THE PLACE OF MATHEMATICS IN ENGINEERING PRACTICE

By Sir W. H. White.

The foundations of modern engineering have been laid on mathematical and physical science; the practice of engineering is now governed by scientific methods applied to the analyses of experience and the results of experimental research. The Charter of the Institution of Civil Engineers defines engineering as the "art of directing the great sources of power in Nature for the use and convenience of man." Obviously such direction can only be accomplished by engineers who possess an adequate acquaintance with "natural knowledge"—with the laws which govern these great sources of power. Obedience to natural laws is a condition essential to the full utilisation of the great sources of power. It is true, no doubt, that notable achievements in engineering were accomplished during the last century by men whose education was imperfect, whose mathematical and scientific knowledge was small, whose appeal to past experience gave little assistance in the solution of new problems. Their successors now enjoy greatly superior educational advantages; they can profit by enormous advances made in all departments of science and manufacture; they can study and criticise works done by their predecessors in the light of long subsequent experience; but even now there is room for surprise, if not for wonder, when one realises the great success attained by these early engineers.

The advantages obtainable by the combination of scientific training with practical experience are, however, in no way depreciated because, through force of circumstances, the pioneers of engineering had to do their work as best they could. Not a few of these men recognised the serious disadvantages resulting from their lack of scientific training and gave valuable assistance to a movement which ultimately led to the existing methods of training. George Stephenson, for example, who grew to full manhood practically uneducated, took care to secure for his son Robert Stephenson the advantages of a good elementary education which was followed by a period of practical training and then by a course of scientific study at Edinburgh University. The careers of both father and son were greatly influenced by this action, and it is of interest to note that the scheme which Stephenson framed and carried out for the professional training of his son—including the alternation of scholastic and engineering work—was, in its essential features, identical with that recommended in the Report of a representative Committee of British Engineers over which I had the honour to preside eight years ago. That Report has been approved by the leading Engineering Institutions of the United Kingdom and is now largely influencing the education of engineers.

The fundamental idea underlying the accepted system of training engineers consists in the combination of an adequate knowledge of the sciences which bear upon engineering with a thorough practical training on actual engineering works. No man is now entitled to admission as a Corporate Member of the Institution of Civil Engineers unless and until he has given proof of the possession of both these qualifications. Neither kind of training standing alone, or when developed disproportionately, can be regarded as satisfactory, or as meeting the needs of engineering practice. Formerly undue prominence was given to practical training and experience; while the facilities for scientific training were at first non-existent and for a long period were inadequate. Then came a better appreciation of scientific method and a great development of technical education, departments being established for the teaching of engineering science in Universities and University Colleges. The utilisation of these opportunities for instruction by considerable numbers of young men not unnaturally brought about a swing of the pendulum which went beyond reasonable limits. For a time there was a tendency to exalt unduly scientific education, and to depreciate the value of practical training. The hard pressure of experience has done much to adjust that disproportion. University graduates when they enter upon actual work soon discover that degrees in engineering, valuable as they undoubtedly are, require to be supplemented by thorough practical training. On the other hand, men who begin their engineering careers as pupils or assistants to practising engineers or as members of engineering office-staffs, become convinced that their limit of possible attainment must be low, unless scientific knowledge is added to practical experience. Under existing conditions new and difficult problems continually arise in all branches of engineering practice, and satisfactory solutions can only be found by bringing to bear upon these problems all the resources furnished by natural knowledge, accumulated experience and experimental research.

The full equipment of an engineer must include knowledge of other sciences besides the mathematical, but our present concern is exclusively with the latter. An adequate knowledge of mathematics must be possessed by every educated engineer, because he thus acquires valuable tools, by the use of which he can overcome difficulties that would otherwise be insuperable, as well as habits of thought and methods of rigorous investigation which are invaluable when he has to deal with novel and difficult undertakings. Apart from the employment of mathematics it would not be possible for the engineer to carry out designs and construction of engineering works, of structures and machines, capable of fulfilling their intended purposes and possessing both sufficient strength and durability. The days of blind reliance upon engineering formulae and "rules of thumb" are over. Syllabuses of instruction for the guidance of engineering students, standards established for degrees and diplomas in engineering science, conditions laid down as necessary qualifications for membership of great Engineering Societies, all furnish full recognition of the fact that an adequate knowledge of mathematics is essential to the successful practice of engineering.

It may be asked what range and character of mathematical knowledge can be described as adequate? Answers to this question are to be found in Calendars of Schools of Engineering which set forth detailed courses of study considered necessary for the attainment of degrees or diplomas. Identity of conditions does not exist in these Regulations, but a closer approach to uniformity has been reached as greater experience has been gained, and it is obviously desirable that further progress should be made in that direction. Engineering Degrees ought to be based on a common standard and to represent an equal attainment. These degrees, of course, should be regarded simply as certificates of knowledge of the fundamentals of engineering science; they do not cover all the mathematical knowledge requisite for the practice of particular branches of engineering, and in most branches a greater range of mathematics is necessary. Moreover a degree-course in engineering requires to be supplemented in all cases by subsequent practical training and experience, and in many cases by advanced or specialised courses of mathematical study going beyond the standards associated with degrees. In the settlement of these advanced courses the needs of each branch of engineering must be determined on the basis of experience; and the subject is one to be dealt with satisfactorily only on the basis of conference between practising engineers and mathematicians. The former know the needs which must be met: the latter can advise as to the best methods of meeting requirements.

Differences of opinion have always prevailed, and still exist, in regard to the methods by which mathematics should be taught to engineering students. Some authorities favour the arrangement of specialised courses of instruction—"mathematics for engineers" or "practical mathematics"—and advocate the creation of separate mathematical sections for engineering schools, even when these schools form departments of Universities or Institutions which possess well-organised mathematical Departments. Other persons, whose opinions are entitled to equal respect, believe that purely mathematical instruction is best given to engineering students by mathematicians, and that a similar rule should apply to instruction in other branches of science; because that method must lead to a broader view of science and a greater capacity for original and independent investigation than can be obtained by specialised teaching narrowed down to the known requirements of previous engineering practice. Personal experience and observation—as student, teacher and practical engineer—lead me to rank myself with the supporters of the latter method of teaching mathematics. No doubt, in the actual practice of engineers, there is room for "short cuts" and special methods in the use of mathematics; but I am convinced that during the period of education it is advantageous to follow ordinary methods of teaching and to leave specialisation for the time when the performance of actual work will almost inevitably lead each individual to make his choice of the branch of engineering to be followed, and of the methods which will best economise his labour and time in doing the work of calculation. The trend of professional opinion certainly lies in the direction of utilising, as far as possible, existing mathematical departments for the instruction of engineers. During the last year the subject has been exhaustively considered by the Governors of the Imperial College of Science and Technology, a special Committee having been appointed for that purpose. The Imperial College, as is well known, has been formed by bringing together the Royal College of Science, the Royal School of Mines, and the Engineering College founded and maintained by the City and Guilds of London Institute. The last-mentioned College at the outset had to be necessarily self-contained, and had its own Mathematical Department which was admirably organised and conducted by Professor Henrici over a very long period. The College of Science and Royal School of Mines also included a Department of

Mathematics and Mechanics of which Professor Perry has been the distinguished Director for many years. Both these Departments have justified their existence and done admirable work: but the development of the scheme of the Imperial College rendered it necessary to reconsider the subject of future mathematical instruction in the College as a whole. Alternatives taken into consideration were (1) the continuance of separate provision for engineering students; (2) the creation of a single department to be presided over by a mathematician of distinction, in which engineering students would receive their fundamental instruction in mathematics. After thorough investigation the latter course was preferred and will be carried into effect. Its adoption will in no way interfere with the teaching of special applications of mathematics as parts of the courses of instruction given by professors of engineering; and no one familiar with the training of engineers would consider such a change desirable.

A second example of the opinion which now prevails respecting the teaching of mathematics to engineers may be found in a valuable Paper by Professor Hopkinson of Cambridge University, published this year as one of a Series of Special Reports to the Board of Education on the teaching of Mathematics in the United Kingdom. Professor Hopkinson states that for good reasons, and during a considerable period after the Engineering Department was established at Cambridge, its students with few exceptions "got the whole of their instruction within the walls of the Engineering Laboratory," and "had not the full advantage of their position as students of Engineering Science in a centre of mathematical learning and research." In recent years, however, "the establishment of closer relations between the two studies (Mathematics and Engineering) has made great progress, and at the present time the students of Engineering get their foundation of Mathematics and of Elementary Mechanics from College Teachers, many of whom have graduated in the Mathematical Tripos."

It may be interesting to add that in a recent "Summary Report of the Teaching of Mathematics in Japan," Mr Fujisawa has discussed the "teaching of mathematics in technical education" in a brief but interesting fashion. While advocating the "practical" method of instruction, he is careful to explain that the form of utilitarianism which he recommends "has the potentiality of manifesting its usefulness wherever there is a necessity"—a condition which obviously cannot exist unless the engineer is endowed with a good knowledge of the principles and processes of mathematics.

More than seventy years ago, men who had received a mathematical and scientific training in the first British School of Naval Architecture, wrote as follows in vindication of the necessity for the liberal education of those engaged in the designing of ships:—"The study of naval architecture brings early conviction to the mind of the constructor that he can trust little or nothing to *a priori* reasoning. He uses the exact sciences it is true, but uses them only as a means of tracing the connection between cause and effects, in order to deduce principles that may be applied to his future works, with a certainty of producing the results he contemplates." This utterance may be applied with equal force to the practice of other branches of engineering; and, amongst the exact sciences which play so important a part in successful achievement, mathematics certainly hold the first place. The "complete engineer" of the earlier and simpler periods can exist no longer under modern conditions. Even the ablest men are driven to specialise in practice: but whatever branch of engineering may be selected, the worker will need that fundamental training in mathematics to which allusion has been made. Few engineers engaged in professional work have opportunities of prosecuting mathematical studies systematically, although they are continually using mathematical tools provided during their college careers, and not infrequently have occasion to add to their mathematical equipment in order to meet new demands, or to go beyond precedent and experience. When one considers the great responsibilities which practising engineers have to bear, it is not surprising to find that they, as a class, have made comparatively few contributions to the advancement of mathematical science, although they have been well trained in mathematics and continually apply that knowledge. There are, of course, exceptions to this rule; indeed, I have known engineers who turned to mathematics as a recreation, but these men are exceptional. Another group whose members have done notable work of a mathematical nature have been trained as engineers, but have either passed out of practice to an extent which left them ample leisure or have become professors of engineering. The names and work of engineers like Rankine, Froude and John Hopkinson will always be held in honour by mathematicians as well as by members of the profession which they adorned. The labours of many mathematicians who have devoted themselves to the tuition of engineers, and after becoming acquainted with the problems of engineering have done splendid work in the formulation of mathematical theories on which have been based valuable rules of a practical nature, also deserve and always receive the grateful appreciation of engineers. But, speaking broadly, there is a real and abiding distinction between the engineer, however accomplished he may be in mathematical science, and the mathematician however well informed he may be in regard to engineering practice. The mathematician necessarily regards engineering chiefly from the scientific point of view, and although he may aim at advancing engineering science, he is primarily concerned with the bearing of mathematics thereon. The engineer, being charged with actual design and construction of efficient and permanent works in the most economical manner possible, must put considerations of a practical and utilitarian character in the forefront; and while he seeks to utilise the aids which mathematical and physical science can render, his chief aim must always be to achieve practical and commercial success. There is obviously room for both classes, and their close and friendly collaboration in modern times has produced wonderful results. Fortunately the conditions which formerly prevailed have ceased to exist. Much less is said about the alleged distinction between pure and applied science, or about the comparative untrustworthiness of theory as compared with practical experience. Fuller knowledge has led to a better understanding of what is needed to secure complete success in carrying out great engineering works on which the comfort, safety, economical transport and easy communications of the civilized world so largely depend. The true place of mathematics in engineering practice is now better understood, and it is recognised to be an important place, although not so important as was formerly claimed for it by mathematicians.

The character of the change which has taken place in the use of mathematics in connection with engineering practice may be illustrated by reference to that branch of engineering with which my life-work has been connected. It is probably true to say that no branch of engineering has benefited more from mathematical assistance than naval architecture has done, and naval architects undoubtedly require to have at least as intimate a knowledge of mathematics as any other class of engineers. Moreover it was one of the first branches of engineering for which the foundation of a mathematical theory was attempted in modern times; and these attempts were made by men who in their day and generation were recognised to be in the first rank of mathematicians. The work which they did is now almost forgotten, but it laid the foundation for the science of naval architecture as it exists to-day. To France belongs the honour of having given most encouragement to men of science to attack these problems, and the Academy of Sciences aided the movement greatly by offering prizes which brought into the field not a few of the ablest European mathematicians during the latter half of the eighteenth century. Few of these mathematicians had personal knowledge of the sea or ships, and their investigations were influenced by these limitations. Others had made long voyages; like Bouguer, who (in 1735) proceeded to Peru *pour la mesure de la Terre,* and as a consequence of that experience a very practical tone was given to his famous *Traité du Navire,* which was published in 1745. It would be interesting to sketch the valuable work done by this single mathematician, but time does not allow me to do so. My main purpose at present is to illustrate the change that has since taken place in the use of mathematics in attacking engineering problems; and this may be done better by taking a single problem and showing how it was dealt with in the eighteenth and the nineteenth centuries.

Daniel Bernoulli in 1757 won the prize offered for the second time by the Académie Royale des Sciences for an answer to the question:—What is the best means of diminishing the rolling and pitching of a ship without thereby sensibly depriving her of any of the good qualities which she should possess? His *Mémoire* was published subsequently; it is an admirable piece of work, and deals thoroughly with the stability of ships; but here Bernoulli had been anticipated by Bouguer and made acknowledgment of the fact. Greater originality was shown in a mathematical investigation of the behaviour of ships in a seaway; and in a consideration of the influence of wave-motion upon the conditions of fluid pressure, as well as the determination of the instantaneous position of equilibrium for a ship floating amongst waves. Bernoulli recognised that the particles of water in a wave must be subjected to horizontal as well as vertical accelerations, although in his mathematical expressions he took account of the latter only. He emphasised the important influence which the relative sizes of waves and ships must have upon rolling and pitching motions, and advised that attention should be mainly devoted to cases where ships were small in proportion to the waves they encountered. In this particular he departed from assumptions usually made by his contemporaries and anticipated modern views. Bernoulli also dwelt upon the critical case wherein ocean waves, forming a regular series, have a period synchronising with the period of still-water rolling of the ship which they meet. A gradual accumulation of angular motion was shown to be inevitable in such circumstances, and it was remarked that the consequent rolling motions must be considerable and might possibly become dangerous in their extent. Bernoulli recommended the conduct of experiments to determine the periods of oscillation of ships in still water, and described methods of conducting these experiments. He also insisted upon the necessity for making accurate observations of the rolling of ships when amongst waves, and made other suggestions of much practical value, which have since been repeated by writers unfamiliar with Bernoulli's work and have been practically applied. Unfortunately the neglect by Bernoulli in his mathematical investigations of the horizontal accelerations of particles of water, which he recognised as existing in waves, led to erroneous conclusions in regard to the instantaneous position of equilibrium for a ship when floating amongst large waves and the best means for securing steadiness. Bernoulli considered that when a ship was in instantaneous equilibrium, her centre of gravity and the centre of buoyancy—i.e. the centre of gravity of the volume of water instantaneously displaced by the ship—must lie in the same vertical line. This condition of course holds good for a ship floating at rest in still water, but not for a ship floating amongst waves of large relative dimensions. Bernoulli deduced from his investigations a practical rule for the guidance of naval architects: viz. that in order to minimise rolling, ships should be designed so that their centres of buoyancy when they were upright and at rest should be made to coincide with the centre of gravity. He considered that ships should be made deep, that large quantities of ballast should be used, and that the cross-sections should be approximately triangular in form. This practical rule was misleading, and if applied in a design might be exceedingly mischievous in its effect on the behaviour of ships. Bernoulli himself foresaw that, in certain cases, his rule would work badly, but he considered that these would but rarely occur. It is now known that this view was mistaken.

The detailed mathematical investigations contained in Bernoulli's *Mémoire* are still of much interest; they included the examination of cases in which were assumed widely differing ratios of the natural periods of ships to the period of the waves producing rolling motion. Throughout, the motions of ships were supposed to be unaffected by the resistance of the surrounding water, but Bernoulli did not overlook the steadying effect which water-resistance would exercise on a ship in a seaway; on the contrary he recognised the influence which changes in the underwater forms of ships must have upon the amount and steadying effect of that resistance, and he recommended the use of side-keels in order to minimise rolling. Having regard to the state of knowledge at the time this *Mémoire* appeared, it was undoubtedly a remarkable piece of work and it well deserved the reward bestowed by the Academy. It contained many practical suggestions for experimental enquiry and for guidance in the preparation of designs for ships; but it was essentially a mathematical study and had little influence on the work of naval architects.

A century later the same problems were attacked by William Froude, a graduate of Oxford University and an engineer of experience in constructional work. As an assistant to Isambard Brunel, the attention of Froude had been directed to these subjects in connection with the design and construction of the *Great Eastern*, a ship of relatively enormous dimensions and novel type, respecting whose safety, manageability and behaviour in heavy seas serious doubts had been expressed. Like Bernoulli, of whose work I feel confident Froude had no knowledge, the modern investigator perceived that, amongst waves, there must be considerable variations in the direction and magnitude of the pressure delivered by the surrounding water on the surface of a ship's bottom; and that the instantaneous position of equilibrium for a ship exposed to the action of waves of large relative dimensions must be discovered if a theory of rolling was to be framed. Froude worked out a complete theory of trochoidal wave-motion and enunciated the principle of an "effective wave-slope." In his investigations it was assumed that the resultant water-pressure on the ship at each instant acted through the centre of buoyancy and normally to the effective wave-slope. In the differential equation framed for unresisted rolling, Froude took a curve of sines for the effective wave-slope instead of a trochoid. Having obtained the general solution of that equation, he proceeded to consider the behaviour of ships as influenced by variations in the ratios of their still-water periods of rolling oscillation to the relative periods of the waves encountered. In this manner the particular cases considered by Bernoulli were readily investigated, and many of the broad deductions made a century before were amended. The critical case of synchronism of ship-period and wave-period which Bernoulli had brought into prominence was shown to be that requiring most consideration. For that case the increment of oscillation produced by the passage of each wave of a regular series was determined on the hypothesis of unresisted rolling, and was shown to be about three times as great as the maximum inclination to the horizontal of the effective wave-slope. It was also made clear that apart from the influence of water-resistance, such synchronism of periods must lead to a ship being capsized by the passage of comparatively few waves. Up to this point, the investigation made by Froude was strictly mathematical, and the modern engineer who had received a thorough mathematical training had reached results superior to those obtained by the famous mathematician a century before; becoming, in fact, the founder of a theory for the oscillations of ships amongst waves which has been universally accepted. Like Bernoulli, Froude became impressed with the necessity for experiments which would determine the periods of still-water rolling for ships; and with the desirability for making observations of the rolling of ships in a seaway. In addition he emphasised the necessity for more extensive observations on the dimensions and periods of sea-waves, a subject which had been investigated to some extent by Dr Scoresby and other observers, but had been left in an incomplete state. One great generalisation was made by Froude at an early period in this important work, and it has since become a fundamental rule in the practice of naval architects; viz. that freedom from heavy rolling under the conditions usually met with at sea was likely to be favoured by making the period of still-water rolling of ships as large as was possible under the conditions governing the designs. This rule was the exact converse of that laid down by Bernoulli, as the effect of the latter rule by increasing the stability would have lessened the period. The explanation of this simple rule is to be found in the consideration that the longer the natural period of a ship is, the less likely is she to encounter waves whose period will synchronise with her own.

Purely mathematical treatment of the subject did not satisfy the mind of a trained engineer like Froude. For practical purposes it was essential that the effect of water-resistance to rolling should be determined and brought into the account. Here purely mathematical investigation could not possibly provide solutions; experimental research, conducted in accordance with scientific methods, became necessary. Aided by the Admiralty, Froude embarked upon a series of experiments which extended over several years. Most of these experiments were made on actual ships, but models were employed in special cases. In the analysis of experimental results, mathematics necessarily played a great part; indeed without their employment, the proper deductions could not have been made. On the basis of these analyses, Froude obtained valuable data and determined experimentally "coefficients" of resistance to rolling experienced respectively by the flat and curved portions of the immersed surfaces of ships. Furthermore he demonstrated the fact that the surface disturbance produced by the rolling of ships in still-water accounted for a large part of the extinctive effect which was produced when a ship which had been set rolling in still water was allowed to come to rest. In this way, and step by step, Froude devised methods by means of which naval architects can now calculate with close approximation the extinctive effect of water-resistance for a new design. Finally, Froude produced a method of "graphic integration," the application of which in association with the calculation of the effect of water-resistance, enables a graphic record to be constructed showing the probable behaviour of a ship when exposed to the action of successive waves, not merely when they form a regular series, but when they are parts of an irregular sea. Subsequent investigators have devised amendments or extensions of Froude's methods, but in all essentials they stand to-day as he left them—a monument of his conspicuous ability, and an illustration of the modern method in which mathematics and experimental research are associated in the solution of engineering problems which would otherwise remain unsolved.

In tracing as has been done the contrast between the methods of Bernoulli and Froude, an indirect answer has been given to the question—What is the true place of mathematics in engineering practice? It has been shown that even in the hands of a great mathematician, purely mathematical investigation cannot suffice, and that Bernoulli became convinced, in the course of his study of the behaviour of ships in a seaway, that no complete or trustworthy solution could be found apart from experimental research, as well as careful observations of ocean waves and the rolling of actual ships. Bernoulli was not in a position to undertake, or to lead others to undertake, these experiments and observations. In his mathematical investigations he made, and necessarily made, certain assumptions which are now known to have been incorrect. Even the most accurate mathematical processes, when applied to equations which were framed on imperfect or incorrect assumptions, could not produce trustworthy results; and consequently the main deductions made by Bernoulli, and the rules recommended by him for the guidance of naval architects, would have led to disappointment if they had been applied in practice. On the other hand, Froude, himself a great experimentalist, was fortunately able to impress upon the British Admiralty through the Constructive Staff the importance of making experiments and extensive observations of wave-phenomena and the behaviour of ships. Not merely did Froude devote many years of personal attention to these subjects, but he was aided over a long period by the large resources of the Royal

Navy. Similar work on a very large scale was also done simultaneously by the French Navy. Some of my earliest experiences at the Admiralty forty-five years ago were gained in connection with these observations and experiments, so that I speak from personal knowledge of the influence which Froude exercised, the inspiration of his great devotion and wonderful initiative. As a result of all these efforts, a great mass of experimental data was accumulated; the results of a large number of observations were summarised and analysed; and, in the end, the soundness of the modern theory was established, and the future practice of naval architects was made more certain in their attempts to produce designs for ships which should be steady and well-behaved at sea.

At the risk of making this lecture appear to be chiefly a notice of work done by William Froude, or a summary of the advances made in the science of naval architecture, another illustration will be given of the general principle laid down in regard to the place of mathematics in engineering practice.

Mathematicians, from an early date, were attracted by the subject of the resistance offered by water to the motion of ships and made many attempts to frame satisfactory theories. The earliest investigations were based upon the assumption that the immersed surface of a ship's skin could be treated as if it consisted of an aggregation of elements, each of which was a very small plane area, set at a known angle of obliquity to the direction of motion through the water. For each elementary plane area it was proposed to estimate the resistance independently of the others, and as if it were a small isolated flat plate. The integration of such resistances over the whole surface was supposed to represent the total resistance of the water to the motion of the ship at a given speed. Certain further assumptions were made in regard to the laws connecting the resistance of each unit of area with its angle of obliquity to the direction of motion and with the speed of advance. The effect of friction was, in most cases, neglected; nor was any account taken of surface disturbance produced by the motion of the ship. It is unnecessary to dwell upon the errors and incompleteness of these assumptions. So long as ships were propelled by sails little practical importance attached to an exact determination of the resistance experienced at a certain speed. When steamships came into use it was of primary importance to have the power of making close approximations to that resistance because estimates for the engine-power required to attain a given speed had to be based thereon. The subject received great consideration, as the result of which certain simple rules were framed and commonly employed in making estimates for the engine-power to be provided in new ships. These rules were mainly based on the results obtained by trials of existing vessels; and these trial-results, of course, included not merely the effect of water-resistance—as influenced by the form and condition of the immersed surface of a ship—but were also affected by the varying efficiency of the propelling apparatus and propellers. Many attempts were made to separate these items of performance and to determine the actual amount of the resistance for a ship and the separate efficiencies of her propellers and machinery. Little progress was secured until 1868. Mathematical theories were framed, it is true, for estimating the efficiency of propellers; but while these theories were accurate enough if the assumptions underlying them had been complete and representative of actual phenomena, there was no possibility of fulfilling those conditions since the phenomena were neither fully known nor understood.

In 1868 a special and representative Committee—including Rankine and Kelvin—appointed by the British Association, made a Report on this subject and recommended that towing experiments should be made on full-sized ships. The Committee was almost unanimously of opinion that the only method which would give trustworthy information in regard to the resistances experienced at various speeds was to tow actual ships and not to depend upon models. William Froude dissented from this conclusion and recommended model experiments. Accepting the stream-line theory of resistance which Rankine had introduced, Froude based upon it a system of experiment which dealt separately with frictional resistance and applied to the residual resistance—after friction had been allowed for and deducted—a law of "corresponding speeds" between models and full-sized ships which he had worked out independently. That law had been previously recognised in a more general form by mathematicians, and had been investigated for this particular case by a French mathematician, M. Reech, of whose work Froude was then ignorant. By this happy association of mathematical theory with experimental research, Froude placed in the hands of naval architects the means of solving problems which could not be dealt with either by purely mathematical investigation, or by experience with actual ships. Experimental tanks of the character devised by Froude have now been multiplied in all maritime countries. The latest and in many respects the best of these tanks, which is due to the generosity of Mr Alfred Yarrow, is a Department of the British National Physical Laboratory. The operations of these tanks have resulted in a great addition to natural knowledge and have secured enormous economies of fuel. The success achieved in connection with modern developments of steam navigation and the attainment of very high speeds is chiefly due to tank experiments which have involved relatively small cost, and enabled naval architects to choose for every design the form which gives the least resistance possible under the conditions laid down for a new ship, even when the size or speed required go beyond all precedent. Considerations of stability, carrying capacity, available depths of water, dimensions of dock entrances and other matters, as well as speed and fuel consumption, may limit the designer and narrow the alternatives at his disposal; but ordinarily there is room for considerable variations of form in a new design, and in making the final selection of form it is essential that the designer should know how the resistances of these permissible alternatives compare. Naval architects throughout the world enjoy great advantages in this respect over their predecessors, and owe their position entirely to the genius and persistence of William Froude.

Since the work of Froude in this direction was done, model experiments have become the rule in many departments of engineering and the scientific interpretation of the results has greatly influenced the designs of structures and machines. Prominent amongst these recent applications of experimental research on models stand those relating to air-resistance and wind-pressure on bridges and other structures. In regard to the laws of wind-pressure much has been discovered in recent years, and in connection with the effects of wind-pressure on engineering structures especial reference ought to be made to the work done by Dr Stanton at the National

Physical Laboratory. All engineers owe a debt of gratitude to that distinguished experimentalist and to the Institution where he works; and they recognise the fact that he has demonstrated the trustworthiness of deductions made from tests with small models exposed to the action of air-currents when applied on the full scale to complicated structures for which independent mathematical calculations of the effect of wind-pressure could not be made. The late Sir Benjamin Baker, who was chiefly responsible for the design of the Forth Bridge, was one of the first to appreciate and make use of this experimental system, and no engineer of his time more frankly admitted than he did what a debt engineering practice owed to mathematics when used in the proper manner.

The proper use of mathematics in engineering is now generally admitted to include the following steps. First comes the development of a mathematical theory, based on assumptions which are thought to represent and embody known conditions disclosed by past practice and observation. From these theoretical investigations there originate valuable suggestions for experimental enquiries or for careful and extensive investigations. The results obtained by experimental research or from observation and experience must be subjected to mathematical analysis: and the deductions made therefrom usually lead to amendments or extensions of the original theory and to the device of useful rules for guidance in practice. Purely mathematical theories have served and still serve a useful purpose in engineering; but it is now universally agreed that the chief services of mathematics to engineering are rendered in framing schemes for experimental research, in analysing results, in directing the conduct of observations on the behaviour of existing engineering works, and in the establishment of general principles and practical rules which engineers can utilise in their daily professional employment.

One of the most recent examples of this procedure is to be found in the constitution and proceedings of the Advisory Committee appointed in 1909 by the British Government in connection with the study of Aeronautics. Its membership includes distinguished mathematicians, physicists, engineers and officers of the army and navy, and its President is Lord Rayleigh, Chancellor of the University of Cambridge. The declared intention in establishing this Committee was to bring the highest scientific talent "to bear on the problems which have to be solved" in order to endow the military and naval forces of the British Empire with efficient aerial machines. The Reports published during the last two years are of great value; the work done by the Committee—described as "the scientific study of the problems of flight with a view to their practical solution"—has been accompanied and supplemented by research and experiment carried out by the Director of the National Physical Laboratory (Dr Glazebrook) and his staff in accordance with a definite programme approved by the Committee. These investigations necessarily cover a very wide field in which there is ample room for the operations of all the branches of science and engineering represented on the Committee, and there can be no doubt that already the influence of the work done has been felt in practice. No one who has followed the progress made in aerial navigation, however, can fail to be convinced that although a considerable amount of purely mathematical investigation has been devoted to the problems of flight, it has hitherto had but little influence on practice, in comparison with that exercised by improvements due to mechanical engineering—tending to greater lightness of the engines in relation to their power—and by actual experiments made with models and full-sized flying machines. A stage has been reached, no doubt, where the interpretation by mathematicians of the experimental results available and their suggestions as to the direction in which fuller experimental research can best be carried out are of great importance, and that fact is universally recognised by engineers.

Even when the fullest use has been made of mathematical science applied in the best way and of experimental research there still remain problems which have hitherto defied all efforts at their complete solution, and engineers have to be content with provisional hypotheses. Of the James Forrest Lectures given annually at the Institution of Civil Engineers a long series has been devoted to the description of "Unsolved Problems in Engineering." Mathematicians seeking fresh fields to conquer might profitably study these utterances of practising engineers of repute. On this occasion it must suffice to mention two classes of subjects on which additional light is still needed, although they are now less obscure than they have been in the past, thanks to long years of work and experiment.

The first group has relation to the laws which govern the efficiency of screw-propellers when applied to steamships, and has long engaged the attention of a multitude of writers in all maritime countries. Many mathematical theories have been published, which are of interest and value as mathematics, and are sound if the fundamental assumptions made could be accepted. It is, however, no exaggeration to say that at the present time there exists no mathematical theory which has any considerable influence on the design of screw-propellers and the determination of the form, area and pitch. Experience and experiment are still mainly depended upon when work of that kind has to be undertaken. Of course certain mathematical principles underlie all propeller designs, but the phenomena attending the operation of a screw-propeller at the stern of a ship in motion are too variable and complex to be represented by any mathematical equation even if they were fully known and understood—which they are not. The water in which a propeller works has already been set in motion by the ship before it reaches the propeller, and the "wake" of a ship in motion is in a very confused state. The action of a propeller upon the water "passing through it" and the manner in which its effective thrust is obtained still remain subjects for discussion and for wide differences of opinion between mathematicians and experimentalists who have seriously studied them. Froude initiated a system of model-experiments for propellers, both when working in open water and when attached to and propelling ships or developing an equivalent thrust. His son, Mr R. E. Froude, has done remarkable work in the same direction, and many other experimentalists have engaged in the task: but after more than seventy years of experience in the practical use of screw-propellers we remain without complete or accurate knowledge which would enable the designs of propellers for new ships, of novel types, or of very high speed, to be prepared with a certainty of success. On the whole, naval architects and marine engineers depend largely upon the results of experience with other ships. Although model-experiments are also utilised, there is not the same confidence in passing from results obtained with model propellers to full-sized propellers as there is in passing from model ships to full-sized ships. Probably this fact is chiefly due to essential differences in the reactions between the water and the surfaces of models and the surfaces of full-sized screws moving at the rates of revolution appropriate to each. These matters are receiving and have already received careful study not merely by the Superintendents of Experimental Tanks, but by practising engineers like Sir Charles Parsons and Sir John Thornycroft. The phenomenon of "cavitation"—which has been described as the breaking-away of water from the screw surfaces when the rate of revolution of the screw exceeds certain limits, and when the thrust per unit of area on the screw exceeds certain values—is one which has given much trouble in the cases of vessels of exceedingly high speed such as destroyers and swift cruisers. It is being investigated experimentally, but up to the present time no general solution has been found. In existing conditions surprising differences in the efficiency of propellers have been produced by what appeared to be small changes in design. On the whole the largest improvements have been obtained as the result of full-scale trials made in ships, although model-experiments have been of service in suggesting the direction in which improvements were probable. In my own experience very remarkable cases have occurred, and not infrequently it has been difficult even after the event to explain the results obtained. One such case may be mentioned as an illustration. A large cruiser obtained the guaranteed speed of 23 knots on trial with a development of about 30,000 horse-power. I had anticipated a higher speed. Progressive trials made at various speeds showed that the "slip" of the propellers became excessive as the maximum speed was approached, although the blade area given to the propellers on the basis of past experience was adequate for the power and thrust. The blade-area was increased about 20 per cent., the diameter and pitch of the screw-propellers being but little changed. With these new propellers the maximum speed became 24 knots and 23 knots was obtained with a development of 27,000 horse-power, as against 30,000 horse-power required with the original screws. The increase of blade area necessarily involved greater frictional losses on the screws, yet the effective thrust was increased, a higher maximum speed was attained, and the power required at all speeds became less than in the earlier trials down to 15 knots. This incident could be paralleled from the experience of many naval architects, and it illustrates the uncertainties which still have to be faced in steamship-design when unprecedented speeds have to be guaranteed.

This open confession of lack of complete knowledge, made in the presence of the professors of an exact science such as mathematics, may be thought singular. It is the fashion to criticise, if not to condemn, designers of ships and their propelling apparatus on the ground that after long experience there ought to be a complete mastery of these problems. That criticism, however, is hardly fair; because it overlooks the fact that throughout the period of steam navigation there has been incessant change in the dimensions, forms and speeds of ships and in the character of the propelling apparatus.

Knowledge is also still incomplete, and possibly complete knowledge will never be attained, in regard to the stresses experienced by the structures of ships at sea, when driven through waves and made to perform rolling, pitching and heaving movements simultaneously. The subject has long engaged the attention of mathematicians and naval architects. Early in the last century Dr Young made a study of the causes of longitudinal bending in wood-built ships, and presented a Memoir to the Royal Society. The eminent French geometrician Charles Dupin also dealt with the subject; which had great practical interest at a date when serious "hogging" or "arching" of ships was a common occurrence. Since iron and steel have been available for ship-construction—and as a consequence the dimensions, speeds and carrying powers of ships have been enormously increased—questions of a similar character have arisen on a larger scale, and have been carefully studied. There is much in common between ship construction and bridge construction under modern conditions; and because of this resemblance engineers practising in works of a constructional nature on land have been brought into close relation with the structural arrangements of ships. Sir William Fairbairn, who was associated with the younger Stephenson in the construction of the Menai and Conway tubular bridges, and Isambard Brunel, whose chief work was on railways, but who designed the famous steamships *Great Western, Great Britain* and *Great Eastern,* are amongst the men of this class who have most influenced shipbuilding. There are, however, obvious and fundamental differences between the conditions of even the greatest bridge founded on the solid earth, and those holding good in the case of self-contained floating structures carrying great loads across the sea, containing powerful propelling apparatus, and necessarily exposed to the action of winds and waves. In the former case bending moments and shearing stresses which must be provided against can be closely estimated, and ample strength can be secured by adopting proper "factors of safety." In the case of ships no similar approximations are possible; because their structures are stressed not only by the unequal distribution of weight and buoyancy, but have to bear rapidly varying and compound stresses produced by rolling and pitching motions, by external water-pressure and by the action of the propelling apparatus, as well as to resist heavy blows of the sea. Inevitably, therefore, the naval architect has to face the unknown when deciding on the "scantlings" of various parts of the structure of a new ship the design of which goes beyond precedent.

Mathematicians have had the courage to attack these problems and to propound theories respecting them. Professor Kriloff of the Imperial University, St Petersburg, has been one of the latest workers in this field, and has probably carried the mathematical theory furthest: but his work, like that of his predecessors, has had little effect on the practice of naval architects. Indeed it seems too much to expect that even the most accomplished mathematician can deal satisfactorily with the complex conditions which influence the variable stresses acting, from moment to moment, on the structure of a ship at sea. In these circumstances naval architects have been compelled to fall back upon experience with ships which have been long in service at sea, and to obtain the best guidance possible from the application of mathematics to the analysis of that experience and to the device of rules of a comparative nature. In general this procedure has led to the construction of ships which have possessed ample strength, although the actual margin of strength in excess of the permissible minimum has not been ascertained. In the comparatively few cases wherein weaknesses have been brought to light on service, scientific analysis has enabled even more valuable lessons to be deduced. But it cannot be said that purely mathematical investigation has been of great service to this branch of engineering.

Rankine many years ago proposed to base comparisons of the longitudinal bending moments and shearing stresses of ships amongst waves on the hypothesis that the distribution of weight and buoyancy should be determined for two extreme cases: first when a ship was momentarily resting in equilibrium on the crest of a wave having a length equal to her own length, and a height (hollow to crest) as great as was likely to occur in a seaway—say one-twentieth of the length of the wave. Second when she was momentarily floating astride a hollow of such waves. It was recognised, of course, that these cases were purely hypothetical, but the hypothesis has proved of great value in practice. Attempts have been made to carry the calculations further, and to take account of the effects of rolling, pitching and heaving motions, and of variations in the direction and amount of water-pressure consequent on wave-motion. Practice has been influenced but little by these attempts, which have necessarily been based on more or less arbitrary assumptions themselves not free from doubt. On the other hand Rankine's method has been widely used by naval architects; and the accumulated results of calculations obtained for ships whose reputations for strength and seaworthiness were good, are now available for reference. For new designs calculations of a similar character are made, and by comparison of results with those obtained for completed ships, most closely approaching the new design in type and dimensions, the principal scantlings are determined for various parts of the structure. In calculating the strengths of ships, they are usually treated as hollow girders exposed to the action of forces tending to cause longitudinal bending. The bending moments and shearing stresses calculated for the two extreme hypothetical conditions above described, are used in order to estimate the maximum stresses corresponding thereto in any members of the new ship's structure. A comparison of these maximum stresses (per unit of area of material) with the corresponding figures for successful ships is taken as a guide for determining the sufficiency or insufficiency of the scantlings proposed for the new vessel. In providing for adequate transverse strength and for margins of strength to meet local requirements, naval architects make separate calculations, but in these cases also are guided chiefly by comparisons based on actual experience with other ships. Mathematicians may regard this procedure as unsatisfactory: but they may be assured that any suggestions for improved or more exact methods which may be made will be welcomed by naval architects provided they command confidence and are capable of practical application.

In considering the relation of mathematics to engineering practice one important fact should always be borne in mind. The mission of engineers as a class is to produce results, "to do things," which shall be of practical service to humanity, and shall ensure safety of life and property. Complete solutions of problems, in the mathematical sense, are not usually to be found by engineers; they have to be content, in many cases, with partial solutions and fairly close approximations. It may be taken for granted that engineers desire to perform efficiently the duties laid upon them and that they are ready to avail themselves of all assistance which can be rendered by contemporary science, and by mathematicians in particular. On their behalf it has been my endeavour on this occasion to make suitable acknowledgment of the debt which engineering already owes to mathematics, and to indicate a few of the many problems in which further assistance is needed. All members of the engineering profession will endorse my expression of the hope that the close and friendly relations which have long existed between mathematicians and engineers, and which have yielded excellent results during the past century, will always continue and in future be productive of even greater benefits.