BOUNDARY PROBLEMS IN ONE DIMENSION

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§ 1. *Introduction.*

By a boundary problem in one dimension I understand primarily the following question:

To determine whether an ordinary differential equation has one or more solutions which satisfy certain terminal or boundary conditions, and, if so, what the character of these solutions is and how their character changes when the differential equation or the boundary conditions change*. This is the central problem, of which various modifications are possible. In its simplest forms this question is as old as the subject of differential equations itself. By the end of the nineteenth century it already had a considerable literature, which since that time has expanded rapidly. I shall try during the present hour to indicate some of the greatest advances made both as to results attained and methods used. In thus trying to get a brief and yet comprehensive survey of a large subject, the desirability of a thorough correlation of the parts becomes doubly apparent, and I trust that you will find that in this respect I have succeeded at a few points in adding something to what was to be found in the literature. The older results will be discussed in detail only so far as may seem necessary to make the scope and importance of the more recent ones intelligible.

The subject is so large that I must limit myself to certain central aspects of it by leaving out of consideration almost entirely

(1) Non-linear boundary problems, that is cases in which the differential equation or the boundary conditions or both are non-linear.

(2) Cases in which two or more parameters enter. (Klein's theorem of oscillation with its extensions.)

(3) Cases where we have to deal not with a single differential equation but with systems of differential equations.

(4) Cases in which the differential equation has singular points in or at the ends of the interval with which we deal, or, what is essentially the same thing, cases in which this interval extends to infinity.

All of these cases are of the highest importance.

* The question of finding effective means for computing the solutions in question is also one which might well be considered here.

An even more sweeping restriction than any of these is indicated by the very title of the lecture. This restriction to one dimension, i.e. to ordinary rather than partial differential equations, is made absolutely necessary by the time at my disposal if we are actually to reach the deeper lying parts of the subject. Fortunately the one-dimensional case may be regarded to a very large extent as the prototype of the higher cases; but in this simple case methods are available which enable us to go far beyond the point which we can hope at present to reach for partial differential equations.

The problem with which we deal is, then, this:

A linear differential equation which, for the sake of simplicity, I write as of the second order,

$$P(u) \equiv \frac{d^2u}{dx^2} + p_1 \frac{du}{dx} + p_2 u = r \quad \text{.......................(1)},$$

has coefficients p_1, p_2, r which are continuous functions of the real variable x in the finite interval

$$a \leqq x \leqq b \quad \text{..................................(X)}.$$

We wish to solve this equation subject to the linear boundary conditions

$$\left.\begin{aligned} W_1(u) &\equiv \alpha_1 u(a) + \alpha_1' u'(a) + \beta_1 u(b) + \beta_1' u'(b) = \gamma_1 \\ W_2(u) &\equiv \alpha_2 u(a) + \alpha_2' u'(a) + \beta_2 u(b) + \beta_2' u'(b) = \gamma_2 \end{aligned}\right\} \quad \text{.........(2)},$$

where the α's, β's, γ's are constants.

Why is this problem an important one? The most obvious answer is that it is one of which special cases come up constantly in applied mathematics; that even its special cases are of sufficient difficulty to have demanded the serious attention of the best mathematicians for nearly two hundred years; that in connection with this problem methods and results of large scope have been developed. From another and more abstract point of view also this problem may claim importance: it is one of the simplest and most natural generalizations of that most central of all subjects, the theory of a system of linear algebraic equations. This is a fact which has been known ever since John and Daniel Bernoulli in their treatment of vibrating strings replaced the uniform string by a massless one weighted at equal intervals by heavy particles. The effect of this was to replace the differential equation for determining the simple harmonic vibrations of the string, which is a special case of (1), and the boundary conditions, which come under (2), by a system of linear algebraic equations.

The idea involved in this physical example may be formulated more generally as follows:

We may replace (1) by a *difference* equation of the second order:

$$L_i u_{i+1} + M_i u_i + N_i u_{i-1} = R_i \qquad (i = 1, 2, \dots n-1) \quad \text{...........}(\bar{1}),$$

and the boundary conditions by

$$\left.\begin{aligned} A_1 u_0 + A_1' u_1 + B_1 u_{n-1} + B_1' u_n &= C_1 \\ A_2 u_0 + A_2' u_1 + B_2 u_{n-1} + B_2' u_n &= C_2 \end{aligned}\right\} \quad \text{.................}(\bar{2}).$$

The equations $(\bar{1})$ and $(\bar{2})$ taken together form a system of $n+1$ linear algebraic equations for determining $u_0, u_1, \dots u_n$. If now we allow n to become infinite, causing the coefficients of $(\bar{1})$ and $(\bar{2})$ to vary in the proper way, we easily obtain the system (1), (2) as the limiting form.

In the same way the linear boundary problem for a differential equation of the nth order may be regarded as the limit of a linear boundary problem for a difference equation of the nth order, that is, again, of a system of linear algebraic equations.

It goes without saying that this relation yields a fertile source of suggestions both as to the facts in the transcendental case and as to possible methods of proof. It was indeed the unpublished method which Sturm originally used in his fundamental investigations*. On the other hand, the passage to the limit may be rigorously carried through, as was done by Cauchy in his proof of the fundamental existence-theorem for differential equations (not merely in the linear case). This proof was completed in 1899 by Picard and Painlevé by showing that the solution of the difference equation approaches that of the differential equation uniformly not only in a certain small neighbourhood of the point where the initial conditions are given, but throughout any closed interval about this point in which the solution in question of the differential equation is continuous. With this fact at our disposal there is no longer any difficulty in carrying through rigorously the passage to the limit from the difference equation to the differential equation in other cases of boundary problems, as was shown in a sufficiently general case by Porter† more than ten years ago. Thus we may regard this method of passage to the limit as one of the well-established methods, both heuristic and otherwise, of approaching boundary problems.

This linear boundary problem for difference equations has, however, also distinct interest in itself apart from any assistance it may give us in the transcendental case. During the last few years great interest has been awakened in the theory of difference equations from a very different side by the remarkable work of Galbrun, Birkhoff, and Nörlund. It seems therefore an opportune time that this side of the subject should be also further developed. I shall return to this matter presently.

§ 2. *Generalities. Green's Function.*

A special case of the general linear boundary problem (1), (2) is the *homogeneous* boundary problem in which $r \equiv 0$, $\gamma_1 = \gamma_2 = 0$:

$$P(u) = 0 \quad \text{...} (1'),$$

$$W_1(u) = 0, \quad W_2(u) = 0 \quad \text{..........................} (2').$$

This system we shall call the *reduced system* of (1), (2). If this system has no solution except the trivial solution $u = 0$, I call it incompatible. If it has essentially only one solution, I call it simply compatible; if it has two linearly independent solutions, I call it doubly compatible. If we have to deal with a differential equation of the nth order, we may have compatibility of order as high as n. One of the most fundamental theorems here, and yet one which, I believe, has been enunciated and proved

* For a reconstruction of this work see the paper by Porter cited below and Bôcher, *Bull. Amer. Math. Soc.* vol. 18 (1911), p. 1.

† *Annals of Mathematics*, 2nd series, vol. 3 (1902), p. 55. This was more than two years before Hilbert, in 1904, took a similar step for integral equations.

only within the last few years*, is that *a necessary and sufficient condition that the general boundary problem* (1), (2) *have one and only one solution is that the reduced problem be incompatible.* It should be noticed that this is the direct analogue of a familiar theorem concerning linear algebraic equations.

It is readily seen that the general case is that in which the reduced system is incompatible. The case in which the reduced system is compatible, so that the complete system has either no solution or an infinite number of solutions, we may therefore speak of, for brevity, simply as the *exceptional case.* This exceptional case will always occur when the boundary conditions (2′) are linearly dependent. It may however occur in other cases too, and it is from this fact that the most interesting and important questions relating to boundary problems arise.

Of all the boundary problems by far the simplest and most important is what we may call the *one-point problem* in which all the β's or all the α's are zero, so that the boundary conditions (2) involve only one of the end-points of (X). If in this case conditions (2′) are linearly independent, equations (2) may be solved for $u(a)$ and $u'(a)$ (we assume for definiteness that the β's are zero) and thus be written in the form

$$u(a)=\delta_1, \quad u'(a)=\delta_2.$$

Now the most fundamental existence-theorem in the theory of differential equations tells us that there always exists one and only one solution of (1) which satisfies these conditions. This existence-theorem may then be regarded as the answer to our boundary problem in this case, and phrased as follows: *In the one-point boundary problem the exceptional case can occur only when conditions* (2′) *are linearly dependent.*

So far as we have yet gone there is no necessity for the two points which enter the boundary conditions (2) to be precisely the end-points a, b of (X); they may instead be any two points x_1, x_2 of this interval. Moreover, we may make a further generalization by considering in place of (2) conditions of the form

$$\alpha_1 u(x_1)+\alpha_1' u'(x_1)+\alpha_2 u(x_2)+\alpha_2' u'(x_2)+\ldots+\alpha_k u(x_k)+\alpha_k' u'(x_k)=\gamma,$$

which involve not two points but k; and we may at the same time consider differential equations of the nth order. This is a subject which has hardly been touched upon in the literature so far, but which seems likely to become of importance. The one result which I find in the literature is that if the boundary conditions consist in giving at each of the k points the value of u and of a certain number of its earliest derivatives, *and if the k points are sufficiently near together* the problem always has one and only one solution. This fact was established (not merely in the linear case) by Niccoletti† as a generalization of some methods and results of Picard for certain non-linear differential equations of the second order.

Still another direction in which we may generalize the boundary problem, either in connection with the generalization last mentioned or independently of it, is to admit in connection with the equation of the nth order more than n boundary conditions. We shall have occasion to mention some cases of this sort later.

* Cf. Mason, *Math. Ann.* vol. 58 (1904), p. 532; *Trans. Amer. Math. Soc.* vol. 7 (1906), p. 340; and Bôcher, *Annals of Math.* ser. 2, vol. 13 (1911), p. 71.

† *Turin Atti*, vol. 33 (1898), p. 746.

One occasionally finds the boundary conditions (2) replaced by conditions which involve definite integrals and which, on their face, are not boundary conditions at all*. Such conditions may however often be reduced to precisely the form (2). As an example of this we mention the problem of solving the equation

$$\frac{d^2u}{dx^2} - Gu = r$$

subject to the condition $$\int_a^b \Phi(x)\, u(x)\, dx = C,$$

where Φ is a given continuous function, and C a given constant.

Let $\phi(x)$ be any solution of the equation

$$\frac{d^2\phi}{dx^2} - G\phi = \Phi.$$

By combining this with the equation for u we readily find the formula

$$\int_a^b \Phi u\, dx = \Big[\phi' u - u'\phi\Big]_a^b + \int_a^b r\phi\, dx.$$

The above integral condition may therefore be replaced by

$$-\phi'(a)\, u(a) + \phi(a)\, u'(a) + \phi'(b)\, u(b) - \phi(b)\, u'(b) = C - \int_a^b r\phi\, dx,$$

a condition of precisely the form (2).

If we approach the subject from the point of view of difference equations, this simply means that if we have in place of the boundary conditions *general* linear equations between $u_0, u_1, \dots u_n$, these conditions can by using the difference equation be reduced to the ordinary four (or if we prefer three) term boundary condition form, —an obvious algebraic fact.

Let us leave these generalizations, however, and return to the case in which the conditions (2) involve merely two points, the end-points of the interval (X). While what I am about to say may readily be extended to equations of the nth order, I will again, for the sake of simplicity, speak merely of the equation of the second order, i.e. of the system (1), (2) in precisely the form in which we wrote it at first.

If, as is in general the case, the reduced system (1′), (2′) is incompatible, we are led to the important conception of the Green's Function by trying to find a function not identically zero satisfying (2′) and which comes as near as possible to being a solution of (1′)—it is to fail in this only through a finite jump of magnitude 1 at a point ξ of (X) in its first (or in the case of equations of the nth order in its $(n-1)$th) derivative. Such a function, $G(x, \xi)$, always exists and is uniquely determined when (1′) and (2′) are incompatible. A characteristic property of this function and one upon which its importance depends is that when (1′), (2′) are incompatible, the solution of the semi-homogeneous problem (1), (2′), which then exists and is uniquely determined, is given by the formula

$$u = \int_a^b G(x, \xi)\, r(\xi)\, d\xi \quad \text{..............................(3)},$$

* Cf Picone, *Annali della R. Scuola Normale Superiore di Pisa*, vol. 11 (1909), p. 8; and v. Mises, *Heinrich Weber Festschrift* (1912), p. 252.

which, as we mention in passing, includes as a special case (viz. when conditions (2) involve only one of the points a or b) the formula for the solution of (1) obtained by the method of variation of constants.

These Green's Functions may also be regarded, if we wish, as the limits of the Green's Functions for the difference equation, i.e. the solution of the reduced system corresponding to $(\bar{1})$, $(\bar{2})$, except that *for a single value of* i the second member of $(\bar{1})$ is to be taken not as zero but as 1*. The formula (3) then becomes a special case of the obvious one for building up the solution of a general system of non-homogeneous linear algebraic equations of non-vanishing determinant from the solutions of the special non-homogeneous system obtained by replacing one of the second members by 1 while all the other second members are replaced by zero.

So far we have demanded merely the continuity of the coefficients of (1). If in addition we demand the existence and continuity of the first derivative of p_1†, we can add considerably to the properties of the Green's function. When regarded as a function of ξ, it then satisfies the differential equation adjoint to (1′)

$$Q(v) \equiv \frac{d^2 v}{d\xi^2} - \frac{d(p_1 v)}{d\xi} + p_2 v = 0 \qquad (1''),$$

except when $\xi = x$. Moreover, still regarding it as a function of ξ, we find that it satisfies a system of homogeneous boundary conditions precisely analogous to (2′) but with different coefficients, these coefficients being however independent of the parameter x just as the coefficients of (2′) are independent of ξ:

$$\left.\begin{aligned} \overline{W}_1(v) &\equiv \bar{\alpha}_1 v(a) + \bar{\alpha}_1' v'(a) + \bar{\beta}_1 v(b) + \bar{\beta}_1' v'(b) = 0 \\ \overline{W}_2(v) &\equiv \bar{\alpha}_2 v(a) + \bar{\alpha}_2' v'(a) + \bar{\beta}_2 v(b) + \bar{\beta}_2' v'(b) = 0 \end{aligned}\right\} \qquad (2'').$$

The system (1″), (2″) is of fundamental importance in the whole theory of linear boundary problems and is called the system adjoint to (1′), (2′). A special case of it was used by Liouville‡ but the general formulation and application of the conception was made for the first time by Birkhoff§ less than five years ago. The reason why even now this conception is not as well known as it deserves to be is that the special cases which have almost exclusively absorbed the attention of mathematicians belong to the class of *self-adjoint* systems where not only the equation (1′) is self-adjoint but the boundary conditions (2″) are also identical with (2′). It is true that a somewhat more general case than this has received a little attention from Hilbert and his pupils‖, namely the case which they call that of "Greenian boundary conditions"¶ where (2′) and (2″) are identical without (1′) being self-

* Cf. Bôcher, *Annals of Math.* 2nd series, vol. 13 (1911), p. 71, where other references for the literature of Green's Functions will be found.

† For the equation of the nth order

$$\frac{d^n u}{dx^n} + p_1 \frac{d^{n-1}u}{dx^{n-1}} + \dots + p_n u = 0,$$

the requirement would be the existence and continuity of the first $n-i$ derivatives of p_i.

‡ Liouville's *Journal*, vol. 3 (1838), p. 604.

§ *Trans. Amer. Math. Soc.* vol. 9 (1908), p. 373. See also for the relation to Green's functions, Bôcher, *Bull. Amer. Math. Soc.* vol. 7 (1901), p. 297 and *Annals of Math.* vol. 13 (1911), p. 81.

‖ See for instance Westfall, *Zur Theorie der Integralgleichungen* (dissertation), Göttingen, 1905, p. 19.

¶ It remains to be seen whether this case is really of sufficient importance to deserve a name.

adjoint. The general case, however, in which (2′) and (2″) are different is, apart from Birkhoff's fundamental paper, only just beginning to receive attention.

Here too the analogies for difference equations are interesting and simple. In place of the adjoint system (differential equation and boundary conditions) we now have the system of homogeneous linear algebraic equations whose matrix is the conjugate (transposed) of the original system; it is this system which the Green's function of the difference equation satisfies when regarded as a function of its second argument. The self-adjoint case now becomes the case in which the matrix of the system of linear equations is symmetric or can be made symmetric by a combination of rows and columns. Such expressions as

$$\int_a^b vP(u)\,dx$$

which occur in Green's Theorem

$$\int_a^b \Big[vP(u) - uQ(v)\Big]\,dx = \Big[T(u,v)\Big]_a^b \quad \dots\dots\dots\dots\dots\dots(4)$$

(where T is a homogeneous bilinear differential expression of order one less than P), have as their analogues, in the case of difference equations, bilinear forms. I shall not go into these analogies in detail, since they have become very familiar during the last eight years in the similar case of linear integral equations as developed by Hilbert and his pupils. I wished however to say enough to make it clear that we can get to a large extent the satisfaction and the benefit of these analogies in the case of linear differential equations, without going to the subject of integral equations, by simply regarding the differential equation (of any order) as the limit of a difference equation. This same remark applies equally well to those parts of the subject upon which I have not yet touched, and I shall not in general think it necessary to repeat it.

§ 3. *Small Variations of the Coefficients.*

All the deeper lying parts of the theory of boundary problems depend directly or indirectly on the effect produced by changes in the coefficients of the differential equation or of the boundary conditions or of both. Such changes are frequently, indeed usually as the literature of the subject now stands, produced by supposing these coefficients to depend on one or more parameters. The more general point of view, however, is to consider arbitrary variations in these coefficients; and here, before coming to the deeper lying questions, it is essential to know under what conditions *small* variations of this sort will produce a small variation in the solution of the problem. The fundamental fact here is*

I. *If the reduced system* (1′), (2′) *is incompatible, it remains incompatible after a variation of the coefficients of* (1) *and* (2) *which is uniformly sufficiently small; and*

* I have not found this fact in the literature. In the special case in which only one of the end-points appears in the boundary conditions I proved it in *Trans. Amer. Math. Soc.* vol. 3 (1902), p. 208 and *Amer. Journ. of Math.* vol. 24 (1902), p. 315. The general theorem may be deduced from this special case by following the general lines of the reasoning given by me in *Annals of Math.* vol. 13 (1911), p. 74. Indeed the case of the equation of the nth order where the boundary conditions involve k points (cf. § 2) presents no difficulty here.

such a variation produces a variation in the solution of (1), (2) *and in its first two derivatives which is uniformly small throughout* (X).

It is merely a special case of this if we assume the coefficients of (1) to be continuous functions of (x, λ) when x is in (X) and the parameter λ lies in any one or two dimensional region Λ of the complex λ-plane. The coefficients of (2) we then also assume to be continuous functions of λ in Λ. An immediate corollary of the above theorem is then:

II. *If for a certain point* λ_0 *of* Λ *the system* (1′), (2′) *is incompatible, the same will be true throughout a certain neighbourhood of* λ_0, *and throughout this neighbourhood the solution of* (1), (2) *and its first two derivatives are continuous functions of* (x, λ).

Something essentially new is, however, added if we demand that the coefficients be analytic functions of λ and wish to infer the analytic character of the solution. Here the facts are these:

III. *If when* x *lies in* (X) *and* λ *in a certain two-dimensional continuum* Λ *of the* λ*-plane the coefficients of* (1) *are continuous functions of* (x, λ) *and analytic functions of* λ, *and if the coefficients of* (2) *are analytic in* λ *throughout* Λ, *and if* λ_0 *is a point in* Λ *such that when* $\lambda = \lambda_0$, (1′) *and* (2′) *are incompatible, then the same will be true throughout a certain neighbourhood of* λ_0 *and the solution of* (1), (2) *throughout this neighbourhood is, together with its first two derivatives with regard to* x, *continuous in* (x, λ) *and analytic in* λ*.

If the coefficients depend on a parameter λ, as in cases II and III, the values of λ for which (1′), (2′) are compatible are readily seen to be precisely the roots of the equation

$$\begin{vmatrix} W_1(y_1) & W_1(y_2) \\ W_2(y_1) & W_2(y_2) \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

where y_1 and y_2 are any pair of solutions of (1′) which do not become linearly dependent for any value of λ with which we are concerned. This equation we call the characteristic equation and its roots the *characteristic parameter values* (Eigenwerte), or characteristic numbers. In case III it is clear that (5) may be taken as analytic in λ, so that in this case, provided (5) is not identically satisfied, the characteristic numbers are all isolated though there may be an infinite number of them with cluster-points on the boundary of Λ. These characteristic numbers are the only singularities of the solution of (1), (2) regarded as a function of λ, and also of the Green's function of (1′), (2′), and it may be shown that these functions can have no other singularities there than poles. In special cases the solution of (1), (2) may have no singularity at some of these points. Those characteristic numbers for which (1′), (2′) become simply compatible we call simple characteristic numbers, those for which they become doubly compatible, double characteristic numbers, and so on in the higher cases when we are dealing with equations of higher order than the second.

In all that has been said so far no restrictions have been made concerning the

* In the special case in which (2) involves only one of the points a or b the proof of this theorem follows from the uniform convergence of the method of successive approximations. The general case may be inferred from this as indicated in a similar case in the preceding foot-note.

reality of the quantities used except that x be real. In particular the coefficients of (1) may be complex. If the system (1), (2) is real, then when (1′), (2′) is incompatible, the solution of (1), (2) is real; while if (1′), (2′) is compatible, it has a *real* solution not identically zero. In this case we can add various further facts to those already mentioned in this section, of which I mention the following immediate consequence of II and III.

IV. *If for a certain real range* Λ *of values of* λ *the coefficients of* (1) *are real continuous functions of* (x, λ) *and the coefficients of* (2) *real continuous functions of* λ*; if there is no characteristic value of* λ *in* Λ*; and if for no point in* Λ *the solution* u *of* (1), (2) *vanishes at* a *or* b*, or at any interior point where its derivative also vanishes; then* u *has the same number of roots in* (X) *for all values of* λ *in* Λ *and these roots are continuous functions of* λ.

If we add to our hypothesis that the coefficients of (1), (2) *be analytic in* λ*, we may add to the conclusion that the roots are analytic functions of* λ.

It must not be inferred from what I have said so far that the theory of boundary problems consists wholly, or even chiefly, in establishing existence-theorems or in proving by the exact methods of modern analysis facts which a hundred years ago would have seemed self-evident to any mathematician. Some applied mathematicians make it a reproach to pure mathematics that it has come now to a state where it is interested solely in questions of this sort. If this were so it would indeed be a cause for reproach; but it should perhaps rather be regarded as a warning of whither certain extreme tendencies in modern pure mathematics might lead us if allowed to get too much the upper hand. The good old-fashioned view that it is the main object of mathematics to discover essentially new facts is, however, hardly in danger of becoming obsolete in a generation which has just witnessed the splendid achievements of Poincaré. In the subject of boundary problems, while we need as a foundation the existence-theorems, and exact proofs of facts which in themselves are quite to be expected, these are *only* a foundation. We wish not merely to be able to say: under such and such conditions there exists a solution of the boundary problem which is continuous (or analytic) but also to be able to say what this solution is like and what can be done with it. However incomplete the theory still is, we can make important statements of this sort, as we shall now see.

§ 4. *Sturm's Fundamental Results and their Recent Extensions.*

Sturm's great memoir of 1836, which forms to a certain extent the foundation of our whole subject, produces on most superficial readers the effect of being complicated and diffuse. Nothing could be a greater mistake. The paper is very rich in content, and, while it would no doubt be possible to present the material more compactly than Sturm has done, there is by no means the repetition of which one gets the impression on a first reading owing to similarity in appearance of theorems which are really very different. It must be confessed, however, that Sturm does fail to emphasize sufficiently his really fundamental results.

Sturm, throughout whose work all quantities used are assumed real, takes the differential equation in the self-adjoint form

$$\frac{d}{dx}\left(K\frac{du}{dx}\right) - Gu = 0, \quad (K > 0) \qquad\qquad (6),$$

where K has a continuous first derivative, to which any homogeneous linear differential equation of the second order can readily be reduced. Perhaps the most fundamental result of the whole paper is the one which when stated roughly says that if the solutions of (6) oscillate in (X), they will oscillate more rapidly when G or K is decreased. The precise statement is this:

If we consider the two differential equations

$$\left.\begin{aligned} \frac{d}{dx}\left(K_1\frac{du}{dx}\right) - G_1 u = 0 \\ \frac{d}{dx}\left(K_2\frac{du}{dx}\right) - G_2 u = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

where throughout (X)

$$0 < K_2 \leqq K_1, \qquad G_2 \leqq G_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

and if a solution u_1 of the first equation has two successive roots at x_1 and x_2, then every solution u_2 of the second will vanish at least once in the interval $x_1 < x < x_2$ provided both equality signs in (8) *do not hold at every point of this interval.*

If we note that this theorem tells us that if u_2 is a solution which vanishes with u_1 at x_1, then it vanishes again *before* u_2 vanishes for the first time, we see the appropriateness of the statement that the solutions of the second equation oscillate more rapidly than those of the first.

The proof of this theorem is made to depend by Sturm on the formula

$$[K_2 u_1 u_2' - K_1 u_2 u_1']_{c_1}^{c_2} + \int_{c_1}^{c_2} (G_1 - G_2)\, u_1 u_2\, dx + \int_{c_1}^{c_2} (K_1 - K_2)\, u_1' u_2' dx = 0 \ldots(9),$$

where u_1 and u_2 are any solutions of the first and second equations (7) respectively and c_1, c_2 any points of (X). This formula, which may be regarded as merely a special application of Green's Theorem*, yields an immediate and extremely brief proof of the theorem we are considering in the special, but very important, case $K_1 \equiv K_2$. In the general case the proof is by no means so easy, it being necessary then to introduce a parameter so as to pass over continuously from K_1, G_1 to K_2, G_2, and to consider carefully the effect of small changes of this parameter. Simpler methods have therefore since been devised for treating the general case, of which I will mention the extremely elegant one recently given by Picone†. This consists in using in place of (9) the formula

$$\int_{c_1}^{c_2} (K_1 - K_2)\, u_1'^2 dx + \int_{c_1}^{c_2} (G_1 - G_2)\, u_1^2 dx + \int_{c_1}^{c_2} K_2 \left(u_1' - u_2' \frac{u_1}{u_2}\right)^2 dx$$
$$+ \left[\frac{u_1}{u_2}(K_2 u_1 u_2' - K_1 u_2 u_1')\right]_{c_1}^{c_2} = 0 \quad \ldots\ldots(10),$$

which may be deduced without difficulty from the differential equations. In applying this formula we must assume that u_2 does not vanish between c_1 and c_2, and vanishes at one or both of these points only if u_1 vanishes there. By means of this formula the proof of Sturm's theorem is immediate.

* Cf. Dunkel, *Bull. Amer. Math. Soc.* vol. 8 (1902), p. 288.

† *Annali della R. Scuola Normale Superiore di Pisa*, vol. 11 (1909), p. 1, where however only special cases of (10) are used. Another brief proof, based on the use of Riccati's resolvent of (6), had been previously given by me: *Trans. Amer. Math. Soc.* vol. 1 (1900), p. 414.

I have insisted somewhat at length on this one simple result of Sturm both on account of its great importance and because it represents a direction for investigation which, I believe, might well be pursued farther. The question is: What changes in K and G will cause the solutions of (6) to oscillate more rapidly? Sturm's theorem gives one answer to this question. There are, however, many other changes in K and G besides a decrease in one or both which will have this same effect. Further theorems can of course be obtained by multiplying (6) before and after the change by different constants, or by making a change of independent or of dependent variable. All these results, while they may be formally more general, may be said not to go *essentially* beyond Sturm's classical theorem. An illustration of this which will be of some importance for us is the following:

The special case of equation (6) where $G = l - \lambda g$, $K = k$, where g, l, k are continuous functions of x independent of the parameter λ,

$$\frac{d}{dx}\left(k\frac{du}{dx}\right) + (\lambda g - l)\,u = 0 \quad \text{.......................(11)},$$

has been much considered ever since Sturm's time. If $g \geqq 0$, the equality sign not holding at all points with which we are concerned, an increase of λ will produce a decrease of G and consequently it is merely a special case of Sturm's theorem in its simplest form to infer that if for one value of λ a solution oscillates, the solutions will oscillate more rapidly for a larger value of λ. Precisely the reverse is clearly true if $g \leqq 0$. During the last few years, however, another case of (11) has also been considered by several authors using various methods, namely the case $l \geqq 0$, while g changes sign. An increase in λ then causes G to decrease for some parts of (X) and to increase for others. It looks as though we had here a case going decidedly beyond that of Sturm. If, however, we divide (11) by $|\lambda|$, we get an equation in which

$$K = \frac{k}{|\lambda|}, \quad G = \frac{l}{|\lambda|} - (\operatorname{sgn}\lambda)\,g.$$

Consequently an increase in $|\lambda|$ (λ retaining one sign) produces a decrease in K while G either decreases or remains constant, and we see that we have precisely Sturm's case.

I know of no published result* which goes in this direction, and in the sense I have explained, essentially beyond Sturm's.

By the side of this theorem I will recall to you another one even simpler and better known and which Sturm proved by the same methods. It may indeed be regarded as a limiting case of the above theorem.

The roots of two linearly independent real solutions of a real homogeneous linear differential equation of the second order separate each other.

These theorems perhaps hardly come within the subject of boundary problems if we take the term in a strict sense, since no particular boundary conditions are laid

* From a verbal communication of Professor R. G. D. Richardson I understand that in a paper shortly to appear in the *Mathematische Annalen* he has made progress in this direction in the case of (11) when g changes sign and l is negative at some or all points of (X). This would appear to be a case really different from Sturm's.

down, but they are so fundamental for all work whose object is to determine the nature of the solutions of boundary problems that they could not be omitted here. Other theorems of the same sort contained in Sturm's memoir refer to the roots of u' or more generally of functions of the form $\phi_1 u - \phi_2 u'$, where ϕ_1 and ϕ_2 are given functions satisfying certain conditions*.

To the same category of theorems, preliminary, so to speak, to true boundary problems, are the various tests which have been given, some of which are contained in or follow readily from Sturm's memoir, for the equation (1) being oscillatory in (X), that is possessing solutions which vanish more than once there†.

In all of these cases we have theorems whose extension to equations of higher order is by no means easy, not merely because of essentially new difficulties which may be and doubtless are involved in the proofs, but still more because it is not easy to surmise what the character of the analogous theorems will be. The only investigation in this direction with which I am acquainted is a recent paper by Birkhoff‡ in which theorems concerning the roots of the real solutions of real homogeneous linear differential equations of the third order are obtained. The method used is one which, while familiar in other parts of the theory of linear differential equations, had never, I think, been used in treating boundary problems or questions relating to them. It consists in interpreting a fundamental system of solutions, u_1, u_2, u_3, as the homogeneous coordinates of a point in a plane. As x varies, this point traces out a curve whose shape is characteristic for the oscillatory properties of the solutions. I mention as a sample one of the simpler results obtained, from which it will be evident that we really have to deal with an extension of the results of Sturm mentioned above. Birkhoff proves that in an interval (X), where q and its derivative q' are real and continuous, the equation

$$u''' + qu' + \tfrac{1}{2} q'u = 0,$$

to which every self-adjoint equation of the third order may be reduced, always has real solutions which do not vanish, but that if two real solutions do vanish, their roots separate each other either singly or in pairs. Moreover, if q is increased, the maximum number of roots in (X) increases.

We have here a field worthy of further cultivation.

§ 5. *Boundary Problems as Treated by Sturm.*

Sturm's memoir may perhaps best be divided from a logical point of view into three parts, though this division is by no means followed out by the author in his method of exposition. We have

First those parts of the memoir which do not involve any boundary conditions. These we have already sufficiently considered.

* This part of Sturm's memoir, while extensive, is rather incomplete. Much more general results have been obtained by another method by Bôcher, *Trans. Amer. Math. Soc.* vol. 2 (1901), p. 428.

† Bôcher, *Bull. Amer. Math. Soc.* vol. 7 (1901), p. 333. Of a somewhat different character is Kneser's paper, *Math. Ann.* vol. 42 (1893), p. 409, since it deals with an infinite interval. The question there is essentially the behaviour of solutions in the neighbourhood of a singular point.

‡ *Annals of Math.* vol. 12 (1911), p. 103.

Secondly those theorems that refer to what we have called one-point boundary conditions, viz. $u(a) = \gamma_1$, $u'(a) = \gamma_2$. Since the existence-theorem here was well known, being merely the fundamental existence-theorem for differential equations, the theorems concern (a) the *character* of the solution of the boundary problem and (b) the *changes* produced in it by changes in the differential equation or in the boundary conditions. What is most essential here is contained in what I have called Sturm's two Theorems of Comparison.

Thirdly there comes a special kind of two-point boundary problem, the boundary conditions being the so-called Sturmian Conditions:

$$\left.\begin{aligned} \alpha u(a) + \alpha' u'(a) = 0, \qquad |\alpha| + |\alpha'| \neq 0 \\ \beta u(b) + \beta' u'(b) = 0, \qquad |\beta| + |\beta'| \neq 0 \end{aligned}\right\} \qquad \text{...............(12)},$$

characterised by the fact that each involves only one end-point of the interval. Here all three aspects of boundary problems are considered: (a) the existence of characteristic numbers; (b) the nature of the characteristic functions; (c) the changes produced in the characteristic numbers and functions by changes in the differential equation or the boundary conditions. The main result here is the Theorem of Oscillation, or perhaps it would be more correct to say the Theorems of Oscillation, since a variety of these may be formulated.

The first theorem of comparison may be roughly but sufficiently characterised by saying that it tells us that a decrease of G, or K, or $K(a)\,u'(a)/u(a)$ causes all the roots of u in (X) to decrease; while the second theorem of comparison tells us that under the same conditions the value of $K(b)\,u'(b)/u(b)$ will decrease provided the number of roots of u has not been changed. Both of these theorems are proved by Sturm by means of formula (9), which may, when K_1 and K_2 are not identically equal, be advantageously replaced by (10).

I shall not stop to enunciate Sturm's theorem of oscillation in any very general form. The general case would be that in which K or G or both are functions of (x, λ) which decrease as λ increases, while the ratio $K(a)\,u'(a)/u(a)$ may also decrease with λ. I enunciate, however, merely two special cases in which λ does not enter the boundary conditions and where the differential equation has the form (11).

I. *If $g \geqq 0$, the equality sign not holding throughout* (X), *and if α, α', β, β' are constants, there exist an infinite number of real characteristic numbers for the system* (11), (12). *These are all simple and have no cluster-point except $+\infty$. If, when arranged in order of increasing magnitude, they are denoted by $\lambda_0, \lambda_1, \lambda_2, \ldots$ and the corresponding characteristic functions by $u_0, u_1, u_2, \ldots$, then u_n has exactly n roots in the interval $a < x < b$.*

This is the best known special case of the theorem of oscillation. Another special case which, after division by $|\lambda|$, follows with exactly the same ease is this:

II. *If g changes sign in* (X) *and*

$$l \geqq 0, \qquad \alpha\alpha' \leqq 0, \qquad \beta\beta' \geqq 0,$$

there exist an infinite number of real characteristic numbers for the system (11), (12). *These are all simple and have $+\infty$ and $-\infty$ as cluster-points. If the positive and* *negative characteristic numbers arranged each in order of increasing numerical value are denoted by*

$$\lambda_0^+, \quad \lambda_1^+, \quad \lambda_2^+, \ldots$$

and

$$\lambda_0^-, \quad \lambda_1^-, \quad \lambda_2^-, \ldots$$

and the corresponding characteristic functions by

$$u_0^+, \quad u_1^+, \quad u_2^+, \ldots$$

and

$$u_0^-, \quad u_1^-, \quad u_2^-, \ldots,$$

then u_n^+ and u_n^- have exactly n roots in the interval $a < x < b$.

I doubt if it has been noticed before that this theorem is substantially contained in Sturm's results. It has been re-discovered three times during the last few years*.

That in the first of these cases there can be no imaginary characteristic numbers had been shown by Poisson by means of a special case of (9). A slight modification of this reasoning establishes this same fact for the second case†.

Sturm thus had both existence-theorems for the characteristic numbers and, in the theorems of oscillation, some rather specific information as to the nature of the characteristic functions. The next thing was to consider the changes produced in the characteristic numbers and functions by changes in the coefficients of the equation or of the boundary conditions. Such questions are also touched upon by Sturm, but we will not enter upon their consideration here.

As has already been said, all of these results including the theorems of oscillation, have their counterparts in the theory of linear difference equations, and it was from this side that the subject was first approached by Sturm. However, these oscillation properties will not hold for all equations of the form

$$L_i u_{i+1} + M_i u_i + N_i u_{i-1} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(13),$$

but only for those for which $L_i N_i > 0$ for all values of i with which we are concerned. As an illustration let us take the theorem that the roots of two linearly independent solutions of (1′) separate each other. In order to get the analogous theorem for (13) we must introduce the conception of *nodes* as follows: Corresponding to the values $i = 1, 2, \ldots n$ let us mark points $x_1, x_2, \ldots x_n$ on the axis of x, whether equally spaced or not is for our present purpose of no consequence. At the point x_i we erect an ordinate equal to u_i and we join the successive points thus obtained by straight lines. We regard the broken line thus formed as representing the solution u_i of (13), and the points where this line meets the axis of x we call the nodes of u_i. If the condition $L_i N_i > 0$ is fulfilled, it is readily seen that a solution of (13) not identically zero corresponds to a broken line which crosses the axis of x at each of its nodes, and here the theorem holds that the nodes of any two linearly independent

* Sanielevici, *Ann. de l'École Normale Supérieure*, 3rd ser. vol. 26 (1909), p. 19; Picone, *loc. cit.* (1909), and Richardson, *Math. Ann.* vol. 68 (1910), p. 279. The mere fact of the existence of an infinite number of positive and also of negative characteristic numbers (proved for instance under certain restrictions in Hilbert's 5th *Mitteilung*) is an even more obvious corollary of Sturm's work, even if no restriction is placed on the sign of l.

† Picone, *loc. cit.* p. 16.

solutions of (13) separate each other*. Without the restriction in question the theorem is false as the example†

$$u_{i+1} - u_i - u_{i-1} = 0$$

shows. Here the solution determined by the initial conditions $u_0 = 0$, $u_1 = 1$ gives for positive values of i Fibonacci's numbers 0, 1, 1, 2, 3, 5, 8, 13, ... with no node; while the solutions determined by $u_0 = -10$, $u_1 = 6$ and by $u_0 = -10$, $u_1 = 7$ both have several positive nodes, but these nodes do not separate each other. In the same way the other more complicated theorems of Sturm are, for the case of difference equations, essentially bound to the inequality in question.

This apparent failure of the analogy is less surprising when we notice that every linear differential equation of the second order may be obtained as the limit of an equation of the form (1) in which *after a certain point in the limiting process* the inequality in question holds. It is therefore only those difference equations that come nearest to the differential equations, so to speak, which share with them the simple oscillation properties. Difference equations of the form (13) in general will have oscillation properties of a very different character concerning which, so far as I know, nothing has been published, though from Sturm's brief remarks it seems possible that he had developed this theory also.

The results of Sturm concerning the oscillatory properties of the solutions of differential equations and the existence of characteristic values have been carried forward in various directions since his time, partly by methods more or less closely related to his own and partly by a number of essentially different methods. Of these there are four which we may describe briefly as

(1) Liouville's method of asymptotic expressions.

(2) The method of successive approximations.

(3) The minimum principle.

(4) Integral equations.

It will be well for us to glance briefly at these methods in succession before proceeding to consider the present state of knowledge of the theory of one-dimensional boundary problems.

§ 6. *Asymptotic Expressions.*

Liouville's greatest contribution to the theory of boundary problems, which had been so brilliantly inaugurated by his friend Sturm a few years before, was first the discovery of asymptotic expressions for the large characteristic values and the corresponding characteristic functions, and secondly the application of these expressions in the theory of the development of arbitrary functions‡. It is the first of these questions which we must now consider.

* E. J. Moulton, *Annals of Math.* vol. 13 (1912), p. 137.

† Or, more generally, the difference equation satisfied by Gauss's symbols $[a_1, a_2, \ldots a_n]$.

‡ Liouville's *Journal*, vol. 2 (1837), p. 16 and p. 418.

Liouville begins by reducing equation (11), in which he assumes $k > 0$, $g > 0$, by a change of both independent and dependent variable to the normal form*

$$\frac{d^2u}{dx^2} + (\mu^2 - \bar{l})\, u = 0, \qquad (\mu^2 = c^2\lambda) \quad \text{.................(14)},$$

where for the sake of simplicity we may suppose that the transformation has been so made that the interval (X) goes over into the interval $(0, \pi)$. It is then sufficient to consider this simpler equation. The boundary conditions (12) may be written

$$\left.\begin{aligned} u'(0) - hu(0) &= 0 \\ u'(\pi) + Hu(\pi) &= 0 \end{aligned}\right\} \quad \text{..........................(12')},$$

provided we assume $\alpha' \neq 0$, $\beta' \neq 0$. If we suppose u multiplied by a suitable constant, the first equation (12′) may be replaced by the two non-homogeneous conditions

$$u(0) = 1, \qquad u'(0) = h \quad \text{..........................(15)},$$

and it is the non-homogeneous boundary problem (14), (15) which Liouville first considers. He shows that its solution satisfies the relation

$$u = \cos \mu x + \frac{h}{\mu} \sin \mu x + \frac{1}{\mu} \int_0^x \bar{l}(\xi)\, u(\xi) \sin \mu (x - \xi)\, d\xi \quad \text{........(16)}.$$

This is of interest as being the first occurrence, so far as is known, of an integral equation of the second kind, and also because it is the first appearance of an integral equation as the equivalent of the system consisting of a differential equation and boundary conditions†.

By means of (16) Liouville readily infers that u and u' may be written

$$\left.\begin{aligned} u &= \cos \mu x + \frac{\psi_1(x, \mu)}{\mu} \\ u' &= -\mu \sin \mu x + \psi_2(x, \mu) \end{aligned}\right\} \quad \text{........................(17)},$$

where ψ_1 and ψ_2 (and all functions which in this section are denoted by ψ) are continuous functions of (x, μ) which for all real values of μ and all values of x in $(0, \pi)$ remain in absolute value less than a certain constant. From (17) we see that u differs when μ is large only in unessential ways from $\cos \mu x$, so that the large characteristic values of μ may be approximately obtained, as is readily shown with entire rigour, by substituting $\cos \mu x$ in the second condition (12′) in place of u. If then we denote the squares of the characteristic numbers arranged in order of increasing magnitude by $\mu_0^2, \mu_1^2, \mu_2^2, \ldots$, we may write for the positive values μ_i the expression $n + i + \gamma_i$, where γ_i approaches zero as i becomes infinite, and n denotes an integer independent of i whose value is as yet unknown. It is worth while to notice that we thus get a new proof, quite independent of Sturm's, of the existence of an infinite number of positive characteristic values, and of a part of the theorem of oscillation, namely that *at least after a certain point* each characteristic function

* Here and in what follows certain conditions of differentiability etc. must be satisfied by the coefficients of (11). Concerning the possibility of removing these restrictions cf. A. C. Dixon, *Phil. Trans.* vol. 211 (1911), p. 411.

† I.e. not only is (16) a consequence of (14), (15), but conversely (14), (15) is a consequence of (16). This last fact, it is true, is not brought out by Liouville.

has just one more root in $(0, \pi)$ than the preceding one. We see also that there are at most a finite number of negative or imaginary values for μ^2.

If, however, we are willing to make use of Sturm's theorem of oscillation, we may readily infer that $n = 0$ and thus get the more specific asymptotic formula

$$\mu_i = i + \gamma_i, \qquad (\lim_{i=\infty} \gamma_i = 0) \quad \text{.......................(18)}.$$

The formulae (17), (18) are merely the roughest kind of asymptotic formulae, and Liouville proceeded to sharpen them by a further application of the integral equation (16). This process was carried a little farther by the same method by Hobson* whose results we record

$$u = \cos \mu x \left[1 + \frac{\psi_3(x, \mu)}{\mu^2}\right] + \sin \mu x \left[\frac{\phi(x)}{\mu} + \frac{\psi_4(x, \mu)}{\mu^2}\right] \quad \text{......(17')},$$

where ϕ is continuous in $(0, \pi)$;

$$\mu_i = i + \frac{c}{i} + \frac{\delta_i}{i^2}, \qquad \pi c = h + H + \frac{1}{2}\int_0^\pi \bar{l}(x)\, dx \quad \text{...........(18')},$$

and the constants δ_i are all in absolute value less than a certain constant.

By substituting these asymptotic expressions for μ_i in the asymptotic expressions for u, Liouville and Hobson readily obtain, after certain reductions, asymptotic expressions for the characteristic functions, which need not be here recorded.

All of these formulae, even the simple one (18), require certain modifications† when in the boundary conditions (12) $\alpha' = 0$ or $\beta' = 0$; the method to be used, however, remains the same.

These asymptotic expressions can be indefinitely sharpened. Thus Horn‡, using another method, obtains expressions of the form (17'), (18') except that instead of containing only the first and second powers of $1/\mu$ and $1/i$, the powers from 1 to n enter, where n is an arbitrary positive integer.

This paper of Horn was the starting point for the modern developments of this subject of asymptotic expressions. In a second paper by Horn§ and in papers by Schlesinger|| and Birkhoff¶ similar asymptotic expressions are obtained not only for equations of the second order in which the parameter enters in a more general way and in which the coefficients of the equation are not all assumed real, but also for equations of higher order in similarly general cases. These investigations refer, however, merely to asymptotic expressions of solutions of a differential equation either without special reference to the boundary conditions or else in the case where

* *Proc. London Math. Soc.* 2nd ser. vol. 6 (1908), p. 349.

† This was mentioned on p. 445 of my article II A 7*a* in the *Encyclopädie* and the formulae corresponding to (18) when one but not both of the quantities α', β' are zero were given. By an oversight the case $\alpha' = \beta' = 0$, where (18) must be replaced by

$$\mu_i = 1 + i + \gamma_i,$$

was not mentioned. These cases are considered by Kneser, *Math. Ann.* vol. 58 (1903), p. 136.

‡ *Math. Ann.* vol. 52 (1899), p. 271.

§ *Math. Ann.* vol. 52 (1899), p. 340.

|| *Math. Ann.* vol. 63 (1907), p. 277.

¶ *Trans. Amer. Math. Soc.* vol. 9 (1908), p. 219. The method used in this paper was rediscovered by Blumenthal, *Archiv d. Math. u. Phys.* 3rd ser. vol. 19 (1912), p. 136.

the boundary conditions refer to a single point. The question of characteristic values does not present itself. This question, however, was taken up by Birkhoff in a second paper* in the case of the equation

$$\frac{d^n u}{dx^n} + p_1 \frac{d^{n-1} u}{dx^{n-1}} + \ldots + p_{n-1} \frac{du}{dx} + (p_n + \lambda g)\, u = 0 \quad \ldots\ldots\ldots\ldots (19),$$

where g and the p_i's are continuous functions of x of which all except g may be complex while g is assumed to be real and not to vanish in (X). General linear homogeneous boundary conditions are considered, certain special cases merely (so-called *irregular cases*) being excluded. Under these very general conditions Birkhoff establishes the existence of an infinite number of characteristic values, of which when n is odd all but a finite number are simple, while when n is even an infinite number of multiple characteristic values can occur only in very special cases; and at the same time he obtains an asymptotic expression for them. By means of this result an asymptotic expression for the characteristic functions is obtained.

The question of the reality of the characteristic numbers, even when the coefficients of (19) are real, is not touched upon. Professor Birkhoff, however, calls my attention to the fact that it is possible to treat questions of this sort by the methods there given. For instance, to mention only an obvious case, one sees that if n is odd there can, apart from the irregular cases†, be at most a finite number of real characteristic values.

§ 7. *The Method of Successive Approximations.*

Still another method which goes back to Liouville is the method of successive approximations. Although in his published papers he used this method only in very special cases, it is certain that he was familiar with it in more general forms, though it is impossible now to say to what extent of generality he had carried it. The method may be formulated as follows in order to include the special cases to be found in the literature and many others:

Let us write the homogeneous linear differential expression

$$P(u) = \frac{d^n u}{dx^n} + p_1 \frac{d^{n-1} u}{dx^{n-1}} + \ldots + p_n u$$

in the form

$$P(u) \equiv L(u) - M(u),$$

where L, M are homogeneous linear differential expressions, whose coefficients we assume to be continuous, of orders n and $m < n$. The differential equation (1) may then be written

$$L(u) = M(u) + r \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20).$$

We wish to solve this equation subject to a system of linear boundary conditions which we will write in the form

$$U_i(u) = V_i(u) + \gamma_i, \qquad (i = 1, 2, \ldots n) \quad \ldots\ldots\ldots\ldots (21),$$

where U_i and V_i are homogeneous linear expressions in $u(a)$, $u'(a)$, … $u^{[n-1]}(a)$,

* *Trans. Amer. Math. Soc.* vol. 9 (1908), p. 373.

† To these irregular cases belongs the one treated by Liouville in Liouville's *Journal*, vol. 3 (1838), p. 561.

$u(b), \ldots u^{[n-1]}(b)$. We have thus transposed part of equation (20) and conditions (21) to the second member, and we will suppose this so done that the auxiliary homogeneous system

$$\left.\begin{aligned} &L(u)=0, \\ &U_i(u)=0, \qquad (i=1,2,\ldots n) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(22)$$

is incompatible. We may then start from any function u_0 for which $M(u_0)$ is continuous in (X) and $V_i(u_0)$ $(i=1,\ldots n)$ are defined, and determine a succession of functions $u_1, u_2, \ldots$ by means of the equations

$$\begin{aligned} &L(u_{j+1})=M(u_j)+r, \\ &U_i(u_{j+1})=V_i(u_j)+\gamma_i, \qquad (i=1,2,\ldots n). \end{aligned}$$

If u_j and its first $n-1$ derivatives converge uniformly throughout (X), and this is what we shall mean when we say the process converges, the limit of u_j is precisely the solution of the problem (20), (21). The question whether this process converges or not depends, as was noticed by Liouville in some special cases*, on the characteristic values for the problem

$$\left.\begin{aligned} &L(u)=\lambda M(u), \\ &U_i(u)=\lambda V_i(u), \qquad (i=1,2,\ldots n) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(23).$$

This connection can best be stated by considering the system

$$\left.\begin{aligned} &L(u)=\lambda[M(u)+r_1]+r_2, \\ &U_i(u)=\lambda[V_i(u)+\gamma_i']+\gamma_i'', \qquad (i=1,2,\ldots n) \end{aligned}\right\} \quad \ldots\ldots\ldots(24),$$

where $r_1+r_2\equiv r$, $\gamma_i'+\gamma_i''=\gamma_i$, $M(u_0)+r_1=0$, and $V_i(u_0)+\gamma_i'=0$, so that when $\lambda=1$ (24) reduces to (20), (21). The fact then is this:

The method of successive approximations applied to (24) (*in the same way in which it was applied above to* (20), (21)) *converges for values of* λ *which lie in a certain circle* (*finite or infinite*) *described about* $\lambda=0$ *as centre and diverges outside. If this circle is not infinite, its radius is precisely the absolute value of one of the characteristic values of* (23). *All the characteristic values of* (23) *which lie within this circle are such that for them the system* (24) *has solutions* (*necessarily in infinite number*), *while if all the characteristic values on its circumference are simple roots of the equation* (5), *there is at least one of them for which the system* (24) *has no solution*†.

If there exist no characteristic values, it follows that the method of successive approximations will always converge; and this will, in particular, be the case if only one of the end-points a or b enter in the boundary conditions. The well-known fact that in this case the method of successive approximations surely converges‡ appears thus as a special case of the above general theorem.

In other cases, in which characteristic values do exist, it will be important in applying the theorem to know whether for a given characteristic value of λ the system (24) has solutions or not. Necessary and sufficient conditions of this sort

* Liouville's *Journal*, vol. 5 (1840), p. 356.

† The statement here made goes far beyond anything I have found in the literature, and is sufficient for our purposes, although a considerable generalization is possible. I expect to take up this matter in detail on another occasion.

‡ Cf. Fuchs, *Annali di Matematica*, ser. 2, vol. 4 (1870), p. 36.

have been given by Mason* in fairly general cases for differential equations of the second order. By means of such conditions the above theorem can of course be thrown into other, but equivalent, forms. In all investigations with which I am acquainted where the method of successive approximations is used a special case of this theorem in some of its forms plays a central part. Thus in Picard's well-known application of the method† to the semi-homogeneous problem

$$\frac{d^2u}{dx^2} + \lambda A(x) u = 0, \qquad A > 0$$

$$u(a) = u(b) = 1,$$

the fact upon which the possibility of applying the method depends is that the successive approximations converge or diverge according as $|\lambda|$ is less or greater than the smallest characteristic value; and this is readily seen to be substantially a special case of the above theorem. Again, although the term successive approximation is not used, §§ 9, 10 of Kneser's paper of 1903‡ are in substance an application of this method to the semi-homogeneous problem

$$\frac{d}{dx}\left(k\frac{du}{dx}\right) + (\lambda g - l) u + f = 0,$$

$$\alpha u(a) + \alpha' u'(a) = 0, \qquad |\alpha| + |\alpha'| \neq 0,$$

$$\beta u(b) + \beta' u'(b) = 0, \qquad |\beta| + |\beta'| \neq 0,$$

and the first of these sections may be regarded as a proof of the above theorem so far as it refers to this special case.

A second essential element in almost all applications of the method of successive approximations is constituted by Schwarz's constants which serve the purpose of giving a second test of the range of convergence of the process. It is by a comparison of the inferences drawn from these two methods that the final result is deduced. For details we refer here to the work of Picard and Kneser already cited.

§ 8. *The Minimum Principle.*

That linear boundary problems can be brought into intimate connection with the calculus of variations was first noticed and is still best known in connection with Laplace's equation, where the method involved has received the now almost universally accepted misnomer Dirichlet's Principle. It was pointed out by Weierstrass some fifty years ago that the existence of a minimum is here by no means obvious and that Dirichlet's Principle does not establish rigorously the existence of a solution of the boundary problem in question. This criticism was, however, not generally known in 1868 when H. Weber§ applied a similar method to establish the existence of characteristic numbers for the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0$$

* *Trans. Amer. Math. Soc.* vol. 7 (1906), p. 337. See also *C. R.* vol. 140 (1905), p. 1086. Cf. also for equations of higher order, Dini, *Annali di Mat.* vol. 12 (1906), pp. 240 ff.

† *Traité d'Analyse*, vol. 3, 2nd edition, p. 100.

‡ *Math. Ann.* vol. 58, p. 81. See also for more general cases the paper of Dini just cited.

§ *Math. Ann.* vol. 1, p. 1.

subject to the boundary condition $u = 0$. While this work of Weber thus remained inconclusive, it at least made it clear that *granting the existence of a minimum* a precisely similar method could be carried through in the similar one-dimensional case*. The facts here are these:

Consider the problem of determining the function $u(x)$ with continuous first and second derivatives in (X) which satisfies the conditions

$$u(a) = u(b) = 0,$$

$$\int_a^b Au^2 dx = 1$$

(A being a given function which is everywhere positive), and which minimizes the integral

$$J = \int_a^b u'^2 dx.$$

If we admit that such a function, u_0, exists and call the corresponding (minimum) value of J, λ_0, it is readily proved that λ_0 is a characteristic number for the differential equation

$$\frac{d^2u}{dx^2} + \lambda A u = 0$$

with the boundary conditions $\quad u(a) = u(b) = 0,$

and that u_0 is the corresponding characteristic function. Moreover it is shown that λ_0 is the smallest characteristic number.

To get the next characteristic number we add to the conditions imposed above on u the further one

$$\int_a^b A u_0 u\, dx = 0.$$

The function u_1 satisfying this condition as well as those stated above and minimizing J is the second characteristic function and this minimum value of J is the second smallest characteristic number, λ_1.

By adding to the conditions already imposed the further one

$$\int_a^b A u_1 u\, dx = 0,$$

we get the third characteristic value and function, etc.

After Hilbert's brilliant achievement in 1899 of inventing a method by which in many cases the existence of a minimizing function in problems of the calculus of variations may be established, it was natural to hope that this method might be applied successfully to this problem also. This was in fact done by Holmgren†, but a far simpler method of accomplishing the same result for this special problem, as well as for certain other boundary conditions, had been invented a little earlier by Mason‡ to whom the problem had been proposed by Hilbert. As first given, this

* Cf. Picard, *Traité d'Analyse*, 1st edition, vol. 3 (1896), p. 117, where only a partial account of the matter is given.

† *Arkiv för Mat., Astr. och Fysik*, vol. 1 (1904), p. 401.

‡ Dissertation, Göttingen, 1903. Some serious mistakes contained here were corrected in the abridged version, *Math. Ann.* vol. 58 (1904), p. 528.

method involved the use of some of Fredholm's results in the newly developed theory of integral equations, but it has been subsequently modified by Mason so as to be entirely independent of this theory and at the same time extended to much more general cases*. Still another method of establishing the existence of a minimum was given by Richardson†. This method depends essentially on development theorems in the theory of integral equations.

§ 9. *The Method of Integral Equations.*

We come finally to the method of integral equations which has held such a prominent place in the mathematical literature of the last few years. The central fact here is that a linear differential equation, whether ordinary or partial, together with a system of linear boundary conditions can be replaced by a single integral equation of the second kind. We have already seen how this fact presented itself in a very special case in the early work of Liouville. In the case of the fundamental boundary problem for Laplace's equation it formed the starting point for Fredholm's epoch-making investigations. It was however reserved for Hilbert‡ to bring out this relation clearly in more general cases, and to make use of it in the theory of characteristic numbers of differential equations and of the developments according to their characteristic functions.

The relation of the linear boundary problem for ordinary differential equations to the subject of integral equations is actually established by formula (3) above which may be regarded as an integral equation of the first kind for the function $r(x)$. The integral equation of the second kind originally used by Hilbert in the case of certain self-adjoint systems was

$$f(x) = u(x) + \lambda \int_a^b G(x, \xi)\, u(\xi)\, d\xi \qquad (25),$$

where G is the Green's function of a certain homogeneous system, the differential equation of which we will denote by $L(u) = 0$. Hilbert shows that the reciprocal, $\bar{G}(x, \xi, \lambda)$, of the kernel $\lambda G(x, \xi)$ of this equation (the "solving function") is precisely the Green's function of the equation

$$L(u) + \lambda u = 0 \qquad (26),$$

with the same boundary conditions as before. Since the characteristic numbers for this last system are the poles of its Green's function, we see from one of the most fundamental of Fredholm's results that these characteristic numbers are the values of λ for which the determinant of equation (25) vanishes; that is they are, according to Hilbert's terminology, the characteristic numbers of the homogeneous integral equation

$$u(x) + \lambda \int_a^b G(x, \xi)\, u(\xi)\, d\xi = 0 \qquad (27).$$

A comparison of this equation with the equation of the first kind (3) shows that

* *Trans. Amer. Math. Soc.* vol. 7 (1906), p. 337, also vol. 13, p. 516. For the treatment of the special case here mentioned see *The New Haven Colloquium* of the American Mathematical Society, 1910, p. 210.

† *Math. Ann.* vol. 68 (1910), p. 279.

‡ *Göttinger Nachrichten*, 1904, Zweite Mitteilung, p. 213.

if λ is a characteristic value, every solution of (27) is a solution of (26) which satisfies the given boundary conditions, and *vice versa*. Consequently we have in (27) a homogeneous integral equation of the second kind equivalent to the homogeneous system consisting of (26) and a set of homogeneous boundary conditions independent of λ.

In the cases considered by Hilbert, G is a real symmetric function of (x, ξ), that is we have to deal here with what we have called a real self-adjoint system. Here Hilbert's beautiful theory of integral equations with real symmetric kernels comes into play*, the fundamental theorem in which is that such a kernel always has at least one characteristic number† and can have no imaginary characteristic numbers. It was possible for Hilbert to go at once farther since the kernel G was readily shown to be *closed*, that is to be such that the equation

$$\int_a^b G(x, \xi)\, u(\xi)\, d\xi = 0$$

is satisfied by no continuous function u except zero. For such kernels he had established the existence of an infinite number of characteristic numbers. He thus obtained at one stroke the theorem: Every real self-adjoint system in which the parameter λ does not enter the boundary conditions, and enters the differential equation only in the form (26), has an infinite number of real and no imaginary characteristic numbers‡.

Other applications made by Hilbert, including the theorems concerning the developments according to characteristic functions, will be mentioned later.

Hilbert has sketched at the close of his fifth and in his sixth *Mitteilung*§ still another method for reducing a boundary problem to an integral equation of the second kind. This method, which he carried through in detail only in the case of a special partial differential equation, leads us to a kernel which is not a Green's function, but is formed by means of a *parametrix*, that is a function of (x, ξ) which satisfies the same boundary conditions as the Green's function, and whose $(n-1)$th derivative has the same discontinuity, but which does not satisfy the differential equation.

A linear integral equation may be regarded as the limiting form of a system of linear algebraic equations. This fact, which had been noticed by Volterra and put to essential use by Fredholm, as the very names determinant and minor sufficiently indicate, was made by Hilbert in his first paper the foundation, not merely

* *Göttinger Nachrichten*, 1904, Erste Mitteilung, p. 49. This theory was subsequently put into still more elegant and complete form by E. Schmidt, Göttingen dissertation, 1905, *Math. Ann.* vol. 63 (1907), p. 433.

† In my Tract: *Introduction to the Study of Integral Equations*, Cambridge, England, 1909, p. 47, I erroneously attributed this theorem to Schmidt. This mistake will shortly be corrected in a second edition.

‡ It is no essential generalization, as Hilbert himself points out, to consider the differential equation

$$L(u) + \lambda g u = 0,$$

where g is continuous and does not vanish. The general conception of a self-adjoint system is not formulated by Hilbert, but his work evidently applies to this case.

§ *Göttinger Nachrichten*, 1906, p. 480; 1910, p. 8.

heuristically but also in the way of rigorous deduction, of the theory of integral equations of the second kind. We thus have two methods of treating a boundary problem in one dimension as the limit of an algebraic problem concerning linear equations; first the direct method of difference equations described near the beginning of this lecture, and secondly the indirect method of replacing the boundary problem by an integral equation and regarding this as the limit of a linear algebraic system. Not only do these two methods look very unlike when superficially considered, but they present also a deeper lying difference: the determinant and its minors of the linear algebraic system whose limit is the integral equation approach definite limits, namely the Fredholm determinant and the Fredholm minors of the integral equation; whereas the determinant and its minors of the system of difference equations do not approach any limits as we pass over to the transcendental case. In spite of this apparently essential difference, there is the very closest relation between these two methods of obtaining the transcendental problem as the limit of an algebraic one. This relation was pointed out to me a few days ago in conversation by Dr Toeplitz of Göttingen, and may be briefly stated as follows:

If we use Hilbert's original method of passing from the differential to the integral equation by means of the Green's function $G(x, \xi)$, as explained above, the connection with the system of difference equations may be established by considering the homogeneous linear algebraic system *reciprocal* to the system of difference equations of which $L(u) = 0$ is the limit. This system has as its matrix, as is readily seen, precisely the Green's function of the difference equation, and if we add to the terms in the principal diagonal the quantities

$$\frac{u_0}{\lambda}, \ \frac{u_1}{\lambda}, \ \ldots \frac{u_n}{\lambda},$$

we get the linear algebraic system of which the integral equation (27) is the limit.

On the other hand, if we use the parametrix to pass from the differential to the integral equation, the connection with the difference equation is even more direct. In order to make the determinant and its minors of the difference equation converge when we pass to the limit, it is sufficient to combine the linear algebraic equations into an equivalent system by taking suitable linear combinations with constant coefficients of the equations, and this can be done in an infinite number of ways. The limit of the algebraic system as thus modified is precisely the integral equation of the second kind yielded by the use of the parametrix.

These relations will be explained in detail in Dr Toeplitz's forthcoming book on integral equations.

All the methods which have been devised to treat linear integral equations, for instance Hilbert's method of infinitely many variables, may be regarded as being indirect methods for the treatment of linear boundary problems; but any discussion of such questions would obviously be beyond the scope of this lecture.

§ 10. *The Present State of the Problem.*

The methods discussed in the last three sections have in common the very important advantage that they are capable of generalization without serious difficulty to the case of partial differential equations. It was therefore well worth while for their inventors and others to apply them somewhat systematically to the proofs of theorems in the case of the one-dimensional problem which had been already proved by other methods. The fact that the proofs by the newer methods were almost invariably both less direct and less simple than the earlier proofs leaves these applications of the newer methods still of decided interest, since they pointed the way to be followed in deducing really new results for partial differential equations. As an example in point I mention Richardson's use of the calculus of variations in proving Sturm's theorem of oscillation*.

I wish now, however, to indicate the stage which has been reached in *results* rather than in methods, and in doing this we begin with the case of the differential equation of the second order.

Twelve years ago in writing the article on boundary problems in one dimension for the mathematical Encyclopaedia I was obliged to present as an unsolved, and indeed until then almost unformulated, problem the question of solving the real homogeneous equation of the second order

$$\frac{d^2u}{dx^2} + p(x)\frac{du}{dx} + q(x, \lambda)\, u = 0 \qquad (28),$$

subject to the "periodic" boundary conditions

$$u(a) = u(b), \quad u'(a) = u'(b) \qquad (29).$$

If we assume that as λ increases through the interval

$$l < \lambda < L \qquad (\Lambda),$$

q constantly increases from negative or zero values to values which at least for some part of (X) become positively infinite, and that

$$\int_a^b p\, dx = 0,$$

this problem has since been answered by the following theorem of oscillation which I quote in detail because it really goes beyond Sturm's results and is at the same time simple†:

The problem (28), (29) *has an infinite number of characteristic numbers in the interval* (Λ), *and these have L as their only cluster point. If we indicate these characteristic numbers in order of increasing magnitude by $\lambda_0, \lambda_1, \lambda_2, \ldots$, each double characteristic number being repeated, and the corresponding characteristic functions by $u_0, u_1, u_2, \ldots$, then u_n vanishes an even number of times, namely n or $n+1$ times.*

This theorem was not completely proved until Birkhoff‡ in 1909 established it as a special case of a much more general theorem of oscillation referring to the

* *Math. Ann.* vol. 68 (1910), p. 279.

† Given in my Encyclopaedia article for the special case in which q has the same value at $a+\xi$ as at $b-\xi$ while the values of p at these two points are the negatives of each other (the statement as to p is there incorrectly given); for the case where $q=\lambda g - l\,(g>0)$, by Mason, *C. R.* vol. 140 (1905), p. 1086 (see also Tzitzéica, *ibid.* p. 492); as here stated, by Bôcher, *C. R.* vol. 140 (1905), p. 928, except that it was not there proved that *only two* u's have the same number of roots.

‡ *Trans. Amer. Math. Soc.* vol. 10, p. 259. Special cases of these results were subsequently deduced by another method by Haupt, Dissertation, Würzburg, 1911.

general real self-adjoint homogeneous problem for the differential equation of the second order*. These results of Birkhoff, which he obtains by a natural extension of Sturm's methods, may be regarded as on the whole the high-water-mark of our subject so far as theorems of oscillation are concerned. They do not, however, at present include Sturm's theorems of oscillation when K depends on λ or even the special case mentioned above where q is replaced by $\lambda g - l$, g changing sign, but $l \geqq 0$.

It is perhaps of interest to return for a moment to the periodic case in order to remark that if we seek there not the truly periodic solutions (on the supposition that p and q have the period $b-a$) but periodic solutions of the second kind, i.e. if we seek to make $u(b)$ and $u'(b)$ merely proportional to $u(a)$ and $u'(a)$, we are imposing not a linear but a quadratic homogeneous boundary condition, viz.

$$u(a)\,u'(b) - u'(a)\,u(b) = 0.$$

This example of a quadratic boundary problem is interesting because of its relative simplicity—the problem always has one, and in general two, linearly independent solutions. It was considered explicitly by Floquet† in 1883, but is essentially the problem of Riemann and Fuchs concerning the existence of solutions of an analytic linear differential equation which behave multiplicatively when we go around a singular point. Concerning this quadratic problem and its relations to the linear problem (28), (29) reference should also be made to the work of Liapounoff‡.

We come next to a series of interesting but rather special investigations concerning the equation of the fourth order. The equations here considered are of the self-adjoint form

$$\frac{d^2}{dx^2}\left(k\frac{d^2u}{dx^2}\right) + \lambda g u = 0, \qquad k > 0, \quad g > 0.$$

In 1900, and more generally in 1905, Davidoglou§ treated this equation by the method of successive approximations, the boundary conditions being the very special ones which present themselves in the theory of the vibrating rod. By using Picard's methods it was shown that Sturm's theorem of oscillation may be transferred without change to this case, multiple roots for the characteristic functions never occurring *between* the points a and b. This same differential equation has since been treated by Haupt (*loc. cit.*) subject to more general, but still very special, real homogeneous self-adjoint boundary conditions; the method used being to consider the effect on the characteristic numbers and functions of continuous changes in the differential equation—a method, it will be seen, not unlike in spirit, however it may differ in detail, from the methods used by Sturm.

In all the cases mentioned so far only self-adjoint problems have been considered. Liouville‖, in 1838, considered a special real but not self-adjoint homogeneous

* This requires $p \equiv 0$ in (28), but this is no essential restriction.

† *Annales de l'École Normale Supérieure*, 2nd ser. vol. 12, p. 47.

‡ *Memoirs of the Academy of St Petersburg*, 8th ser. vol. 13 (1902), No. 2, where references to some earlier work by the same mathematician will be found.

§ *Annales de l'École Normale Supérieure*, 3rd ser. vols. 17 and 22, pages 359 and 539.

‖ Liouville's *Journal*, vol. 3, p. 561.

equation of the nth order with boundary conditions of a rather special form* to which special methods were applicable resembling those used in establishing Fourier's theorem concerning the number of real roots of algebraic equations. In this way a theorem of oscillation precisely like Sturm's was established. Liouville noticed that the characteristic values were the same for this problem and its adjoint, and that the corresponding characteristic functions for these two problems have the same number of roots.

Finally we note a very recent paper by v. Mises† who reverts to Sturm's original method of obtaining the differential equation as the limiting form of a difference equation to treat the equation (11) either under the assumption $g > 0$ or $l > 0$ and with the boundary conditions

$$\int_a^b Au\,dx = 0, \qquad \int_a^b Bu\,dx = 0,$$

where A and B are given functions. From what was said in § 2 it will be seen that these are equivalent to conditions of the form (2′), where, however, the coefficients are in general functions of λ of a special kind.

The only other result of a general character which has been obtained is Birkhoff's proof, already mentioned, of the existence of an infinite number of characteristic numbers for the general (not necessarily real or self-adjoint) boundary problem in which the parameter does not enter the boundary conditions, and enters the differential equation only in the form indicated in (19), g being real and positive, and his asymptotic expressions in this case. A similar result of Hilb‡ deserves notice, although it refers only to special equations of the first and second orders, because it involves non-homogeneous differential equations with $n+1$ instead of n non-homogeneous boundary conditions; a case, however, which may readily be reduced to the type of problem we have been considering (i.e. a homogeneous system involving n boundary conditions) provided we are willing to admit the parameter into the coefficients of one of the boundary conditions.

§ 11. *The Sturm-Liouville Developments of Arbitrary Functions.*

Almost as old as linear boundary problems themselves, and indeed one of the chief causes for the importance of and continued interest in these problems, is the question of developing a more or less arbitrarily given function $f(x)$ in the form of a series whose terms are the characteristic functions of such a problem. The simplest case here is that of the system (11), (12), with which alone we shall be concerned in this section§. Moreover we assume $g > 0$. Denoting the characteristic functions by $u_0, u_1, \ldots$, we have the problem of determining the coefficients $c_0, c_1, \ldots$ so that the development

$$f(x) = c_0 u_0 + c_1 u_1 + \ldots \qquad \text{.............................(30)}$$

* Namely $n-1$ homogeneous conditions involving a and one homogeneous condition involving b. Liouville writes, to be sure, n non-homogeneous conditions at a, but they are, for his purposes, equivalent to $n-1$ homogeneous ones.

† H. Weber, *Festschrift*, 1912, p. 252.

‡ Crelle's *Journal*, vol. 140 (1911), p. 205.

§ We will assume that neither α' nor β' is zero. These are exceptional cases which require a separate treatment which presents no difficulty.

shall be valid. By means of (9) we readily see that the u_i's satisfy the relation

$$\int_a^b g u_i u_j dx = 0, \qquad (i \neq j) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(31),$$

by means of which the formal determination of the coefficients of (30), precisely as in the case of Fourier's series, is effected, namely

$$c_i = \frac{\int_a^b g f u_i dx}{\int_a^b g u_i^2 dx}, \qquad (i = 0, 1, \ldots) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(32).$$

Liouville* set himself the problem of considering this formal development of Sturm and proving first that it converges, and secondly that its value is $f(x)$, but though he invented methods of great importance and got some valuable results, he did not succeed in carrying his treatment even for the simplest functions $f(x)$ to a successful conclusion.

Let us first consider the question of showing that if $f(x)$ is continuous and the series (30) with coefficients (32) converges uniformly in (X), its value must be precisely $f(x)$. Liouville by a simple and ingenious process showed that under these conditions the function represented by the series coincides with the function $f(x)$ for an infinite number of values of x in (X), but did not perceive that this was not sufficient. A rigorous proof was first given by Stekloff† in 1901 (modified and simplified in 1903 by Kneser‡) by the method of successive approximations. Further proofs have since been given, namely one by Hilbert§ completed by Kneser|| by means of integral equations, and a very simple one by Mason¶ by means of the calculus of variations.

If we turn to the question of the convergence of the series, we find that Liouville accomplished decidedly more than in the matter just considered, since he proved by a method, which when examined in the light of our modern knowledge proves to be essentially rigorous, that if $f(x)$ is continuous and consists of a finite number of pieces each of which has a continuous derivative, the series will converge uniformly. This he did by means of the asymptotic expressions of § 6. Finally Kneser** in his remarkable papers of 1903 and 1905, which so far as we have not already described them depend essentially on the use of asymptotic values, gave a comprehensive, rigorous, and simple treatment of this whole subject which applies to functions satisfying Dirichlet's conditions throughout the region (X), and even establishes the uniform convergence of the development in any portion of (X) where $f(x)$ is continuous. Thus, with Kneser's papers, all the more fundamental questions concerning the development of an arbitrary function in a Sturm-Liouville series were completely and satisfactorily settled.

* Liouville's *Journal*, vol. 1 (1836), p. 253; vol. 2 (1837), p. 16 and p. 418.

† *Ann. de la Faculté des sciences de Toulouse*, ser. 2, vol. 3, p. 281.

‡ *Math. Ann.* vol. 58, p. 81.

§ *Göttinger Nachrichten*, 1904, 2te Mitteilung, p. 213.

|| *Math. Ann.* vol. 63 (1907), p. 477.

¶ *Trans. Amer. Math. Soc.* vol. 8 (1907), p. 431.

** *Math. Ann.* vol. 58, p. 81 and vol. 60, p. 402.

It was however of interest to accomplish the same thing in other ways, and two other methods essentially distinct from Kneser's and from each other have since been developed. The first of these was Hilbert's remarkable application of integral equations to this development problem*, while the second by A. C. Dixon† involved Cauchy's method of residues.

The subject was not however hereby exhausted. There remained, for instance, the question of showing that, as in the case of Fourier's series, the convergence of the development at a particular point depends, roughly speaking, only on the behaviour of $f(x)$ in the neighbourhood of this point, a question which was successfully treated by Hobson‡. One could, however, hardly have anticipated that there was still room for such an extensive advance as was to be made by Haar§ in two papers which seem to have such a degree of finality that we must consider them in some detail.

Haar's work, like almost all other work on this subject, involves the reduction of the differential equation to the normal form (14) by means of Liouville's transformation, and, for the sake of simplicity, it is only of this normal form I shall speak. Moreover we will assume that the characteristic functions have been multiplied by such constants as to make the denominators of the coefficients (32) have the value 1.

From the earlier work on the development of functions we need merely assume as known that the very simplest kind of functions, say analytic functions, are represented uniformly by their Sturm-Liouville development.

Let us now denote by $s_n(x)$ and $\sigma_n(x)$ the sums of the first $n+1$ terms of the Sturm-Liouville and of the cosine development of $f(x)$ respectively:

$$s_n(x) = \int_0^\pi f(\alpha) \sum_{i=0}^{n} u_i(\alpha)\, u_i(x)\, d\alpha,$$

$$\sigma_n(x) = \int_0^\pi f(\alpha) \left[\frac{1}{\pi} + \frac{2}{\pi}\sum_{i=1}^{n} \cos i\alpha \cos ix\right] d\alpha.$$

We have then

$$s_n(x) - \sigma_n(x) = \int_0^\pi f(\alpha)\, \Phi_n(\alpha, x)\, d\alpha \quad \text{.................(33)},$$

where

$$\Phi_n(\alpha, x) = \sum_{i=0}^{n} u_i(\alpha)\, u_i(x) - \frac{1}{\pi} - \frac{2}{\pi}\sum_{i=1}^{n} \cos i\alpha \cos ix.$$

Now the central fact discovered by Haar, from which everything else flows with the greatest ease, is that *whatever continuous function $f(x)$ represents*

$$\lim_{n=\infty} [s_n(x) - \sigma_n(x)] = 0 \quad \textit{uniformly}.$$

* *Göttinger Nachrichten*, 1904, 2te Mitteilung, p. 213, where, however, the conditions imposed on $f(x)$ were extremely restrictive. The matter was treated more generally by Kneser, *Math. Ann.* vol. 63 (1907), p. 477.

† *Proc. London Math. Soc.* ser. 2, vol. 3 (1905), p. 83.

‡ *Proc. London Math. Soc.* ser. 2, vol. 6 (1908), p. 349.

§ *Zur Theorie der orthogonalen Funktionensysteme.* Göttingen dissertation (1909). Reprinted *Math. Ann.* vol. 69 (1910), p. 331. Also a second paper, *Math. Ann.* vol. 71 (1911), p. 38. See also Mercer, *Phil. Trans.* vol. 211 (1911), p. 111.

The proof consists of three steps of which I give all but the first completely:

(a) By means of the asymptotic expressions for u_i it is shown that there exists a constant M (independent of α, x, n) such that

$$|\Phi_n(\alpha, x)| < M.$$

(b) If $f(x)$ is analytic we know that $s_n(x)$ and $\sigma_n(x)$ both approach $f(x)$ uniformly. Consequently in this case, by (33),

$$\lim_{n=\infty} \int_0^\pi f(\alpha)\,\Phi_n(\alpha, x)\,d\alpha = 0 \quad \textit{uniformly.}$$

(c) Whatever be the continuous function $f(x)$, form a sequence $\phi_1(x)$, $\phi_2(x)$, ... of analytic functions which approach $f(x)$ uniformly. We may write

$$s_n(x) - \sigma_n(x) = \int_0^\pi [f(\alpha) - \phi_m(\alpha)]\,\Phi_n(\alpha, x)\,d\alpha + \int_0^\pi \phi_m(\alpha)\,\Phi_n(\alpha, x)\,d\alpha.$$

Since ϕ_m approaches f uniformly, we see by (a) that m may be so chosen that for all n's and x's the first of these integrals is in absolute value less than $\frac{1}{2}\epsilon$. Having thus fixed m, we see by (b) that the second integral can be made in absolute value less than $\frac{1}{2}\epsilon$ by taking n sufficiently large. This completes the proof.

It is now merely restating a *part* of what we have just proved if we say:

The Sturm-Liouville development of any continuous function $f(x)$ in the case of the normal system (14), (12′) *converges or diverges at any point of* (X) *according as the cosine development of $f(x)$ converges or diverges there. It diverges to $+\infty$ $(-\infty)$ when and only when the cosine development does this. It converges uniformly through a portion of* (X) *when and only when this is true of the cosine development.*

If we now denote by $S_n(x)$ and $\Sigma_n(x)$ the arithmetic means of the first n s's and σ's respectively, we may infer easily from the fact that $s_n - \sigma_n$ approaches zero uniformly, the further fact that

$$\lim_{n=\infty} [S_n(x) - \Sigma_n(x)] = 0 \quad \textit{uniformly.}$$

Consequently, since Fejér has proved that the cosine development of a continuous function of x is always uniformly summable by the method of the arithmetic mean to the value of the function, it follows that the same is true of the Sturm-Liouville development of any continuous function.

The extension to the development of discontinuous functions is not at all difficult and leads, as is indicated by Haar, to analogous results.

Finally in his second paper Haar shows how still other theorems concerning trigonometric series, namely those established by Riemann and his followers, can be carried over to the Sturm-Liouville developments with only very slight changes.

§ 12. *Other Developments.*

The most immediate and natural extension of the Sturm-Liouville developments is to the development according to the characteristic functions of a system which consists of the differential equation (11), in which $g > 0$, and in place of the Sturmian conditions (12) a more general pair of real self-adjoint conditions, thus including, for instance, the periodic conditions (29). The formal work in these cases is the same as before, since the relation (31) is still satisfied. Some cases of this sort were treated by Hilbert in his second *Mitteilung* (1904) by the method of integral equations but only under very restrictive conditions on the function $f(x)$ to be developed, namely the continuity of its first and second derivatives, besides the further fact that $f(x)$ must satisfy the same boundary conditions as the characteristic functions in terms of which it is to be developed. Shortly after, the general case here described was treated by A. C. Dixon, in the paper referred to above, by Cauchy's method of residues, the restrictions to be placed upon $f(x)$ being very much less restrictive.

Here again an essential advance was made by Birkhoff* in 1908. Even more significant here than the generalization to equations of the nth order of the form (19)† is the fact that the condition of reality is dropped and that the system considered is no longer required to be self-adjoint. This last generalization makes, as Liouville had already noticed in a special case‡, an essential change even in the formal work of expansion, since formula (31) is no longer valid. It is desirable now to consider by the side of the given problem the adjoint problem. This has, as we know, the same characteristic values as the original system, and if we denote the corresponding characteristic functions first of the original system and then of the adjoint system by

$$u_0,\ u_1,\ u_2,\ \ldots$$

$$v_0,\ v_1,\ v_2,\ \ldots$$

respectively, we have the relation

$$\int_a^b g u_i v_j dx = 0, \qquad (i \neq j) \quad \ldots\ldots\ldots\ldots\ldots\ldots(34),$$

which reduces to (31) when the system is self-adjoint. We have then essentially not an orthogonal but what is known as a biorthogonal system. By means of this equation the coefficients may be formally determined by the expression

$$c_i = \frac{\int_a^b g f v_i dx}{\int_a^b g u_i v_i dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(35),$$

where, however, the question of the possible vanishing of the denominator must be further considered. This formal work, which had been given by Liouville in a special case, is the basis of Birkhoff's paper.

At a characteristic number λ_i the Green's function $G(x, \xi)$ has in general a pole of the first order whose residue Birkhoff finds to be given by the formula

$$\frac{u_i(x)\, v_i(\xi)}{\int_a^b g u_i v_i dx}.$$

* *Trans. Amer. Math. Soc.* vol. 9, p. 373. A very special case of Birkhoff's result was subsequently obtained by essentially the same method by Hilb, *Math. Ann.* vol. 71 (1911), p. 76.

† Westfall had in 1905 (Göttingen dissertation) considered the real self-adjoint case where the equation is of even order, where, however, no essentially new features occur. The method used was Hilbert's and the restrictions imposed on f were correspondingly great.

‡ Liouville's *Journal*, vol. 3 (1838), p. 561.

This when multiplied by $f(\xi)$ and integrated from a to b is precisely the general term of the formal development of $f(x)$ according to the functions u_i. Consequently the sum of the first $n+1$ terms of this formal development may readily be expressed as a contour integral in the λ-plane whose path surrounds the first $n+1$ characteristic numbers $\lambda_0, \lambda_1, \ldots \lambda_n$. Birkhoff then evaluates the limit of this contour integral as n becomes infinite by means of the asymptotic expressions for the characteristic functions u_i, v_i, and thus establishes at one stroke in fairly general cases both the convergence of the series and the fact that it represents the function $f(x)$.

A similar treatment has since been given by Hilb in the case of two special non-homogeneous systems mentioned at the end of § 10.

The Sturm-Liouville developments have also been generalized in one other direction, namely to the case where in the equation of the second order (11) the function g changes sign while $l \geqq 0$. The results here are still very incomplete, only the real case with certain special self-adjoint boundary conditions having been so far treated. The first treatment was by Hilbert* in 1906, when by means of his theory of polar integral equations he succeeded in establishing the validity of the development under very special restrictions including the continuity of the first four derivatives of the function to be developed. Mason's proof by means of the calculus of variations, referred to above, that if f is continuous and the series converges uniformly, the development represents the function, is valid in this case also.

The numerous important contributions which have been made during the last few years to the theory of series of orthogonal or biorthogonal functions in general all have a direct bearing on the questions here considered, and some of them give, even in the special cases we are here concerned with, essentially new results. It would, however, lead us too far if we should attempt to follow up these more general investigations.

§ 13. *Conclusion.*

The questions we have been considering may be classified roughly as (a) Existence Theorems, (b) Oscillation Properties, (c) Asymptotic Expressions, (d) Development Theorems. For the Sturm-Liouville system (11), (12) the investigation of all of these questions has been carried to a high degree of perfection, although even here the field is not yet exhausted. In the real self-adjoint case for the equation of the second order (11) where $g > 0$ results of a fair degree of completeness in all these directions have also been attained. In most other cases, however, the ground has only just been broken and nearly everything is still to be done.

Of the methods invented during the last few years undoubtedly that of integral equations is the most far-reaching and powerful. This method would seem however to be chiefly valuable in the cases of two or more dimensions where many of the simplest questions are still to be treated. In the case of one dimension where we now have to deal with finer or more remote questions other, in the main older, methods have so far usually proved to be more serviceable. It is only fair to mention

* *Göttinger Nachrichten*, 5te Mitteilung, p. 473. Cf. also Fubini, *Annali di Mat.* ser. 3, vol. 17 (1910), p. 111, where Hilbert's restriction that g vanish only a finite number of times in (X) is removed.

here the very important treatment given by Weyl of cases in which singular points occur at a or b. The development theorems here, where we have frequently not series but definite integrals, or even mixed forms, have so far been handled only by the use of integral equations. Apart from this, it may fairly be said that the greatest advances of recent years in the theory of boundary problems in one dimension, I recall for instance Birkhoff's three important contributions, have been made by other methods, largely indeed by methods more or less closely analogous to the original methods of Sturm and of Liouville. If my lecture to-day can serve to emphasize not the historical importance but the present vitality of these methods it will have served one of its main purposes.