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MANIFOLDS AND HOMOTOPY THEORY (*)

by WILLIAM BROWDER (**)

If one considers the problem of classifying manifolds, as the dimension increases one soon finds even the homotopy type classification to be impossibly complex. For example, any finitely presented group is the fundamental group of some closed manifold for any dimension ≥ 4. Thus one is led to consider the problem of " relative classification ", such as (*a*) classifying up to diffeomorphism all the smooth manifolds of one fixed piecewise linear (*PL*) type, or (*b*) classifying up to homeomorphism all manifolds of one fixed homotopy type, etc. The prototype of such a theory is the theory of (*a*), which began with the work of Milnor on differential structures on spheres, and culminated in the smoothing theory developed by Hirsch, Mazur, Lashof and Rothenberg. Their theory may be described briefly as follows:

Given a *PL* manifold M^m, it has a *PL* stable tangent bundle τ_M, which is induced from the universal *PL* bundle over the classifying space B_{PL} by a map $f: M \to B_{PL}$. The classifying space for stable linear bundles B_0 maps into B_{PL}, $p: B_0 \to B_{PL}$, and if M has a smooth structure γ compatible with its *PL* structure, then the linear tangent bundle of the smooth M_γ defines a lift of f to $f': M \to B_0$ such that $pf' = f$.

THEOREM. — M has a compatible smooth structure if and only if τ_M has a linear structure, i. e., f lifts to $f': M \to B_0$, such that $pf' = f$. Furthermore, *concordance* classes of such structures correspond one to one to homotopy classes of lifts f' of f (homotopies lying over f).

COROLLARY. — If M is a smooth manifold, concordance classes of smooth structures on M compatible with a C^∞-triangulation are in 1-1 correspondence with elements in the homotopy set $[M, PL/0]$, where $PL/0$ is the fibre of the map $p: B_0 \to B_{PL}$.

(Two smooth structures on M are called concordant if there is a smooth structure on $M \times [0, 1]$ which restricts to the two structures at the two ends $M \times 0$ and $M \times 1$).

It remains a difficult problem to calculate the homotopy set $[M, PL/0]$, and in fact the calculation of $\pi_m(PL/0)$ depends on the homotopy groups of spheres. However, the neat and closed form of the result is attractive and useful for many applications. One would like to describe a similar theory for the problem of classifying manifolds

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in a fixed homotopy type, and I will describe the analogous theory, and where the analogies break down.

A Poincaré pair is a pair $(X, \partial X)$ which satisfies Poincaré duality, i. e., there is an element $[X] \in H_n(X, \partial X)$ such that $[X] \cap : H^q(X) \to H_{n-q}(X, \partial X)$ is an isomorphism for all q. The *dimension* of X is defined to be n. If $\partial X = \emptyset$, X is called a Poincaré space, and for a Poincaré pair $(X, \partial X)$ of dimension n, it follows that ∂X is a Poincaré space of dimension $n - 1$.

Instead of a tangent bundle for a Poincaré pair $(X, \partial X)$, we define the Spivak normal fibre space of $(X, \partial X)$ which is the analog of the normal bundle of a smooth manifold $(M^m, \partial M) \subset (D^{m+k}, S^{m+k-1})$. If $(X, \partial X)$ is a connected Poincaré pair of dimension m, for $k > m + 1$ there is a $(k - 1)$ spherical fibre space ξ^k over X, and a pair of maps

$$(f, f_0) : (E_0(\xi), E_0(\xi \mid \partial X)) \to (Y, Y_0)$$

(where E_0 denotes the total space of the $(k - 1)$-spherical fibrations) such that

1) the pair $(X \bigcup_\pi E_0(\xi) \bigcup_f Y,\ \partial X \bigcup_\pi E_0(\xi \mid \partial X) \bigcup_{f_0} Y_0) = (A, B)$ is homotopy equivalent to (D^{m+k}, S^{m+k-1}), and

2) the map $f_{0*} : H_{m+k-2}(E_0(\xi \mid \partial X)) \to H_{m+k-2}(Y_0)$ is zero.

This fibre space is called the Spivak normal fibre space of $(X, \partial X)$ and it is unique up to fibre homotopy equivalence.

There is a classifying space B_G for stable spherical fibrations and maps

$$B_0 \to B_{PL} \to B_{\mathrm{Top}} \to B_G$$

(where B_{Top} is the classifying space for stable euclidean space bundles). If there is a smooth (PL, Top) manifold of the homotopy type of X then the classifying map of the Spivak normal fibre space ξ lifts to $B_0(B_{PL}, B_{\mathrm{Top}})$, but the converse is not true in general, which leads to a rich theory.

Note first that if one lift of ξ to B_H exists ($H = 0$, PL or Top) then the homotopy classes of lifts (homotopies covering a constant map into B_G) correspond 1-1 to elements of the set of homotopy classes of maps $[X, G/H]$, where G/H is the fibre of the map $B_H \to B_G$.

Let us define the set of concordance classes of homotopy H-structures ($H = 0$, PL or Top) on $X = \mathscr{S}^H(X)$ as follows. Consider pairs (M, h) where M is a manifold (in the category of H) and $h : (M, \partial M) \to (X, \partial X)$ is a homotopy equivalence of pairs. Two pairs (M_i, h_i), $i = 0, 1$, are concordant if there is a cobordism W, $\partial W = M_0 \cup M_1 \cup V$, $\partial V = \partial M_0 \cup \partial M_1$ and a homotopy equivalence of pairs

$$k : (W, V) \to (X \times [0, 1], \partial X \times [0, 1])$$

with

$$h(x) = (h_i(x), i) \qquad \text{for} \qquad x \in M_i .$$

Then $\mathscr{S}^H(X)$ is the set of concordance classes of such pairs.

The development of theory of surgery by Milnor, Kervaire, S. P. Novikov, the author, and Sullivan has culminated in the following theorem:

THEOREM. — Let X be a 1-connected Poincaré space of dimension $n \geq 5$ and suppose that its Spivak normal fibre space admits an H-structure ($H = 0$, PL or Top). Then there is an exact sequence of sets

$$P_{n+1} \xrightarrow{\omega} \mathscr{S}^H(X) \xrightarrow{\eta} [X, G/H] \xrightarrow{\sigma} P_n$$

where

$$P_n = \begin{cases} 0 & n \text{ odd} \\ Z & n = 4k \\ Z_2 & n = 4k + 2 \end{cases}$$

Here ω is defined if $\mathscr{S}^H(X) \neq \emptyset$, and in that case there is an action of P_{n+1} on $\mathscr{S}^H(X)$ such that $\eta(x) = \eta(x')$ if and only if x, x' are in the same orbit of the action.

In the case of pairs we have the result of Wall:

THEOREM. — If $(X, \partial X)$ is a Poincaré pair of dimension $m \geq 6$, with X, ∂X 1-connected, $\partial X \neq \emptyset$, and suppose the Spivak normal fibre space admits an H-structure ($H = 0$, PL or Top). Then $\mathscr{S}^H(X) = [X, G/H]$.

(The techniques used were proved first in the smooth case ($H = 0$), and extended to the PL case using the smoothing theory of PL manifolds above, and recently extended to the topological case using the work of Kirby and Siebenmann).

Thus we see an exact analogy with the smoothing theory of PL manifolds where $\partial X \neq \emptyset$, but in case $\partial X = \emptyset$ there is an obstruction to getting the analogous result, an obstruction lying in the group P_n. The underlying reasons for the difference in the theories arise from transversality. One has a Thom transversality theorem for either linear or PL bundles (or even Top bundles for higher dimensions) and this makes possible the exact correspondence between smoothings and lifts. But transversality fails for spherical fibre spaces and this failure is what creates the obstruction groups P_n. This relation has been precisely described in recent work of Levitt, which gives an obstruction theory to transversality for a map of a manifold M into a spherical fibre space, with values in cohomology $H^{j+1}(M : P_j)$.

The whole theory has been generalized by Wall to the non-simply connected case, where one assumes Poincaré duality with local coefficients, and other properties. Then one gets a similar exact sequence as above, and the obstruction groups depend only on the fundamental group system of X, ∂X and are again periodic of period 4. These obstruction groups are algebraically defined, for example, for $\partial X = \emptyset$, as certain Grothendieck groups of quadratic forms over $Z\pi$ or automorphisms of forms. This is analogous to the simply connected case where P_n is the Grothendieck group of even, unimodular Z-forms for $n = 4k$, or non-singular Z_2-quadratic forms for $m = 4k + 2$. The calculation of these groups (over $Z\pi$) has proven very difficult, and there is much work going on in this direction by both geometers and algebraic K-theorists.

For the term in the exact sequence $[X, G/H]$, the calculation is very difficult for $H = 0$, because again the homotopy groups of spheres are closely related. For $H = PL$ however, the homotopy properties of G/PL have been very well analyzed by Sullivan, and the work of Kirby-Siebenmann has enabled one to extend Sullivan's results to G/Top. The results make possible the explicit description of [X, G/Top], in terms of the cohomology and real K-theory of X, and have been used in the topological and PL classification of homotopy projective spaces, lens spaces, and many other manifolds.

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