STRUCTURE THEORY OF RINGS AND ALGEBRAS

POWER-ASSOCIATIVE ALGEBRAS

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An *algebra* is a mathematical system $\mathfrak{A}$ consisting of an n-dimensional vector space over a field $\mathfrak{F}$ and a product xy which is a bilinear function of its arguments x and y. When every element x of $\mathfrak{A}$ generates an associative subalgebra $\mathfrak{F}[x]$ of $\mathfrak{A}$ the algebra $\mathfrak{A}$ is said to be *power-associative*. All of the major classes of algebras which have been studied so far are power-associative. The classes are, of course, the associative, alternative, Lie, and Jordan classes of algebras.

Lie algebras of characteristic not two are defined by the identities

$$xy = -yx, \qquad x(yz) + y(zx) + z(xy) = 0. \tag{1}$$

The structure theory for this class of algebras is due to Elie Cartan,[1] W. Landherr, and N. Jacobson.[2] It exists only for the case where $\mathfrak{F}$ has characteristic zero. Lie algebras are trivially power-associative since $x^2 = 0$, $x^n = 0$ for $n \geqq 2$. Indeed such algebras are *nilalgebras*, that is, all elements are nilpotent. As a consequence there is a sharp divergence between the methods used in the structure theory for Lie algebras and those used in the other major theories.

Alternative algebras may be defined by the identity $(yx)x = y(xx)$ and the *flexible* law $x(yx) = (xy)x$. The structure theory is due to M. Zorn[3] and, like the associative theory, has been developed for the case where $\mathfrak{F}$ can have characteristic $p \neq 0$. The alternative laws imply the *theorem of Artin*[4] which states that $\mathfrak{A}$ is alternative if and only if the subalgebra $\mathfrak{F}[x, y]$ generated by any two elements of $\mathfrak{A}$ is associative. The end result of the theory is that *all simple algebras* of the class *are either associative or are eight-dimensional Cayley algebras.*

A much richer theory exists in the case of the algebras of P. Jordan.[5] These algebras are defined by the identities

$$xy = yx, \qquad x(yx^2) = (xy)x^2. \tag{2}$$

[1] The structure theory is due to Killing and Cartan and was published in Cartan's thesis, Paris 1894. Real simple Lie algebras were determined by Cartan in his paper *Les groupes réels simples et continues*, Ann. École Norm. vol. 31 (1914) pp. 263–265.

[2] Some of the principal classes of simple Lie algebras over an arbitrary field of characteristic zero were determined by W. Landherr in his papers *Über einfache Liesche Ringe*, Abh. Math. Sem. Hamburgischen Univ. vol. 11 (1937) pp. 41–64 and *Liesche Ringe von Typsus* A, Abh. Math. Sem. Hansischen Univ. vol. 12 (1938) pp. 200–241. The remaining classes were determined by N. Jacobson, *A class of normal simple Lie algebras of characteristic zero*, Ann. of Math. vol. 38 (1937) pp. 508–517.

[3] *Theorie der alternativen Ringe*, Abh. Math. Sem. Hamburgischen Univ. vol. 8 (1930) pp. 123–147.

[4] A simple proof of a generalization of this theorem was given by M. Smiley, *The radical of an alternative ring*, Ann. of Math. vol. 49 (1948) pp. 702–709.

[5] These algebras were first defined by P. Jordan in his paper entitled *Über eine Klasse nichtassoziativer hyperkomplexer Algebren*, Göttingen Nachrichten, 1932, pp. 569–575.

The first structure theory of such algebras was given in 1934 by Jordan, E. Wigner, and J. von Neumann.[6] They made the highly restrictive hypotheses that $\mathfrak{F}$ is the field of all real numbers and that $\mathfrak{A}$ is formally real, that is, a sum $x_1^2 + \cdots + x_r^2 = 0$ only if $x_1 = \cdots = x_r = 0$. These hypotheses imply that $\mathfrak{A}$ contains no nilpotent elements and can have no radical. The present structure theory, which is due to myself,[7] G. Kalisch and N. Jacobson,[8] assumes only the basic identities (2) and that $\mathfrak{F}$ has characteristic zero. However the properties of a Jordan nilalgebra[7] were derived for the case where $\mathfrak{F}$ has any characteristic $p \neq 2$.

The reason why the present theories of Jordan and Lie algebras are restricted to the characteristic zero case is that the basic tool in both theories is a trace argument. In the characteristic p case of the associative theory the structure theorems are proved by using the characterization of the radical as the set of all properly nilpotent elements[9] of the algebra. The resulting arguments are strictly associative, and all attempts at extending the characterization to obtain a tool useful for the study of power-associative systems have been fruitless. It is for this reason that a trace argument has remained the only available tool for so long a time.

I have recently analyzed the trace arguments which have been used in the studies of algebras similar to associative algebras, and have carried out[10] an abstraction of the theory as follows. A power-associative algebra $\mathfrak{A}$ over a field $\mathfrak{F}$ is said to be *trace-admissible* if there exists a bilinear function $\tau(x, y)$, with arguments x and y in $\mathfrak{A}$ and values in $\mathfrak{F}$ (an *admissible trace function* for $\mathfrak{A}$), such that

I. τ is *symmetric*, that is, $\tau(x, y) = \tau(y, x)$;

II. τ is *associative*, that is, $\tau(x, yz) = \tau(xy, z)$;

III. $\tau(e, e) \neq 0$ if $e^2 = e \neq 0$;

IV. $\tau(x, y) = 0$ if xy is zero or is nilpotent.

Define the *radical* $\mathfrak{N}$ of $\mathfrak{A}$ to be the maximal nilideal of $\mathfrak{A}$. Then the principal structure theorems will hold for all power-associative trace-admissible algebras and so a great deal of structure theory may be deleted. Let us now see what structure theorems are like and what the results are in the trace-admissible case.

[6] *On an algebraic generalization of the quantum mechanical formalism*, Ann. of Math. vol. 35 (1934) pp. 29–64.

[7] *A structure theory for Jordan algebras*, Ann. of Math. vol. 48 (1947) pp. 546–567.

[8] The principal classes of special simple Jordan algebras over an *arbitrary* field of characteristic zero were determined by G. Kalisch, *On special Jordan algebras*, Trans. Amer. Math. Soc. vol. 61 (1947) pp. 482–494, together with F. D. Jacobson and N. Jacobson, *Classification and representation of semi-simple Jordan algebras*, Trans. Amer. Math. Soc. vol. 65 (1949) pp. 141–169.

[9] Cf. Chapter 2 of my *Structure of algebras*, Amer. Math. Soc. Colloquium Publications vol. 24, New York, 1939.

[10] *Trace-admissible algebras*, Proc. Nat. Acad. Sci. U. S. A. vol. 35 (1949) pp. 317–322.

When the radical $\mathfrak{N}$ of a power-associative algebra $\mathfrak{A}$ is defined to be the maximal nilideal of $\mathfrak{A}$, all Lie algebras become radical algebras and their study is then excluded. Every algebra $\mathfrak{A}$ not a radical algebra contains an idempotent element $e = e^2 \neq 0$ and we may define $\mathfrak{A}_e(\lambda)$ to be the subspace of $\mathfrak{A}$ consisting of all elements x in $\mathfrak{A}$ such that $ex + xe = 2\lambda x$ for λ in F. It is known[11] that $\lambda = 0$, 1 or $\frac{1}{2}$ and that $\mathfrak{A}$ is the supplementary sum

$$\mathfrak{A} = \mathfrak{A}_e(1) + \mathfrak{A}_e(\tfrac{1}{2}) + \mathfrak{A}_e(0) \tag{3}$$

of the three subspaces $\mathfrak{A}_e(\lambda)$. Moreover $ex = xe = \lambda x$ for $\lambda = 0$, 1 and x in $\mathfrak{A}_e(\lambda)$, the subspaces $\mathfrak{A}_e(1)$ and $\mathfrak{A}_e(0)$ are *orthogonal* and *are subalgebras of* $\mathfrak{A}$ *in the commutative case*. The desired basic structure theorems for all power-associative algebra theories are then:

THEOREM 1. *If e is principal (that is, $\mathfrak{A}_e(0)$ contains only nilpotent elements), then $\mathfrak{A}_e(\frac{1}{2}) + \mathfrak{A}_e(0) \subseteq \mathfrak{N}$.*

THEOREM 2. *If $\mathfrak{A}$ is semisimple, that is, $\mathfrak{N} = 0$, then $\mathfrak{A}$ has a unity quantity.*

THEOREM 3. *Every ideal of a semisimple algebra is semisimple.*

Theorems 1 and 2 are actually equivalent. Of course the final goal of any structure theory is a theorem stating the nature of the simple algebras.

The decomposition properties stated above are easy to prove for all power-associative rings $\mathfrak{R}$ whose characteristic is prime to 30 and which are such that $\frac{1}{2}x$ is a unique element of $\mathfrak{R}$ for every x of $\mathfrak{R}$. The mysterious integer 30 enters because of our use of the property that *a commutative ring $\mathfrak{R}$ is power-associative if and only if $x^2x^2 = (x^2x)x$*, and this property holds[12] if and only if the characteristic of $\mathfrak{R}$ is prime to 30. However the decomposition and its properties are now known[13] also for rings whose characteristic is merely prime to two if we assume associativity of fifth and sixth powers as well as the associativity of fourth powers. The decomposition is extended from the commutative to the non-commutative case by the following observation. Let $\mathfrak{A}$ be any power-associative algebra over a field $\mathfrak{F}$ whose characteristic is not two. Then there is an attached commutative algebra $\mathfrak{A}^{(+)}$ which is the same vector space as $\mathfrak{A}$ and is defined relative to a product $x \cdot y$ expressible in terms of the product xy of $\mathfrak{A}$ by $2x \cdot y = xy + yx$. The algebra $\mathfrak{A}^{(+)}$ is power-associative when $\mathfrak{A}$ is, and indeed powers in $\mathfrak{A}$ coincide with powers in $\mathfrak{A}^{(+)}$. The provable properties of $\mathfrak{A}$ are then derivable from those of $\mathfrak{A}^{(+)}$ by using the linearized form

[11] See Theorem 2 of my *Power-associative rings*, Trans. Amer. Math. Soc. vol. 64 (1948) pp. 552–593.

[12] *On the power-associativity of rings*, Summa Brasiliensis Mathematicae vol. 2 (1948) pp. 21–33.

[13] Mr. Louis Kokoris has proved these results as a part of an investigation in which he is trying to extend all of my theorems on commutative power-associative algebras to algebras of characteristic 3 and 5.

$$x(yz + zy) + y(zx + xz) + z(xy + yx) = (yz + zy)x + (xz + zx)y + (xy + yx)z, \tag{4}$$

of the identity $xx^2 = x^2x$.

Let us now return to the results in the trace-admissible case. When $\mathfrak{A}$ is trace-admissible, it has been shown that $\mathfrak{A}$ and $\mathfrak{A}^{(+)}$ have the same radical and so $\mathfrak{A}$ is semisimple if and only if $\mathfrak{A}^{(+)}$ is semisimple. Also the difference algebra $\mathfrak{A}-\mathfrak{N}$ is trace-admissible. When $\mathfrak{A}$ is semisimple, it has a unity quantity e and $\tau(x, y)$ is an admissible trace function for $\mathfrak{A}^{(+)}$ as well as for $\mathfrak{A}$. Then it is easy to show that $\mathfrak{A}^{(+)}$ is a Jordan algebra, that $\mathfrak{A}$ is *flexible*, and that $\mathfrak{A}$ is simple if and only if $\mathfrak{A}^{(+)}$ is simple. The simple trace-admissible algebras are then known if one knows the nature of all algebras $\mathfrak{A}$ such that $\mathfrak{A}$ is flexible and $\mathfrak{A}^{(+)}$ is a known simple Jordan algebra. Such algebras have actually been determined[14] and I shall describe the result later. Let us now list all known simple power-associative algebras which are not nilalgebras.

I have not stated yet what I mean when I say that an algebra is *simple* and I always mean *more* than the obvious assumption that it has no nontrivial ideals. In the associative case it is customary to add the hypothesis that $\mathfrak{A}$ is not a one-dimensional zero algebra, that is, if $\mathfrak{A} = u\mathfrak{F}$, then $u^2 \neq 0$. This hypothesis of the associative case is equivalent in that case to the assumption that $\mathfrak{A}$ is not a simple nilalgebra, and we shall adjoin the assumption that $\mathfrak{A}$ is not a nilalgebra to our definition of a simple power-associative algebra. When $\mathfrak{A}$ is a Jordan algebra, this assumption reduces to the assumption about one-dimensional algebras as in the associative case, and so we are led to our first important unsolved question. Do there exist simple commutative power-associative nilalgebras of dimension $n > 1$? The question may be rephrased as follows. If $\mathfrak{A}$ is any algebra we define $\mathfrak{A}^{(\prime)}$ to be the vector subspace of $\mathfrak{A}$ spanned by all products xy for x and y in $\mathfrak{A}$. Then is it true that if $\mathfrak{A}$ is a commutative power-associative nilalgebra, then $\mathfrak{A}$ contains $\mathfrak{A}^{(\prime)}$ properly? This result would imply that a nilalgebra is solvable, that is, that $\mathfrak{A} \supset \mathfrak{A}^{(\prime)} \supset \mathfrak{A}^{(\prime\prime)} \cdots \supset \mathfrak{A}^{(r)} = 0$. A beginning in the study of this question has been made by M. Gerstenhaber who has shown that if x is a nilpotent element of a commutative power-associative algebra of characteristic zero, the linear transformation $a \to ax$ is nilpotent.

The study of simple nonassociative algebras $\mathfrak{A}$ is reducible[15] to the *central simple* case, that is, to the case where every scalar algebraic extension $\mathfrak{K}$ of the ground field $\mathfrak{F}$ yields a simple algebra $\mathfrak{A}_{\mathfrak{K}}$. If $\mathfrak{A}$ is central simple, we define the degree t of $\mathfrak{A}$ to be the maximal number of pairwise orthogonal idempotents in any $\mathfrak{A}_{\mathfrak{K}}$. As yet even central simple power-associative algebras of degrees one and two have not been completely classified. A class of algebras with a unity quantity e and $t = 1$ is the sum $\mathfrak{A} = e\mathfrak{F} + \mathfrak{B}_1 + \cdots + \mathfrak{B}_m$ where $\mathfrak{B}_i = u_i\mathfrak{F}$

[14] See Theorem 5.13 of the paper of footnote 11.

[15] See N. Jacobson, *A note on non-associative algebras*, Duke Math. J. vol. 3 (1937) pp. 544–548.

$+ v_i\mathfrak{F}$, $u_i^2 = v_i^2 = u_iu_j = u_iv_j = 0$ for $i \neq j$, $u_iv_i = -v_iu_i = \alpha_i e$ for $\alpha_i \neq 0$ in $\mathfrak{F}$. These algebras are trivially power-associative since every element $x = \alpha e + y$ where $y^2 = 0$. They are easily shown to be central simple. A class of algebras with $t = 2$ and a unity quantity e are the algebras $\mathfrak{A} = e\mathfrak{F} + u_2\mathfrak{F} + \cdots + u_s\mathfrak{F}$

$$(5) \qquad u_i^2 = \alpha_i e, \qquad u_ju_i = -u_iu_j \qquad (i \neq j; \quad i, j = 2, \cdots, s),$$

where the $\alpha_i \neq 0$ are in $\mathfrak{F}$. These algebras are power-associative for all definitions of the products u_iu_j as long as $u_ju_i = -u_iu_j$. They are also easily seen to be central simple if $s > 2$. They become the central simple Jordan algebras of degree $t = 2$ if $s > 2$ and $u_iu_j = 0$ for $i \neq j$.

The algebras of (5) can be alternative only for the values $s = 1, 2, 4$, and 8 and we shall use the notation $\mathfrak{C}_s$ for these alternative algebras. Every $\mathfrak{C}_s$ has an involution (involutorial anti-automorphism) $x \to \bar{x}$ determined by $\bar{e} = e$, $\bar{u}_i = -u_i$ for $i = 2, \cdots, s$. The algebra $\mathfrak{C}_2$ may be only semisimple. Every $\mathfrak{C}_s$, except the Cayley algebra $\mathfrak{C}_8$, is actually associative.

Let us now turn to a description of what we shall call the classical central simple Jordan algebras of degree $t > 2$. We shall *describe* these algebras *only* in the case where $\mathfrak{F}$ *is algebraically closed* since the description in this case will be needed later. If $\mathfrak{S}$ is a central simple Jordan algebra of degree $t > 2$, there is an attached algebra $\mathfrak{G}$ consisting of all t-rowed square matrices X with elements x_{ij} in one of the alternative algebras $\mathfrak{C}_s$ described above. When $s = 8$, we must use only the value $t = 3$. The algebra $\mathfrak{G}$ has an involution J defined by $X = (x_{ij}) \to X^J = (\bar{x}_{ji})$, and $\mathfrak{S}$ is the subspace of $\mathfrak{G}$ consisting of all $X = X^J$. Indeed $\mathfrak{S}$ is then a subalgebra of $\mathfrak{G}^{(+)}$. The attached algebra $\mathfrak{G}$ is associative except when $s = 8$ and the other central simple Jordan algebras, defined for $s = 1, 2$, and 4, are *special* Jordan algebras, that is, they may be imbedded in a Jordan algebra $\mathfrak{G}^{(+)}$ where $\mathfrak{G}$ is associative. We shall call an algebra $\mathfrak{A}$ a *classical* Jordan central simple algebra if there exists a scalar extension $\mathfrak{K}$ of the ground field $\mathfrak{F}$ such that $\mathfrak{A}_\mathfrak{K}$ is either one of the algebras $\mathfrak{S}$ given above or has degree two. All classical Jordan algebras have been determined by Kalisch and Jacobson, and all central simple Jordan algebras of characteristic zero are classical Jordan algebras.

Our list of simple power-associative non-nil algebras consists, at this point, of the examples of algebras of degrees one and two, and the Jordan algebras. The arbitrary associative simple algebra is the set of all t-rowed square matrices with elements in an associative division algebra. In the case where the ground field is algebraically closed this reduces to a total matric algebra. We now extend the list by including the algebras which arise as a solution of the problem of finding all flexible algebras $\mathfrak{A}$ such that $\mathfrak{A}^{(+)}$ is a classical simple Jordan algebra.

Let $\mathfrak{A}$ be an algebra over a field $\mathfrak{F}$ and let $\lambda \neq \frac{1}{2}$ be in $\mathfrak{F}$. We may then define an algebra $\mathfrak{A}(\lambda)$ which is the same vector space as $\mathfrak{A}$ but is defined relative to the product $(x, y) = \lambda xy + (1 - \lambda)yx$ where xy is the product in $\mathfrak{A}$. An algebra $\mathfrak{A}$ over $\mathfrak{F}$ is now said to be *quasi-associative* if there exists a scalar extension $\mathfrak{K}$ of $\mathfrak{F}$ and an element λ in $\mathfrak{K}$ such that $\mathfrak{B} = \mathfrak{A}_\mathfrak{K}(\lambda)$ is associative. The algebra $\mathfrak{A}$ is central simple if and only if $\mathfrak{B}$ is central simple and these are the algebras $\mathfrak{A} \neq \mathfrak{A}^{(+)}$ such that $\mathfrak{A}$ is flexible and $\mathfrak{A}^{(+)}$ is a simple Jordan algebra. The principal result on trace-admissible algebras $\mathfrak{A}$ of characteristic zero then states that a trace-admissible algebra $\mathfrak{A}$ is simple if and only if $\mathfrak{A}$ is either a simple Jordan algebra or a simple quasi-associative algebra.

The results outlined so far are quite well known and I want to continue with some new results. I am glad to be able to announce here that I have now been able to obtain a structure theory[16] for arbitrary commutative power-associative algebras over a field of characteristic $p \neq 2$, 3, or 5. As might be expected, I define the radical to be the maximal nilideal, and note that this is really no restriction in the case of Jordan algebras of characteristic p, a structure theory which is included in my theory. The theory succeeds because it begins with the study of simple algebras whereas these algebras are usually studied at the end of a structure theory. Assume first that $\mathfrak{A}$ is a commutative power-associative algebra which is not a nilalgebra and thus that $\mathfrak{A}$ contains an idempotent u. Then we have the decomposition $\mathfrak{A} = \mathfrak{A}_u(1) + \mathfrak{A}_u(\frac{1}{2}) + \mathfrak{A}_u(0)$. Let x_λ represent the arbitrary element of $\mathfrak{A}_u(\lambda)$ and write $z = xy$ for any two elements x and y of $\mathfrak{A}$. Then the subspaces $\mathfrak{A}_u(\lambda)$ have multiplicative properties which may be expressed by the formulas

$$x_\lambda y_\lambda = z_\lambda\,, \qquad x_\lambda y_{1-\lambda} = 0, \tag{6}$$
$$x_{\frac{1}{2}} y_{\frac{1}{2}} = z_0 + z_1\,, \qquad x_\lambda y_{\frac{1}{2}} = z_{\frac{1}{2}} + z_{1-\lambda}$$

for $\lambda = 0, 1$. It follows that we may write

$$x_1 y_{\frac{1}{2}} = y_{\frac{1}{2}}[S_{\frac{1}{2}}(x_1) + S_0(x_1)], \tag{7}$$
$$x_0 y_{\frac{1}{2}} = y_{\frac{1}{2}}[T_{\frac{1}{2}}(x_0) + T_1(x_0)],$$

where $S_{\frac{1}{2}}(x_1)$ and $T_{\frac{1}{2}}(x_0)$ are linear transformations[17] of $\mathfrak{A}_u(\frac{1}{2})$ as well as linear functions of x_1 and x_0 respectively, $S_0(x_1)$ is a linear mapping of $\mathfrak{A}_u(\frac{1}{2})$ into $\mathfrak{A}_u(0)$, $T_1(x_0)$ is a linear mapping of $\mathfrak{A}_u(\frac{1}{2})$ into $\mathfrak{A}_u(1)$. The linearized form of the relation $x^2x^2 = (x^2x)x$ may then be used to show that

$$S_{\frac{1}{2}}(x_1y_1) = S_{\frac{1}{2}}(x_1)S_{\frac{1}{2}}(y_1) + S_{\frac{1}{2}}(y_1)S_{\frac{1}{2}}(x_1), \tag{8}$$

$$\tfrac{1}{2}S_0(x_1y_1) = S_{\frac{1}{2}}(x_1)S_0(y_1) + S_{\frac{1}{2}}(y_1)S_0(x_1), \tag{9}$$

$$S_{\frac{1}{2}}(y_1)T_{\frac{1}{2}}(x_0) = T_{\frac{1}{2}}(x_0)S_{\frac{1}{2}}(y_1), \tag{10}$$

$$2w_{\frac{1}{2}}S_{\frac{1}{2}}(y_1)T_1(x_0) = [w_{\frac{1}{2}}T_1(x_0)]y_1\,. \tag{11}$$

The relation (8) implies that the mapping

$$x_1 \rightarrow 2S_{\frac{1}{2}}(x_1)$$

[16] The results appear in a paper entitled *A theory of power-associative commutative algebras*, Trans. Amer. Math. Soc. vol. 69 (1950) pp. 503–527.

[17] In the case where $\mathfrak{A}$ is a ring, $S_{\frac{1}{2}}$ and $T_{\frac{1}{2}}$ are endomorphisms and S_0 and T_1 are additive mappings. The results are actually derived in the ring case but their extension to the algebra case is immediate.

is a homomorphism of $\mathfrak{A}_u(1)$ onto the special Jordan algebra consisting of the linear transformations $S_{\frac{1}{2}}(x_1)$. The kernel of this homomorphism is an ideal $\mathfrak{B}_u$ of $\mathfrak{A}_u(1)$, and $\mathfrak{B}_u^2 = \mathfrak{C}_u$, where $\mathfrak{C}_u$ is the ideal of $\mathfrak{A}$ of all elements x_1 such that $y_{\frac{1}{2}}x_1 = 0$ for every $y_{\frac{1}{2}}$ of $\mathfrak{A}_u(\frac{1}{2})$. When $\mathfrak{A}$ is simple, the ideal $\mathfrak{C}_u = 0$. It is these inner structural properties that have enabled me to prove the following rather remarkable ring theorem.

THEOREM. *Let $\mathfrak{A}$ be a simple commutative power-associative ring whose characteristic is prime to* 30, *and let $\mathfrak{A}$ contain a pair of orthogonal idempotents whose sum is not the unity quantity of $\mathfrak{A}$. Then $\mathfrak{A}$ is a Jordan ring.*

The proof of this result begins with a proof of the property that if u and v are orthogonal idempotents of a power-associative ring $\mathfrak{R}$, then $(au)v = (av)u$ for every a of $\mathfrak{R}$. Let $\mathfrak{A}$ be simple and e_1 and e_2 be orthogonal idempotents of $\mathfrak{A}$. Then $\mathfrak{A} \subseteq \mathfrak{R}$ where $\mathfrak{R}$ is a commutative power-associative ring with a unity quantity e and the same characteristic as $\mathfrak{A}$. Moreover $\mathfrak{R} = \mathfrak{A}$ or every nonzero ideal of $\mathfrak{R}$ contains $\mathfrak{A}$. In either case $e = e_1 + e_2 + e_3$ for pairwise orthogonal idempotents e_i and we may show that $\mathfrak{R} = \mathfrak{R}_{11} + \mathfrak{R}_{22} + \mathfrak{R}_{33} + \mathfrak{R}_{12} + \mathfrak{R}_{13} + \mathfrak{R}_{23}$ where $\mathfrak{R}_{ii} = \mathfrak{R}_{e_i}(1)$, and $\mathfrak{R}_{ij}$ is the intersection of $\mathfrak{R}_{e_i}(\frac{1}{2})$ and $\mathfrak{R}_{e_j}(\frac{1}{2})$ for $i \neq j$. Also $\mathfrak{R}_{ij}\mathfrak{R}_{jk} \subseteq \mathfrak{R}_{ik}$. We then show that if $g = e_i + e_j$, the intersection of the kernel $\mathfrak{B}_g$ and $\mathfrak{R}_{ij}$ is zero. It follows readily that

$$\mathfrak{B} = \mathfrak{B}_{e_1+e_2} + \mathfrak{B}_{e_1+e_3} + \mathfrak{B}_{e_2+e_3}$$

is an ideal of $\mathfrak{R}$, and that $\mathfrak{B} \supseteq \mathfrak{A}$ if $\mathfrak{B} \neq 0$. This is easily seen to be impossible and so $\mathfrak{B} = 0$, the subrings $\mathfrak{R}_{e_i+e_j}(1)$ are Jordan rings. A computation using the property $x^2x^2 = (x^2x)x$ will then yield the theorem.

The result of this theorem and further arguments about ideals may be used to show that *every simple commutative power-associative algebra has a unity quantity*. Moreover, every such algebra of degree $t > 2$ is a classical Jordan algebra and also every Jordan algebra of degree $t = 2$ is a classical Jordan algebra. The major structure theorems for commutative power-associative algebras then follow readily. However two important unsolved problems remain in the study of algebras of low-degree and we shall present them now.

The first of these problems is that of the nature of a commutative power-associative algebra $\mathfrak{A}$ with a unity quantity e over an algebraically closed field $\mathfrak{F}$. Assume that e is primitive so that $x = \alpha e + y$, where α is in $\mathfrak{F}$ and y is nilpotent for every x of $\mathfrak{A}$. I have shown that if $\mathfrak{A}$ is simple and $\mathfrak{F}$ has characteristic zero, then $\mathfrak{A} = e\mathfrak{F}$. I have also proved this result for *special* Jordan algebras of characteristic p. There remains the general case of algebras of characteristic p.

The second problem is concerned with the nature of simple commutative power-associative algebras of degree $t = 2$ *even in the case of algebras of characteristic zero*. It may be true that all such algebras are classical Jordan algebras, but there is no strong indication that this is actually true.

The results we have described are, in a sense, negative results. For the simple algebras are really the end results of any structure theory of a class of algebras, and the quasi-associative algebras, which are the only nonclassical algebras we have obtained, are only minor distortions of associative algebras. It therefore seems reasonable to propose the question as to whether there are any simple algebras behaving like associative algebras in respect to the existence of idempotents and which are also power-associative. The first real attack on this question might then be an attempt to find all power-associative algebras $\mathfrak{A}$ such that $\mathfrak{A}^{(+)}$ is a central simple Jordan algebra. I have solved this problem and obtained[18] the construction given in the following theorem.

THEOREM. *Let $\mathfrak{A}$ be a power-associative algebra over a field $\mathfrak{F}$ of characteristic prime to* 30, $\mathfrak{S} = \mathfrak{A}^{(+)}$ *be a central simple special Jordan algebra so that there exists a scalar extension $\mathfrak{K}$ of $\mathfrak{F}$ and an associative algebra $\mathfrak{G}$ over $\mathfrak{K}$ such that $\mathfrak{S}_{\mathfrak{K}}$ is the set of all* J*-symmetric elements of $\mathfrak{G}$. Then $\mathfrak{K}$ may be selected so that there exists a linear mapping* T *of the set of all* J*-skew elements of $\mathfrak{G}$ into $\mathfrak{S}_{\mathfrak{K}}$ such that the product $x \cdot y$ of $\mathfrak{A}$ is expressible in terms of the product xy of $\mathfrak{G}$ by the formula*

$$x \cdot y = \tfrac{1}{2}(xy + yx) + (xy - yx)\mathrm{T}. \tag{12}$$

Conversely if $\mathfrak{G}$ is an associative algebra attached to a central simple Jordan algebra $\mathfrak{S}$ and $\mathfrak{A}$ is the vector space $\mathfrak{S}$ of all J*-symmetric elements of $\mathfrak{G}$, the algebra $\mathfrak{A}$ defined by* (12) *is a central simple power-associative algebra.*

It should be evident that powers in $\mathfrak{A}$ coincide with powers in $\mathfrak{G}$ and in $\mathfrak{S}$. In the case where $\mathfrak{S} = \mathfrak{A}^{(+)}$ is not a *special* Jordan algebra the algebra $\mathfrak{G}$ is not associative, $xx^2 \neq x^2x$, in $\mathfrak{G}$ even for the elements x of $\mathfrak{S}$. It is then necessary to adjoin the hypothesis that T shall annihilate diagonal skew elements of the algebra $\mathfrak{G}$ of all three-rowed matrices of Cayley elements.

In the case where $\mathfrak{F}$ is the real number field and $\mathfrak{S}$ is a formally real algebra the algebras defined by (12) are also formally real. Such algebras might have physical applications. In any event the new classes of algebras defined by (12) should have interesting properties and provide a starting point for new problems of structure and representation for power-associative algebras.

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[18] These results will appear in a paper entitled *New simple power-associative algebras* which will be published in Summa Brasiliensis Mathematicae.