$, $\beta = 0$ is an H-straight as QP, Fig. 7, perpendicular to the x_3 axis. A point L on it is moved to L', while P goes to P'. As distances are unaltered $\overline{LP} = \overline{L'P'}$ in





H-measure. The points L at a given distance from the x_3 axis form an H-circle C whose centre is Q. If we rotate the plane α about the x_3 axis, the locus of C is a sort of torus. The equation of this surface is

$$A(\zeta_1^2 + \zeta_2^2) + B(\zeta_3^2 - \zeta_4^2) = 0, \tag{47}$$

for $\zeta_1^2 + \zeta_2^2$ is unaltered by the rotation about the x_3 axis, that is, by the following:

(48)
$$\begin{array}{c|cccc} & \zeta_1 & \zeta_2 & \zeta_3 & i\zeta_4 \\ \hline \zeta_1' & \cos\theta & \sin\theta & 0 & 0 \\ \zeta_2' & -\sin\theta & \cos\theta & 0 & 0 \\ \zeta_3' & 0 & 0 & 1 & 0 \\ i\zeta_4' & 0 & 0 & 0 & 1, \end{array}$$

and $\zeta_3^2 - \zeta_4^2$ is unaltered for (46). If we apply both (46) and (48) a point on the surface (47) describes a screwlike motion upon it. This surface is the analogue of the celebrated *Clifford Surface.* The rectilinear generators or Clifford parallels are here imaginary.