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ON TOPOLOGICAL OBSTRUCTIONS TO INTEGRABILITY

by Raoul BOTT (1)

§ 1. Introduction.

In this lecture I would like to describe the state of the art in the problem of " foliating " a manifold or, as I prefer to view it, the problem of constructing integrable fields on a manifold. This subject has seen some interesting developments in the past two years and is also contemporary in the sense that, as you will see, it leads to " huge spaces ". By a huge space I mean here simply one whose homotopy groups are not finitely generated in every dimension. In the past we—and I think quite rightly—have shied away from such objects, but recently they have cropped up in various contexts: notably in the index theory associated to Von Neumann algebras of type II, and also in the localization of spaces at a given prime, and I am confident that in the future these " huge " spaces will enter into many of the analysis inspired problems in topology.

§ 2. Integrability.

Let me start by recalling the basic facts concerning the local theory of integrability. Consider a C^∞-manifold M and let TM denote its field of tangent planes. By a section of TM one means a smooth function $p \to X_p$ which attaches to each $p \in M$ a tangent vector at p. These are therefore the " vector-fields " or " infinitesimal motions " of M. If x, y are any two such sections their Lie bracket $[x, y]$ is again a well determined vector-field on M and the bracket operation satisfies the Jacobi-identity:

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]]. \tag{2.1}$$

By a field of tangent k-planes on M one means a smooth family $E = \{ E_p \,;\, p \in M \}$ of k-subspaces of T_pM. In short a k-dimensional " sub-bundle " of TM, and such a field is called integrable if its space of sections is closed under the bracket:

$$x, y \in \Gamma(E) \Rightarrow [x, y] \in \Gamma(E). \text{ (2)} \tag{2.2}$$

The term integrable is here justified by the well-known theorem of Frobenius [7], Clebsch-Deahna to the effect that if E is integrable, then locally E is generated by

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(2) $\Gamma(E)$ denotes the set of smooth sections of E.

parallell translation—relative to some coordinate system—from a fixed k-plane E_0. Quite equivalently this may also be put in the following way:

There exists a covering $\{ U_\alpha \}$ *of* M *by coordinate patches* U_α, *with coordinates* $\{ x_1^\alpha, \ldots, x_n^\alpha \}$ *such that on* U_α, E *consists of the planes tangent to the slices*

$$x_{k+1}^\alpha = c_1 \ldots x_n^\alpha = c_q, \qquad q = n - k.$$

These slices are therefore local integral manifolds of maximal dimension, which fiber U_α into submanifolds of codimension q.

It follows that if one defines

$$f_\alpha : U_\alpha \to \mathbb{R}^q$$

by the formula

$$f_\alpha(p) = \{ x_{k+1}^\alpha(p), \ldots, x_n^\alpha(p) \}$$

then f_α defines a " submersion " of U_α in $\mathbb{R}^q$, in the sense that the differential of f_α,

$$df_\alpha : T_p U_\alpha \to T_{f(p)}\mathbb{R}^q$$

is *onto at each point of* U_α, and our previous slices now are simply the *fibers*, $f_\alpha^{-1}(p)$, of f_α.

The $\{ f_\alpha \}$ may therefore be thought of as a system of maximal local integrals of E, which completely describe E.

Now using the implicit function theorem, it is easy to see that because f_α and f_β are both submersions, one can, for each $x \in U_\alpha \cap U_\beta$, find diffeomorphisms:

$$g_{\alpha\beta}^x : W_\beta^x \to W_\alpha^x,$$

of a neighborhood of $f_\beta(x) \in \mathbb{R}^q$ into a neighborhood of $f_\alpha(x) \in \mathbb{R}^q$, such that near x

$$g_{\alpha\beta}^x \circ f_\beta = f_\alpha. \tag{2.3}$$

Finally, it follows from (2.3) and again the submersion property of f_α that for points near $x \in U_\alpha \cap U_\beta \cap U_j$:

$$g_{\alpha\beta}^x \circ g_{\beta j}^x = g_{\alpha j}^x. \tag{2.4}$$

I have written these equations mainly for future reference. At this point, I want you essentially only to understand that integrable subbundles E of TM can either be described by the integrability condition (2.2), or by a system of local integrals $\{ f_\alpha \}$ of E which are local submersions of M in $\mathbb{R}^q$. Then, in particular any global submersion $f : M \to N$ of one manifold on the other defines an integrable field or " foliation " on M. Thus, for instance, if f is a fibration, then the field of tangents along the fiber is always integrable. Integrable fields generated in this way may be thought of as the most trivial examples.

To show you what may happen in more interesting cases let me remind you of two classical examples.

The first is the foliation on the torus $\mathbb{R}^2/\mathbb{Z}$ induced by the " foliation " of $\mathbb{R}^2$ by lines of a given slope m:

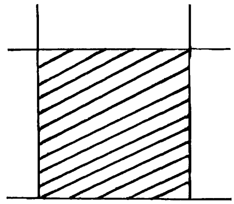

FIG. 1.

Thus, here I am drawing the " leaves ", i. e., *the maximal integral submanifolds* of the line field. If m is rational these leaves are all circles. If m is irrational they are all dense in T.

Next let me show you the beautiful Reeb foliation of the three sphere: First foliate the strip $|y| \leq 1$ in $\mathbb{R}^2$ as indicated in Figure 2:

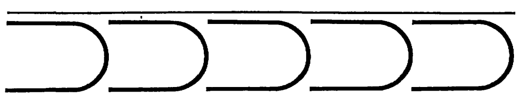

FIG. 2.

Next rotate this figure about the x-axis to obtain a foliation of a cylinder. There after identity points which differ by a integer x coordinate.

The result is a foliation of the anchor-ring,

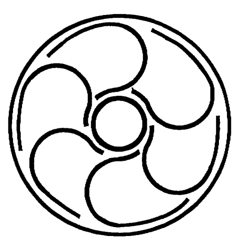

FIG. 3.

whose leaves are either planes coiling up to the bounding torus, or the bounding torus itself. Now if we take, for S^3 the set in complex 2-space $\mathbb{C}_2$, given by

$$|z_1|^2 + |z_2|^2 = 1,$$

it is easy to see that S^3 is the union of two anchor rings

$$S^3 = A_1 \cup A_2,$$

given by the equations $|z_1| \leq |z_2|$ and $|z_2| \leq |z_1|$ which intersect in the torus $T = \{ z_1| = |z_1| = 1/2 \}$. The foliations just described on A_1 and A_2 therefore fit together to form a foliation of S_3, which has one compact leaf, namely the torus T. All the other leaves are non-compact and curl up around this torus in opposite directions as we approach T from outside and inside. One may use this fact to show that this foliation though C^∞, is not *analytic.*

Concerning the higher spheres we know very little, in fact, we do not know whether any odd sphere S^n, of dim ≥ 3 admits an integrable $(n-1)$ field (*). One only has A. Haefliger's beautiful result that: *analytic integrable $(n-1)$-fields exist on a compact n-manifold only if its fundamental group is infinite.*

Another question which arises immediately in connection with this example is the existence of a compact leaf, and in this regard we have another beautiful result, due to Novikov, which asserts that every integrable 2-field on S^3 has a compact leaf. For 1-fields on S^3 it is not known whether a compact leaf has to exist. In fact, this is the famous Seifert problem. But these interesting and deep questions are really not pertinent to the problem (2.5) and I will have to leave them without further comment.

§ 3. On the nature of the global problem.

It is clear from the preceding that locally one can always construct integrable q-fields on a manifold M. The question therefore arises as to what difficulties one encounters in trying to construct a global field.

Now first of all, observe that difficulties will arise, because in general M does not admit a q-field, integrable or not. For instance, as is very well known, the 2-sphere S^2 admits no smooth line-field. On the other hand, the nature of this first question " does M admit a q-field ? " has been understood and much studied for many years. In particular, it has been converted into a purely homotopy-theoretic question.

Let me describe this translation to you, as it also points the way for our more refined question.

Please keep in mind during this development, that the homotopy theorist is a most singleminded person who treats only questions which can be phrased in terms of homotopy classes of continuous maps. Hence to please him we must convert all our geometric information into spaces and maps. In the present context this is not hard to do.

First of all one forms the Grassmanian variety

$$G_m(\mathbb{R}^N) = \{ A \subset \mathbb{R}^N \} \tag{3.1}$$

consisting of the set of m-subspaces of $\mathbb{R}^N$, topologized by the requirement that two such subspaces A and B are close, if and only if the unit spheres of A and B are close in $\mathbb{R}^N$. Next one includes $\mathbb{R}^N \subset \mathbb{R}^{N+1}$ in a standard manner and takes the limit of the compact spaces $G_m(\mathbb{R}^N)$ under the induced inclusions, to obtain the space

$$G_m = \lim_{N \to \infty} G_m(\mathbb{R}^N). \tag{3.2}$$

(*) Added November 10, 1970. Quite recently B. Lawson has constructed such foliations on all spheres of dimension $2^k + 3$.

This " infinite Grassmanian " is of fundamental importance in topology, because it *classifies* the vector-bundle functor; that is, there is a natural m-vector bundle E_m over G_m, with the property that for any reasonable space X the set of isomorphism classes of m-vector bundles over X, say $\mathrm{Vect}_m(X)$, is naturally in one to one correspondance with the homotopy classes $[X, G_m]$ of maps of X into G_m.

$$\mathrm{Vect}_m(X) \simeq [X, G_m]. \tag{3.3}$$

This correspondence assigns to a map $f: X \to G_m$ the pullback $f^{-1}E_m$ of E_m to X.

In case some of you are lost at this point, let me describe for you a particular consequence of (3.3) in quite elementary terms.

First of all note that an imbedding of M in a Euclidean space, $M \subset \mathbb{R}^N$ induces a map

$$\gamma: M \to G_m. \tag{3.4}$$

Indeed, simply let $\gamma(p)$ equal the subspace of $\mathbb{R}^N$ *parallel to the tangent plane to* M *at* p.

Now it turns out that these maps all belong to the same homotopy class $\gamma_M \in [M, G_m]$, and that this homotopy class which we refer to as the Gauss map of M, corresponds to the tangent bundle under the isomorphism (3.3).

The class γ_M is the first and fundamental homotopy theoretic invariant of the differentiable structure on M. Incidentally γ_M, also serves to define the *Pontrjagin ring* of M. This is the image of the cohomology $H^*(G_m; \mathbb{Q})$ under γ_M^* in $H^*(M; \mathbb{Q})$. In fact, quite generally, if E is any vector bundle over X, one defines its rational Pontrjagin ring by the formula

$$\mathrm{Pont}(E) = f_E^* H^*(G_m; \mathbb{Q}) \tag{3.5}$$

where $f_E: X \to G_m$ is the map corresponding to E under the isomorphism (3.3).

But to return to our problem of finding a k-plane field on M. The class γ_M is very pertinent to this question because, as is actually not hard to see, *constructing a k-field on M amounts to giving a " lifting " $\tilde{\gamma}$ of the Gauss map in the following diagram:*

$$\begin{tikzcd} & G_k \times G_{m-k} \arrow[d, "\pi"] \\ M \arrow[ur, dashed, "\tilde{\gamma}"] \arrow[r, "\gamma_M"'] & G_m \end{tikzcd}$$

Here π is induced by the direct sum maps

$$G_k(\mathbb{R}^N) \times G_{m-k}(\mathbb{R}^{N'}) \to G_m(\mathbb{R}^{N+N'})$$

sending (A, B) to $A + B$.

Problems of the type

$$\begin{tikzcd} & Z \arrow[d] \\ X \arrow[ur, dashed] \arrow[r] & Y \end{tikzcd} \tag{3.7}$$

where the solid arrows are given homotopy classes of maps and a map from X to Z is sought which makes the diagram homotopy commutative, are called lifting problems and one has by now quite standard methods of treating them. Because, as we have just noted, the problem of constructing a k-field on M can be translated into such a lifting problem for the Gauss map, it is natural to ask whether our more refined question concerning the existence of integrable fields has a similar translation into a further lifting of γ_M. Now in the last two years Haefliger [5] and Milnor [7], using different approaches, but both based on deep results of Phillips [8, 9 and 10] and more generally Gromov [3], have essentially clarified the status of this question. Let me very briefly summarize Haefliger's point of view here.

Recall that an integrable E gave rise to *local* submersions

$$f_\alpha : U_\alpha \to \mathbb{R}^q,$$

and transition functions $g_{\alpha\beta}$ satisfying the equations (2.3) and (2.4). Haefliger now drops the condition that f_α be *submersions*, and considers more general systems $(f_\alpha, g_{\alpha\beta})$ satisfying only (2.3), and (2.4). Under a suitable equivalence relation, these systems give rise to a set-valued functor $\mathscr{H}_q(M)$, which one should think of as homotopy classes of foliations with singularities. The virtue of this construction is first of all that $\mathscr{H}_q$ makes sense on *all-spaces* (not just on manifolds!) is homotopy invariant and satisfies the " Meyer-Vietoris " condition of E. Brown [2]. Hence by Brown's general existence theorem there exists a space $B\Gamma_q$ which " classifies " $\mathscr{H}_q$. That is, there is a natural correspondence:

$$\mathscr{H}_q(X) = [X, B\Gamma_q]. \tag{3.8}$$

The space $B\Gamma_q$ thus plays the same role *relative* to $\mathscr{H}_q$ as the space G_q plays relative to the isomorphism classes of vector-bundles $\mathrm{Vect}_q(X)$. Furthermore, passing from Haefliger's " cocycle " $\{ f_\alpha, g_{\alpha\beta} \}$ to the differential $dg_{\alpha\beta}$ gives rise to a map

$$\nu : B\Gamma_q \to G_q \tag{3.9}$$

which expresses the fact that each element of $\mathscr{H}_q(M)$ has an associated " quotient-bundle ".

The construction of $\mathscr{H}_q$ and hence $B\Gamma_q$ now naturally leads to the questions:

A. How does the functor $\mathscr{H}_q(M)$ differ from the classes of integrable fields on M under a suitable equivalence relation?

B. To what extent does the homotopy of $B\Gamma_q$ differ from that of G_q?

For both these problems the Phillips-Gromov generalization of the Smale-Hirsch immersion theory it of fundamental importance. Essentially is enables one to push all the singularities of a " Haefliger structure " on open manifolds off to infinity. As a consequence on open manifolds any Haefliger structure compatible with the Gauss map is homotopic to an honest foliation! The precise result is as follows:

THEOREM I (Haefliger, Milnor). — *Let $\mathscr{E}_q(M)$ denote the classes of integrable plane fields on M of codimension q; under the following equivalence relation: two such fields E and E′ are equivalent if and only if there exists a field $\mathscr{E}$ of codimension q on $M \times I$,* *which is transversal to all the slices* $M \times$ const, *and induces* E (resp. E') *on the slice* $M \times 0$ (resp. $M \times 1$).

Then on open manifolds

(3.10) $$\mathscr{E}_q(M) = \textit{homotopy classes of liftings of } \gamma_M$$

in the diagram

(3.11)
$$\begin{tikzcd} & G_{m-q} \times B\Gamma_q \arrow[d, "1 \times \nu"] \\ & G_{m-q} \times G_q \arrow[d, "\pi"] \\ M \arrow[r, "\gamma_M"'] \arrow[uur, dashed] & G_m \end{tikzcd}$$

Concerning the second problem these same methods lead to the result.

THEOREM II (Haefliger, Milnor). — *The map* $\nu : B\Gamma_q \to G_q$ *induces isomorphisms in homotopy in dimension* $\leq q$ *and is onto in dimension* $\leq q + 1$.

Thus, in particular, combining these two theorems we see that if M is open and of the homotopy type of a complex of dimension $\leq q + 1$ then every plane field of codimension q on M is homotopic to an integrable one.

To summarize the situation, these developments show that first of all on open manifolds our problem reduces to a lifting problem, and secondly that in low dimensions integrability induces no new difficulty. In short, these theorems are both of the existence type.

I would finally like to report on the meager crop of *nonexistence* theorems which are at present known.

§ 4. Some global obstructions to integrability.

Classical obstruction-theory teaches one that a complete understanding of the obstructions to lifting a map from X to Y,

$$\begin{tikzcd} & Y \arrow[d, "\pi"] \\ M \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

involves, first of all, the homotopy groups of the " homotopy-theoretic fiber " of π. This is the space F which occurs as the inverse image of a point p in X under π, when π is replaced by a fibering in its homotopy class.

For instance if $F\Gamma_q$ denotes this fiber for the map $\nu : B\Gamma_q \to G_q$, so that we have the exact " sequence ":

(4.1) $$F\Gamma_q \to B\Gamma_q \to G_q,$$

then Theorem II is quite equivalently expressed by the statement

(4.2) $$\pi_r(F\Gamma_q) = 0, \quad \text{for} \quad r \leq q.$$

The homotopy groups of the fiber are important because there is no impediment to lifting over successive skeletons as long as these homotopy groups are zero, while, in general the obstruction to lifting from t to the $(t+1)st$ skeleton is in $H^{t+1}(M, \pi_t(F))$. Of particular interest therefore is the first nonvanishing homotopy group of F.

Now in many of the classical lifting problems one could get at this information because the universal spaces X and Y were given explicitly by some relatively easy constructions. For instance, in the classical problem $G_k \times G_{m-k} \xrightarrow{\pi} G_m$ all the spaces can be treated directly.

In the present instance, and this is really typical of all the more subtle modern universal spaces such as B_{Top}, B_{PL}, etc., the space $B\Gamma_q$ is not really known to us in any manageable manner, and one can therefore get at this type of information only by very roundabout methods.

At present only the following results are known about the higher homotopy of $F\Gamma_q$. First of all, J. Mather [6] has very recently constructed a surjection (*):

$$\mathrm{Diff}_0(\mathbb{R}^1) \to \pi_2(F\Gamma_1) \to 0. \tag{4.3}$$

On the other hand, one can use the integrability condition which I noticed two years ago to prove that:

(4.4) *For $q \geq 2$, some $\pi_k(F\Gamma_q)$ is nonzero, and in fact not finitely generated.*

Let me remark briefly how this first nonexistence—or obstruction—result comes about.

First I recall the integrability criterion alluded to earlier [1].

INTEGRABILITY CRITERION: *A sub-bundle E of the tangent bundle TM is integrable only if the ring* Pont (T/E) *generated by the rational Pontrjagin classes of T/E vanishes in dimension greater than* $2 \times \dim (T/E)$

$$\mathrm{Pont}^k\,(T/E) = 0 \qquad \text{if} \qquad k > 2 \dim T/E. \tag{4.5}$$

The proof of this proposition is very direct, provided only that one uses the geometric definition due to Pontrjagin, Chern, Weil of the Pontrjagin classes as real cohomology classes represented by differential forms. Indeed, to give a clue to the initiated in this geometric framework, the infinitesimal integrability condition can be exploited to define a connection on T/E which is flat *along the leaves*, and then the result follows immediately. Essentially the same construction can be used to strengthen this criterion as follows:

THEOREM III. — *The homomorphism*

$$\nu^* : H^*\{G_q;\mathbb{Q}\} \to H^*(B\Gamma_q;\mathbb{Q}) \tag{4.6}$$

is zero in dimensions greater than $2q$.

Now the rational cohomology of G_q is well known to be a polynomial algebra $\mathbb{Q}[P_1,\ldots,P_{[q/2]}]$ in the universal Pontrjagin classes $P_i \in H^{4i}(G_q, \mathbb{Q})$, and is therefore,

(*) $\mathrm{Diff}_0(\mathbb{R}^1)$ denotes the group of diffeomorphisms of $\mathbb{R}^1$ with compact support.

in particular, non-trivial in positive dimensions provided $q \geq 2$. By a standard spectral sequence argument it follows therefore that $\pi_k(F\Gamma_q)$ must be nontrivial for some k. To obtain the nonfinite generation, one still has to show that if one uses $\mathbb{Z}_p$ coefficients then:

$$(4.7) \qquad \nu^* : H^*(G_q, \mathbb{Z}_p) \rightarrow H^*(B\Gamma_q ; \mathbb{Z}_p)$$

is injective.

To prove this one merely has to construct many examples of integrable fields E whose quotient bundles T/E have large mod p Pontrjagin rings, and such examples are easy to construct by taking E to be the horizontal space of flat vector bundles.

A question which seems to me of great interest is whether some of the groups $\pi_k(F\Gamma_q)$ are uncountable or not. In particular, one can relativize the integrability criterion to obtain homomorphisms of certain homotopy groups of $F\Gamma_q$ into the Reals and I would dearly like to know whether they are onto. The first case of interest occurs when $q = 3$ and in this situation the relative invariant gives rise to a homomorphism

$$(4.8) \qquad \theta : \pi_7(F\Gamma_3) \rightarrow \mathbb{R}.$$

Let me now conclude with a very brief remark about the complex analytic case, where some of these questions can be settled.

As is pointed out in Haefliger's paper [5], the space $B\Gamma_q$ should be thought of as the classifying space associated to the groupoid of *germs of diffeomorphisms of* $\mathbb{R}^q$. (Recall that the $g^x_{\alpha\beta}$ were local diffeomorphisms of $\mathbb{R}^q$). A corresponding construction for germs of *complex-analytic automorphism of* $\mathbb{C}^q$ is possible, and leads to a space $B\Gamma_q\mathbb{C}$. One also has a corresponding fibering

$$(4.9) \qquad F\Gamma_q\mathbb{C} \rightarrow B\Gamma_q\mathbb{C} \xrightarrow{\nu_{\mathbb{C}}} G\mathbb{C}_q$$

where now $G\mathbb{C}_q$ denotes the Grassmanian of complex subspaces of $\mathbb{C}^\infty$.

In this situation one can compute the relative invariants alluded to earlier and is then led to the

THEOREM IV. — *The homomorphism*

$$(4.10) \qquad \nu^*_{\mathbb{C}} : H^*\{ G\mathbb{C}_q ; \mathbb{R} \} \rightarrow H^*(B\Gamma_q\mathbb{C} ; \mathbb{R})$$

is zero in $\dim \geq 2q + 1$.

Furthermore there exists a relative invariant θ_q *which maps* $\pi_{2q+1}(F\Gamma_q\mathbb{C})$ *onto* $\underbrace{\mathbb{C} \times \ldots \times \mathbb{C}}_{d(q)}$

$$(4.11) \qquad \overset{0}{\pi_2}(F\Gamma_q\mathbb{C}) \xrightarrow{\theta_q} \underbrace{\mathbb{C} \times \ldots \times \mathbb{C}}_{d(q)} \rightarrow 0$$

where $d(q) = \dim_{\mathbb{R}} H^{2(q+1)}(G\mathbb{C}_q, \mathbb{R})$.

In this case at least, I have therefore fulfilled my promise to introduce you to some genuinely huge spaces.

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