Actes, Congrès intern. math., 1970. Tome 1, p. 191 à 199.

ALGEBRAIC K-THEORY

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I will give here a brief account of the history of algebraic K-theory and some of its main ideas and problems. Some of the work being done in this field at the present time will then be discussed in more detail.

1. Origin and basic results.

Although some early work of J. H. C. Whitehead [41] [42] and G. Higman [19] was later recognized as properly belonging to algebraic K-theory, the subject really began with Grothendieck's work on the Riemann-Roch theorem [9]. In this work, Grothendieck introduced the functor K, now known as K_0. For the case of rings, this functor may be described as follows. If R is a ring with unit, $K_0(R)$ is the abelian group with one generator $[P]$ for each finitely generated projective R-module P, and a relation $[P] = [P'] + [P'']$ for each short exact sequence $O \to P' \to P \to P'' \to 0$. The definition obviously extends to other categories, eg. sheaves, vector bundles, etc. Aside from its use in the Riemann-Roch theorem, this functor has found a number of applications to topology and algebra. For example, Wall [39] showed that if, X is a connected space dominated by a finite CW complex, there is a well defined obstruction $w \in K_0(Z\pi_1(X))$ such that X has the homotopy type of a finite complex if and only if $w = 0$. This result was then used by Siebenmann [31] to give a similar obstruction to the possibility of adding a boundry to an open manifold. This result, together with a calculation of $K_0(Z\pi)$ for a free abelian group π, was then used to prove the important Splitting theorem for manifolds [32]. A more algebraic application may be found in [36]. If G is the cyclic permutation group acting on 47 indeterminates x_i, then the fixed field of $Q(x_1, \ldots, x_{47})$ under G is not a pure transcendental extension of Q.

Probably the best known application of Grothendieck's functor K is the topological K-theory of Atiyah and Hirzebruch [3]. These authors consider a topological space X and define $K^0(X)$ by using vector bundles on X in place of the projective modules considered above. By applying this functor to suspensions of X, they define $K^n(X)$ for $n \leqslant 0$. Bott periodicity shows that $K^n(X) \approx K^{n+8}(X)$ and this is used to define $K^n(X)$ for all $n \in Z$. The resulting functors K^n constitute a cohomology theory, i. e. they satisfy the exactness, excision, and homotopy axioms of [10]. The resulting topological K-theory has found many important applications, for example in Adams' solution of the vector field problem for spheres [1]. A good exposition of this theory may be found in [2].

The next big step in algebraic K-theory was taken by Bass [4]. He tried to find algebraic analogues of the topological functors K^n. By imitating the construction of bundles over a suspension using clutching functions, he found a good definition for the functor $K_1(R)$. This is generated by symbols $[P, \alpha]$ where P is a finitely generated projective R-module and α is an automorphism of P. The relations are $[P, \alpha\beta] = [P, \alpha] + [P, \beta]$ and $[P, \alpha] = [P', \alpha'] + [P'', \alpha'']$ if $O \to P' \xrightarrow{i} P \xrightarrow{j} P'' \to 0$ is exact and $\alpha i = i\alpha', j\alpha = \alpha'' j$. Thie group turns out to be the same as one introduced by J. H. C. Whitehead [41] [42], $K_1(R) = GL(R)/E(R)$ where $E(R) = [GL(R), GL(R)]$ is the subgroup of $GL(R)$ generated by elementary matrices $e_{ij}(r) = 1 + re_{ij}$. Whitehead's theory of simple homotopy types shows that the group $K_1(Z\pi)$ has important topological applications. An example of this is the well known s-cobordism theorem [22].

Bass also succeeded in proving a partial analogue of the exactness and excision properties.

THEOREM 1 (Bass). — *If I is a 2-sided ideal of R, there is an exact sequence*

$$K_1(R, I) \to K_1(R) \to K_1(R/I) \to K_0(R, I) \to K_0(R) \to K_0(R/I).$$

The group $K_0(R, I)$ depends only on I considered as a ring without unit.

The second statement expresses the excision property. For the definition of the relative groups $K_i(R, I)$ and the proof, see [6].

There are two other, essentially equivalent, formulations of this result which avoid the use of the relative groups.

THEOREM 2 (Milnor [26]). — *Let*

$$\begin{array}{ccc} A & \to & A_1 \\ \downarrow & & \downarrow f_1 \\ A_2 & \underset{f_2}{\to} & A' \end{array}$$

be a cartesian diagram of ring homomorphisms such that f_1 or f_2 is onto. Then there is an exact Mayer-Vietoris sequence

$$K_1(A) \to K_1(A_1) \oplus K_1(A_2) \to K_1(A') \to K_0(A) \to K_0(A_1) \oplus K_0(A_2) \to K_0(A').$$

The other formulation, due to Gersten, requires a preliminary definition. If R is a ring without unit, we can adjoin a unit formally to R getting a ring R^+ with unit and a split exact sequence $O \to R \to R^+ \leftrightarrows Z \to 0$. If F is a functor from rings with unit to abelian groups, we extend the definition of F by setting

$$F(R) = \ker [F(R^+) \to F(Z)].$$

This is consistent provided that F preserves finite products [35], in particular for K_0 and K_1. Theorem 2 continues to hold with this extended definition.

THEOREM 3 (Gersten [13]). — *If $O \to A \to B \to C \to 0$ is an exact sequence of rings, there is an exact sequence $K_1A \to K_1B \to K_1C \to K_0A \to K_0B \to K_0C$.*

The hypothesis means that A is a 2-sided ideal of B and $C = B/A$. Therefore A has no unit if $C \neq 0$ (in general).

In addition to the above exact sequences, there is also an exact sequence associated with a localization [6]. This is rather technical and we will not consider it here, but will only mention one of its most important consequences. The following theorem is due to Bass, Farrell and Hsiang [6] [12] and gives a more precise version of an earlier result of Bass, Heller and Swan [7].

THEOREM 4. — *There is a split exact sequence*

$$O \to K_1R \to K_1R[t] \oplus K_1R[t^{-1}] \to K_1R[t, t^{-1}] \to K_0R \to 0$$

Also we have $K_1R[t] = K_1R \oplus \text{Nil } R$ *where* $\text{Nil } R = 0$ *if* R *is regular.*

It follows that $K_1R[t, t^{-1}] = K_1R \oplus K_0R \oplus \text{Nil } R \oplus \text{Nil } R$.

The group Nil R is defined in a manner similar to K_0R using pairs (P, v) where v is a nilpotent endomorphism of P. Details may be found in [6]. This group also has an interesting topological application [11].

2. **Problems.**

One of the most important problems in algebraic K-theory is simply to compute the groups K_iR for various rings R. Group rings $Z\pi$ are particularly important in view of the topological applications. Considerable work has been done on various special cases. Most of the results can be found in [6] [8] [25]. Recently Kervaire and Murthy [23] computed $K_0(Z\pi)$ for π cyclic of prime power order. The computation makes use of classfield theory.

Another important problem is that of finding analogues of algebraic K-theory corresponding to the various classical groups. For $K_1R = GL(R)/E(R)$, the group $GL(R)$ is replaced by orthogonal, symplectic, and unitary groups. For K_0 we consider projective modules with quadratic, symplectic, and Hermitian forms. There is considerable topological interest in this since Wall's surgery obstruction groups are K-functors of this type [40]. Work on this problem has been done by Wall and his students, Bass [5], Milnor [27], Shaneson [30] and M. Stein [34].

A third major problem is that of defining functors $K_n(R)$ for all $n \in Z$. This problem is immediately suggested by the analogy with topological K-theory. A great deal of work is being done on this problem at the present time. I will discuss here some of the results which have been obtained.

It is natural to ask that the functors K_n satisfy the analogues of Theorems 2 and 3, i. e. that the exact sequences extend indefinitely in both directions. We would also like the analogue of Theorem 4 to hold with K_1, K_0 replaced by K_n, K_{n-1}. For $n \leqslant 0$ this determines K_n uniquely. If we have K_n, $K_{n-1}R$ must be the cokernel of the map $K_nR[t] \oplus K_nR[t^{-1}] \to K_nR[t, t^{-1}]$. This definition of K_n for $n < 0$ is due to Bass [6] who shows that it satisfies all the above requirements. Also $K_nR = 0$ for $n < 0$ if R is regular. Details may be found in [6].

For $n \geqslant 2$, we are not so fortunate. A number of definitions for K_nR have been proposed but no analogue of Theorem 2 has been found. This is explained by the following result.

THEOREM 5. — *There is no functor* K_2 *from rings to abelian groups such that for each cartesian diagram*

$$\begin{array}{ccc} A & \to & A_1 \\ \downarrow & & \downarrow f \\ A_2 & \to & A' \end{array}$$

with f a split epimorphism, there is an exact sequence

$$K_2A_1 \oplus K_2A_2 \to K_2A' \to K_1A \to K_1A_1 \oplus K_1A_2$$

We will now consider some of the definitions which have been proposed for K_nR, $n \geqslant 2$.

3. Milnor's K_2.

A very reasonable candidate for K_2R was defined by Milnor [26]. The elementary matrices $e_{ij}(r)$ satisfy certain relations which were found by Steinberg. Milnor considers the group $St(R)$ with generators $x_{ij}(r)$ satisfying the Steinberg relations

$$x_{ij}(r+s) = x_{ij}(r)x_{ij}(s), \ [x_{ij}(r), x_{kl}(s)] = 1$$

if $i \neq 1$, $j \neq k$, and $[x_{ij}(r), x_{jk}(s)] = x_{ik}(rs)$ if $i \neq k$. Define $\varphi : St(R) \to GL(R)$ by $\varphi(x_{ij}(r)) = e_{ij}(r)$. Milnor defines $K_2R = \ker \varphi = \text{center } (St(R))$.

THEOREM 6 (Milnor [26]). — *If*

$$\begin{array}{ccc} A & \to & A_1 \\ \downarrow & & \downarrow f_1 \\ A_2 & \underset{f_2}{\to} & A' \end{array}$$

is a cartesian diagram of ring homomorphisms and both f_1 and f_2 are onto, there is an exact sequence

$$K_2A \to K_2A_1 \oplus K_2A_2 \to K_2A' \to K_1A \to K_1A_1 \oplus K_1A_2 \to K_1A' \to \dots \to K_0A'.$$

This is quite a reasonable approximation to Theorem 2 and about the best one can expect in view of Theorem 5. There is no obvious analogue of Theorem 3 but the sequence of Theorem 1 extends to

$$K_2(R, I) \to K_2R \to K_2(R/I) \to K_1(R, I) \to K_1R \to K_1(R/I) \to \dots \to K_0(R/I).$$

Excision fails for $K_1(R, I)$ by Theorem 5.

It is not known whether the analogue of Theorem 4 holds for K_2 but J. Wagoner [38] has recently shown that $K_2(R[t, t^{-1}]) = K_2R \oplus K_1R \oplus ?$ ([1]). The last summand is still unknown. It is also not known whether $K_2R[t] = K_2R$ for R regular.

The group K_2R is extremely difficult to compute in general. Recently, H. Matsumoto succeeded in computing K_2F for any field F using a very ingenious construction. This leads to some interesting results on algebraic number fields [31].

One further difficulty with K_2 is that there is no obvious definition in terms of categories similar to that of K_0 and K_1. In § 7 we will discuss one possible solution to this problem.

4. Theory of Gersten and Swan.

For convenience, we will work here with rings without unit. If R is a free ring, Gersten [14] and Stallings [33] have shown that $K_0R = K_1R = 0$. This suggests

([1]) This result was also obtained independently by KAROUBI [43].

the requirement $K_nR = 0$ for free R. This resembles the effacability axiom of homological algebra [18] and suggests defining K_n by taking the derived functors of $GL(-)$. Since this is nonabelian, we have to use simplicial methods. A resolution of GL is a functor G_* from rings to simplicial groups together with an augmentation $\varepsilon: G_0 \to GL$ such that $\varepsilon\partial_0 = \varepsilon\partial_1$. We want to choose some such resolution and define $K_nR = \pi_{n-2}(G_*(R))$, for $n \geqslant 3$. We define K_1 and K_2 by the exact sequence $O \to K_2R \to \pi_0 G_*(R) \to GL(R) \to K_1R \to 0$. In [35], I used the theory of acyclic models to find such a resolution. The requirement that $K_nR = 0$ for a free ring R is equivalent to the requirement that G_* be aspherical on models, the models being the free rings. Among all such resolutions, there is a universal one G_* such that if H_* is any resolution which is aspherical on models, there is a map $G_* \to H_*$ unique up to homotopy. Using this G_* we get a series of functors which I will denote by K_n^S. A different approach was investigated by Gersten [15]. He considers the forgetful functor U: Rings $\to$ Sets and its coadjoint $F(S)$ = free ring on S. The composition $T = FU$ is a cotriple on the category of rings which can be used to construct a simplicial ring $T_*(R)$. Gersten uses the resolution $GL(T_*(R)) \to GL(R)$ to define his K functors which I will denote by K_n^G. Gersten's resolution is aspherical on models so there is a map $K_n^S \to K_n^G$. I have recently shown that this is an isomorphism and will therefore use the notation K_n^{GS} for these functors. It is known that $K_1^{GS} = K_1$ and that there is an epimorphism $K_2 \to K_2^{GS}$. This will be an isomorphism if and only if $K_2R = 0$ for free rings R but this has not yet been proved.

The sequence of Theorem 1 is easily extended by converting $G_*(R) \to G_*(R/I)$ into a fibration but it is not known whether there is an exact Mayer-Vietoris sequence for the K_n^{GS} under any reasonable hypothesis, eg. that of Theorem 6. It is also not known whether any of the statements of Theorem 4 hold for the K_n^{GS}. The problem of computing K_n^{GS}, even for a field, seems to be extremely difficult. It is also not known how to extend the definition to categories.

5. Theory of Karoubi-Villamayor.

In [28], Nobile and Villamayor gave another definition of K_n, essentially by defining the " suspension (2) " of a ring. Independently, Karoubi [20] gave a definition of K_n for categories. The two points of view were combined in [21]. The construction of these functors was rather complicated. Recently, Gersten gave a simpler definition using simplicial methods [16]. I will follow Gersten in denoting these functors by $\kappa_n(R)$. The theory has the disadvantage that $\kappa_1(R) \neq K_1(R)$. In fact,

$$\kappa_1(R) \approx GL(R)/U(R)$$

where $U(R)$ is the subgroup generated by all unipotent elements. However, except for this, the theory has many very nice properties. The functors κ_n can be characterized by axioms similar to those of homology theory [16] [21]. To state these, we need some preliminary definitions. We again use the category of rings without unit. A ring homomorphism $f: R \to R'$ will be said to have the covering homotopy property if every $X(t) \in GLR'[t]$ with $X(0) = 1$ can be lifted to $GLR[t]$. We say that f is a fibration is $R[x_1, \ldots, x_m, y_1, \ldots, y_n, y_1^{-1}, \ldots, y_n^{-1}] \to R'[x_1, \ldots, y_1, \ldots, y_1^{-1}, \ldots]$ has the

(2) Karoubi points out that the term " loop space " would be more appropriate.

covering homotopy property for all m, n. (This is somewhat stronger than the property used in [16] [21]). We also say that two ring homomorphisms $f, g: R \to R'$ are homotopic ($f \simeq g$) if there is a ring homomorphism $h: R \to R'[t]$ such that $\partial_0 h = f$, $\partial_1 h = g$, where $\partial_i : R'[t] \to R'$ is given by $\partial_i(p(t)) = p(i)$, $i = 0, 1$.

The functors κ_n are characterized by the following axioms [16] [21].

(1) For each exact sequence $O \to A \to B \to C \to 0$ of ring homomorphisms with f a fibration, there is a natural exact sequence

$$\ldots \to \kappa_n A \to \kappa_n B \to \kappa_n C \to \kappa_{n-1} A \to \ldots \to \kappa_1 B \to \kappa_1 C \to K_0 A \to K_0 B \to K_0 C$$

(2) If $f \simeq g: R \to R'$, then $\kappa_n(f) = \kappa_n(g): \kappa_n R \to \kappa_n R'$.

Axiom 2 is equivalent to the statement that $\kappa_n R[t] = \kappa_n R$. The functors κ_n can be computed as follows [16]. Let $ER = tR[t] = \ker \partial_0 : R[t] \to R$. Then $\partial_1 : ER \to R$ is a fibration. Let $\Omega R = t(t-1)R[t]$ be its kernel. Then Axiom 2 shows that $\kappa_n ER = 0$ so Axiom 1 gives $\kappa_{n+1} R = \kappa_n \Omega R$ for $n \geqslant 1$ and $\kappa_1 R = \ker [K_0 \Omega R \to K_0 ER]$.

To compare κ_n with K_n^{GS} we make one more definition. If F is a functor from rings to abelian groups, we define $\bar{F}(R)$ to be the co-equalizer of $F(\partial_0)$, $F(\partial_1): F(R[t]) \rightrightarrows FR$. This $\bar{F}$ satisfies Axiom 2 and $F \to \bar{F}$ is universal for maps of F into a functor satisfying Axiom 2. Gersten's simplicial definition and the universal property of K_n^S give us a map $K_n^{GS} \to \kappa_n$ and therefore $\bar{K}_n^{GS} \to \kappa_n$. It is not known whether this is an isomorphism for all n, but this is so for $n = 1$. Using a result of Gersten [15], it is easy to see that $\bar{K}_2(R) = \bar{K}_2^{GS}(R) = \kappa_2(R)$ if Nil $R = 0$, eg. if R is regular.

It is not known whether the analogue of Theorem 4 holds for the functors κ_n, but Gersten [17] has shown that this is so when Nil $R = 0$, i. e. $\kappa_n R[t, t^{-1}] = \kappa_n R \oplus \kappa_{n-1} R$ in this case (3). In general, if we define $\kappa_0 = K_0$, then $\kappa_n R[t, t^{-1}] = \kappa_n R \oplus \kappa_0 \Omega^n R$, so Theorem 4 will hold if and only if $\kappa_0 \Omega R = \kappa_1 R$. This is equivalent to the statement that, in the exact sequence of Axiom 1, we can replace K_0 by κ_0. If so, this can be extended to all $n \leqslant 0$ using the functors $\kappa_n = K_n$, $n \leqslant 0$, all of which satisfy Axiom 2.

So far, little progress has been made in computing $\kappa_n R$ for $n \geqslant 2$. Even for the case where R is a field, it is not known whether $\kappa_2 R = K_2 R$. It is quite possible, however, that $\kappa_n R$ will turn out to be easier to compute than K_n. Perhaps a simpler proof of Matsumoto's theorem could be found in this way. Karoubi's definition can be used to define κ_n for categories but this is quite complicated compared to the definitions of K_0 and K_1.

6. Theory of Quillen.

A very interesting topological definition of $K_n R$ was recently proposed by Quillen [29]. I would like to thank Quillen for sending me a detailed account of his work. All rings here will be assumed to have a unit. The definition was suggested by the relation between the homology of the group $GL(R)$ and the functors $K_1(R)$, $K_2(R)$. In fact, $K_1 R = H_1(GL(R))$ and $K_2 R = H_2(E(R))$. Quillen takes the classifying space $BGL(R)$ and attaches 2-cells to kill the subgroup $E(R)$ of $\pi_1 BGL(R) = GL(R)$. This introduces new cycles in dimension 2 but these can all be killed by attaching 3-cells. The result is a space B_R with some very remarkable properties.

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(3) This result was also obtained independently by KAROUBI [43].

THEOREM 7 (Quillen). — *B_R is a homotopy associative, homotopy commutative H-space. The map $BGL(R) \to B_R$ gives an isomorphism of homology.*

Clearly B_R can be regarded as the best H-space approximation to $BGL(R)$. Quillen also considers a functor R (X) which is defined as K_0 of the category of finitely generated projective R-modules with $\pi_1(X)$-action. He shows that the functor $[X, B_R]$ is, in a reasonable sense, the best approximation to R_R by a representable functor. This justifies his definition of $K_n R$ as $\pi_n(B_R)$. We denote these functors here by K_n^Q. Quillen also notes that $K_1^Q = K_1$ and $K_2^Q = K_2$ (Milnor's K_2). This gives further justification for his definition.

Using his calculation of the cohomology of finite linear groups, Quillen can actually compute all of the $K_n^Q(R)$ for the case where R is a finite field.

THEOREM 8 (Quillen). — *For $i > 0$, $K_{2i}^Q(F_q) = 0$ and $K_{2i-1}^Q(F_q) = \otimes\, {}^i\mu_{q^i-1}$, where μ_m is the group of m — th roots of unity in the algebraic closure of F_q.*

To compute $K_n^Q(R)$ for other rings R, the first step would be to calculate $H_*(GL(R))$. For example, it is reasonable to conjecture that $K_n(\underline{Z})$ is a torsion group for all n. This is equivalent to the conjecture that $H_*(GL(Z), Q)$ is trivial. In fact, Mumford and Milnor have conjectured that $H_*(GL_n(\underline{Z}), \underline{Q})$ is trivial for each n. If P_n is the space of positive definite quadratic forms in n variables and $X_n = P_n/GL_n(\underline{Z})$, the above conjecture is equivalent to the conjecture that X_n is acyclic over $\underline{Q}$. Now X_2 is actually contractible and the same seems to be true for X_3 although the proof involves a long calculation which I have not checked. Perhaps X_n is contractible for all n. A result of Magnus [24] shows that $GL_n(\underline{Z})$ is a direct limit of subgroups related to GL_2 and GL_3. It should be possible to use this to reduce the general case to that where $n = 2$ and 3. It would also be interesting to extend the results of Magnus to other rings. Possibly this could be used to prove a stability theorem for K_n^Q analogous to that of Bass for K_1 [6].

It is natural to ask whether $K_n^Q \approx K_n^{GS}$. This would imply that for a free ring R (with unit) we would have $K_n^Q(R) \approx K_n^Q(\underline{Z})$ or, equivalently, that $H_n(GL(R)) \approx H_n(GL(\underline{Z}))$. A proof of this would probably be one of the main steps in showing that $K_n^Q \approx K_n^{GS}$.

One can produce an exact sequence for K_n^Q similar to that of Theorem 1 by converting $B_R \to B_{R/I}$ into a fibration. I do not know whether the analogue of Theorem 4 holds for K_n^Q.

7. Extension to categories.

The groups $K_n^Q(R)$ depend only on the group $GL(R)$. This property is regarded as a disadvantage by Karoubi and Villamayor [21] who would like $K_n(R)$ to reflect properties of the ring R and not just those of $GL(R)$. However, this property suggests a simple way to extend the definition of K_n^Q to categories. If F is a functor from groups to groups, we would like to define $K_F(\mathscr{A})$ for a category $\mathscr{A}$ by taking some sort of direct limit of the groups F (Aut $(\mathscr{A})$) for $A \in \mathscr{A}$. The functor K_1 was treated in this way by Bass in [6]. I will give here one easy way of doing this which is suggested by the definition of K_1.

If E is a short exact sequence $O \to A' \xrightarrow{i} A \xrightarrow{j} A'' \to 0$, we define Aut (E) to be the subgroup of Aut (A') $\times$ Aut (A) $\times$ Aut (A'') consisting of those $(\alpha', \alpha, \alpha'')$ with $\alpha i = i\alpha'$, $j\alpha = \alpha'' j$.

Let F be a functor from groups to groups. We define $K_F(\mathscr{A})$ to be the abelian group with generators $[A, u]$ for $A \in \mathscr{A}$, $u \in F(\mathrm{Aut}(A))$ and with the relations.

(1) $[A, uv] = [A, u] + [A, v]$.

(2) If E, as above, is a short exact sequence and $w \in F(\mathrm{Aut}(E))$ has images $u' \in F(\mathrm{Aut}(A'))$, $u \in F(\mathrm{Aut}(A))$, $u'' \in F(\mathrm{Aut}(A''))$, then $[A, u] = [A', u'] + [A'', u'']$.

For example, if $F(G) = G$ or $G/[G, G]$, then $K_F(\mathscr{A}) = K_1(\mathscr{A})$. If F is a constant functor with value $F(G) = \pi$ for all G, then $K_F(\mathscr{A}) = K_0(\mathscr{A}) \oplus \pi/[\pi, \pi]$.

To get $K_n(\mathscr{A})$, we define F_n by taking BG, killing $[G, G] \subset \pi_1 BG$ by Quillen's method, getting a space BG^+, and setting $F_n(G) = \pi_n BG^+$. We then let $K_n(\mathscr{A}) = K_{F_n}(\mathscr{A})$ (The group Aut (A) should be replaced by $\varinjlim$ Aut (A^n) here).

Now, if $\mathscr{A} = \mathscr{P}_R$ is the category of finitely generated projective R-modules, it is not hard to show that $K_F(\mathscr{P}_R) = F(GL(R))$ provided that F has certain properties. These properties are exactly those which Quillen proves in his work on K_n^Q. Therefore we see that $K_n(\mathscr{P}_R) = K_n^Q(R)$. This gives some justification for our definition. In particular, since $K_2^Q = K_2$, this method gives a reasonable extension of Milnor's K_2 to categories.

If we use only split exact sequences, it should not be too hard to express $K_F(\mathscr{A})$ as a filtered direct limit as in [6]. In this way we could presumably obtain a space B_{A} with $K_n(\mathscr{A}) = \pi_n(B_{\mathrm{A}})$.

A more general form of the above construction is obtained by replacing F (Aut (A)) by K (End A) where K is a functor from rings to groups. In this way, any K-theory could be extended to categories.

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