MODERN DEVELOPMENT OF SURFACE THEORY

By A. D. ALEXANDROV

I will try to give in my lecture an outline of a general theory of surfaces as it has been developed during the past decade by a group of Russian geometers, U. F. Borisov, V. A. Zalgaller, A. V. Pogorelov, U. G. Reshetnak, I. J. Backelman, V. V. Streltsov and myself. This theory arose as a natural generalization of the theory of convex surfaces, the systematic presentation of which was given in my book *The Intrinsic Geometry of Convex Surfaces*, published just 10 years ago. Now this general theory has grown into an extensive branch of geometry with its own concepts, problems, methods and numerous results.

It would be hopeless to try to give here more than a general idea of the theory, so that all details and many results even of a fundamental character must be omitted. In constructing the foundations of the theory my aim was to define and to study the most general surfaces which allow of concepts and results analogous to those of classic Gaussian theory of surfaces. There are, first of all, two basic concepts of Gaussian theory; that of the intrinsic metric of a surface and that of its curvature. We accept an integral point of view according to which the metric is determined not by means of a line-element but by means of the distances between points measured *in* the surface, and the curvature is considered as a set-function, so that we mean integral curvature of a domain on the surface instead of the curvature at a point.

Let a surface S possess the property that any two of its points x, y can be joined by a curve $\widehat{xy}$ which lies in S and has a finite length $s(\widehat{xy})$. We define the intrinsic distance as

$$\rho_s(x, y) = \inf_{\widehat{xy} \subset S} s(\widehat{xy}). \tag{1}$$

It is evident that it satisfies the usual conditions imposed upon the general concept of a metric:

$$(1)\ \rho(x, y) = 0 \text{ if and only if } x = y;$$

$$(2)\ \rho(x, y) + \rho(y, z) \geqslant \rho(z, x).$$

Thus the surface becomes a metric space with the metric ρ_s.

There are two points of view; one may consider a surface as a figure in the space, being interested in its special shape; or the surface may be considered as a metric space with the intrinsic metric ρ. In this case we speak of the intrinsic geometry of the surface, while in the first case we speak of its external geometry.

Corresponding to these two points of view there are two concepts of the curvature of a surface. The external curvature is measured by means of the area of the spherical image and the intrinsic curvature is measured by means of the excesses of geodesic triangles, the excess of a triangle T with the angles α, β, γ being, by definition,

$$\delta(T) = \alpha+\beta+\gamma-\pi.$$

As we are going to consider surfaces with a definite, i.e. finite, curvature we speak of surfaces with bounded curvature. Thus the objects of the purely intrinsic theory are two-dimensional metric manifolds with bounded intrinsic curvature (M.B.C.) and the objects of the external theory are surfaces with bounded curvature (S.B.C.).

The intrinsic and the external theories are not independent; there exists a close connection between them and, first of all, the well-known theorem by Gauss which asserts the equality of the intrinsic and the external curvatures for, at least, sufficiently regular surfaces. Thus we have sketched a certain programme; to give the strict definitions of manifolds of bounded curvature and of surfaces of bounded external curvature, to study their properties and to establish the connection between the intrinsic and the external properties of these surfaces.

A somewhat different, and in some respects even more general, approach to the theory of surfaces may be based upon the concept of parallel translation, which is closely connected with the concept of curvature because of the well-known Gauss–Bonnet theorem. The parallel translation of a vector along a curve on a surface can be defined both in intrinsic and external terms by means of the Levi-Civita construction. If we follow this trend of ideas the objects of the theory are the metric manifolds and the surfaces where parallel translation of vectors is defined for a sufficiently ample set of curves. Such surfaces were studied recently by Borisov, and I will give an account of his results in the last part of my lecture.

1. The definition of M.B.C.

1.1. The intrinsic metric. The length of a curve is defined in any metric space in the usual manner; it is the supremum of the sums of the distances between successive points of the curve. If any two points of a set S in a metric space can be joined by a curve which lies in S and has finite length, we call the set metrically connected. Now we say that the metric of a space is intrinsic in itself, or, simply, intrinsic, provided that the space is metrically connected and the distance between any two points is equal to the infimum of the lengths of curves joining these points.

The introduction of this concept is justified by the following theorem. Let S be a metrically connected set in a metric space R. Then if we define the distance

$$\rho_S(x, y) = \inf_{\widehat{xy} \subset S} s(\widehat{xy}),$$

the metric thus introduced in S proves to be intrinsic in the above sense. Accordingly we speak of the intrinsic metric induced in S by the metric of the surrounding space R. The definition of the metric of a surface given above is a particular instance of this general definition. Thus, our theorem being applied, we see that this metric is intrinsic in itself.

An M.B.C. must be a surface considered from the intrinsic point of view. Therefore it is natural that our first postulate in the definition of an M.B.C. should be the following one. An M.B.C. is a two-dimensional metric manifold with a metric intrinsic in itself.

1.2. The angle. In order to formulate the condition of the boundedness of the curvature by means of the excesses of triangles we have to define a triangle and an angle. (These definitions will be valid for any metric space.) We define, first, the shortest line or segment xy as a curve joining the points x, y and having the length equal to the distance $\rho(x, y)$ between them. Then it is evident what is understood by a triangle or a polygon. We note that in any manifold and even in any locally compact space with an intrinsic metric each point has a neighbourhood any two points of which can be joined by a segment.

The definition of an angle is given as follows. Let L, M be two curves with the common initial point O. Take the variable points $X \in L$, $Y \in M$ $(X, Y \neq O)$ and construct the plane triangle $O'X'Y'$ with sides equal to the distances OX, OY, XY. Let $\gamma(XY)$ be the angle of this triangle at the vertex O'. We define the upper angle between the curves L and M as

$$\bar{\alpha}(LM) = \overline{\lim_{X, Y \to O}} \, \gamma(XY).$$

This angle always exists.

Further, we say that there exists a definite angle between the curves L, M provided that the limit of the angle $\gamma(XY)$ exists, and in this case we define the angle

$$\alpha(LM) = \lim_{X, Y \to O} \gamma(XY).$$

Making use of the upper angle, which always exists, we define the excess of a triangle T as
$$\delta(T) = \bar{\alpha}+\bar{\beta}+\bar{\gamma}-\pi,$$
$\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ being the upper angles between the sides of T.

1.3. The condition of the boundedness of the curvature. Now we are ready to formulate the second and last postulate which defines an M.B.C. Each point has a neighbourhood U such that the sum of the excesses of pairwise non-overlapping triangles lying in U is uniformly bounded above:
$$\Sigma\,\delta(T) < N.$$
N does not depend upon the set of the triangles and depends upon the neighbourhood U only.

Thus, briefly speaking, an M.B.C. is a 2-manifold with an intrinsic metric and with uniformly bounded sums of the excesses of non-overlapping triangles, at least in certain neighbourhoods which cover the manifold. Sometimes we speak of a metric of bounded curvature, which is preferable in the case where we have to consider not only one but many metrics given in the same manifold, i.e. when the manifolds are topologically mapped on to one and the same manifold.

1.4. Curvature. The definition of curvature is quite natural and runs as follows. We define the positive and the negative parts of the curvature of an open set G as the upper and the lower bounds of the sums of the excesses of the pairwise non-overlapping triangles lying in G
$$\omega^{+}(G) = \sup\Sigma\,\delta(T), \quad \omega^{-}(G) = \inf\Sigma\,\delta(T).$$
The curvature itself is defined as
$$\omega(G) = \omega^{+}(G)+\omega^{-}(G),$$
and the absolute curvature
$$\Omega(G) = \omega^{+}-\omega^{-}.$$

After that one can prolong these set-functions on to the ring of Borel sets by following the routine of measure theory. Then the fundamental fact is that these set-functions prove to be totally additive.

Our conditions concerning the excesses of triangles seem to be, in a certain respect, the minimum one has to suppose in order to have the possibility of defining the curvature as a totally additive set-function.

Zalgaller has given an abstract construction of a measure (non-negative totally additive set-function) which covers the definition of Lebesgue's measure, the above definition of the curvature and of many other set-functions which occur in geometry provided the definition starts from certain magnitudes ascribed to such elementary sets as the area of rectangles in the case of Lebesgue's measure or the excesses of triangles in the case of curvature.

1.5. Some other concepts. We define, further, such concepts as the area of a domain, the direction of a curve at a point, and the integral geodesic curvature (the bend) of a curve. For example, two curves are said to have the same definite direction at their common initial point provided the upper angle between them is equal to zero. The angle between two curves depends upon their directions only, i.e. it is the same for all pairs of curves with the same directions. The set of directions at a given point is isometric, with respect to its angular metric, to the set of generators of a cone.

2. The study of M.B.C. by means of approximation

2.1. With the exception of direct methods the first and most fruitful method in the theory of M.B.C. is that of approximation of general M.B.C. by means of polyhedra. First of all we have the following fundamental theorem: Let an intrinsic metric ρ given in a manifold M be the limit of a uniformly convergent sequence of metrics ρ_n with uniformly bounded positive parts of curvatures. Then ρ is a metric of bounded curvature also, and the curvatures of the metrics ρ_n weakly converge to the curvature ω of ρ in the sense that for any continuous function $f(x)$ different from zero on a compact set only,

$$\int f(x)\,\omega_n(dM) \to \int f(x)\,\omega(dM).$$

In particular, the limit of Riemannian metrics with uniformly bounded positive parts of curvature, i.e. $\int_{K>0} K\,dS$, is a metric of bounded curvature.

2.2. The simplest M.B.C. are manifolds with polyhedral metrics, or, in short, polyhedra. A polyhedron is such a manifold with an intrinsic metric, each point of which has a neighbourhood isometric to a cone. This descriptive definition is equivalent to a constructive one: a polyhedron is a manifold with intrinsic metric constructed of plane triangles, or in other words it allows of a subdivision into triangles isometric to plane ones. The curvature of a polyhedron is concentrated in its vertices, i.e. in such points the neighbourhoods of which do not reduce to pieces of the plane. The whole angle θ around such a point is different from 2π and is connected with the curvature ω of the point by the equation

$$\omega = 2\pi - \theta.$$

The curvature of a polyhedron is the sum of the curvatures at the vertices and its positive part is the sum extended over the vertices with the whole angle $\theta < 2\pi$. The convergence theorem above implies that the limit of polyhedral metrics with uniformly bounded positive parts of curvatures is a metric of bounded curvature.

The converse theorem exists in the following form: Any metric of bounded curvature is a limit of a sequence of polyhedral metrics with uniformly bounded absolute curvatures. Or in a more exact form, let P be a compact polygon in an M.B.C., R, and let ρ be the intrinsic metric induced in P by the metric of R. There exists a sequence of polyhedra P_n with uniformly bounded absolute curvatures, and of mappings of these on to P, such that the metrics determined in P by these mappings uniformly converge to ρ. And the positive and the negative parts of their curvatures weakly converge to the corresponding parts of the curvature of the metric ρ. We say that the convergence is regular.

2.3. If we combine this result with the previous convergence theorem, we get a new definition of an M.B.C. as a manifold which is, at least locally, the limit of polyhedra with uniformly bounded positive parts of curvatures. Polyhedra being, obviously, the limits of Riemannian manifolds, an M.B.C. proves to be the limit of Riemannian manifolds with uniformly bounded positive parts of their integral curvatures. In other words, the class of M.B.C. is the closure of the class of Riemannian or of polyhedral manifolds in the sense of uniform convergence of metrics under the condition of the uniform boundedness of positive parts of curvatures, or, what proves to be the same, the boundedness of absolute integral curvatures.

2.4. The above theorems provide the foundations of a method of studying M.B.C. by means of approximation by polyhedra. This method is applied to the study of some fundamental properties of M.B.C. For instance, we define the area of a polygon P in an M.B.C. as the limit of the areas of polyhedra regularly convergent to P.

In order to ensure a standard application of this method we have to supply ourselves with a number of general theorems concerning the convergence of various magnitudes associated with an M.B.C. and figures in it, such as polygons, curves, angles, area, integral geodesic curvature, etc. In fact, we have such theorems at our disposal.

Suppose, now, we are given a problem concerning M.B.C. Then we formulate it for polyhedra, and attempt to solve it for them. Polyhedra being the objects of elementary geometry, the problem reduces to one of a rather intuitive character. And if we succeed in solving the problem for polyhedra it only remains for us to apply suitable convergence theorems in order to obtain the general result. Most of the concrete results in the theory of M.B.C. have been obtained in this way.

3. Analytic characterization of M.B.C.

3.1. Probably the most important result obtained by means of this method is the following theorem by Reshetnak (1953). The metric of each M.B.C. may be determined by means of a line-element of the form

$$ds^2 = \lambda(dx^2+dy^2), \tag{1}$$

where $\log\lambda$ is the difference of two subharmonic functions, and conversely, any metric determined by such a line-element, with the same condition for λ, is a metric of bounded curvature.

More exactly the first part of the theorem may be expressed as follows: Let P be a polygon in an M.B.C., R, homeomorphic to a circle. Then, by means of a map of P on to a domain D of the xy-plane, one can introduce in P co-ordinates x, y in such a way that the length of any curve in P which is the image of a broken line L in D is equal to

$$s = \int_L \sqrt{\{\lambda(dx^2+dy^2)\}}. \tag{2}$$

And if we put $z = x+iy$, $\lambda(x,y) = \lambda(z)$ is representable by means of the following formula

$$\log\lambda(z) = -\frac{1}{\pi}\int_D \log|z-\zeta|\,\omega(dE_\zeta)+h(z), \tag{3}$$

where $\omega(E_\zeta)$ is the curvature of the set in P corresponding to $E_\zeta \subset D$; the integral is understood in the Lebesgue–Radon sense, and $h(z)$ is a suitably chosen harmonic function in D. Since $\omega = \omega^+ + \omega^-$, $\omega^+ \geqslant 0$, $\omega^- \leqslant 0$, the well-known integral representation of subharmonic functions implies that $\log\lambda$ is the difference of two such functions.

This theorem generalizes the well-known fact that the line-element of a regular surface may always be represented in the conformal form (1) and λ is connected with Gaussian curvature by the formula

$$\Delta\log\lambda = -2K\lambda. \tag{4}$$

If we consider this formula as a Poisson equation for $\log \lambda$ we just get the solution in the form (3) with $\omega(dE_\xi) = K\lambda\, d\xi\, d\eta$.

3.2. Reshetnak's theorem adds to the two above definitions of an M.B.C. (i.e. the initial axiomatic one and the constructive one) a third one, the analytic definition. It opens the way for applications of analytic methods to the study of M.B.C. But so far nobody has followed this way to any considerable extent. Almost all results in the theory so far have been obtained by means of geometric methods.

4. Geometrical methods and some results of the theory of M.B.C.

4.1. There are two geometric methods in the theory of M.B.C., that of approximation by polyhedra and the other one which I call the method of cutting and gluing. It is based upon 'the theorem of gluing'. As a polyhedron is constructed or glued up of triangles, so, more generally, an M.B.C. may be constructed of pieces of given M.B.C., for example, of polygons cut out of any M.B.C., by means of gluing them together along segments of their boundaries. The possibility of such a construction under certain conditions imposed upon the boundaries of the glued pieces is ensured by a theorem which I call the 'theorem of gluing'.

In the simplest case when the glued pieces are polygons the theorem reduces to the following statement: Suppose we are given a complex of polygons cut out of some M.B.C.; suppose the complex is a manifold R with a boundary (possibly void) and that the identified segments of the sides of the polygons have equal lengths. Then if we define for any two points $x, y \in R$ the distance as the greatest lower bound of the lengths of curves joining x, y (the lengths being defined by metrics which are already given in each polygon) then R turns out to be an M.B.C.

In the case of more general domains than polygons an additional condition is necessary. It concerns the integral geodesic curvatures of the boundaries, for these curvatures as segment functions should be of bounded variation. For instance the conditions are fulfilled provided we have pieces of regular surfaces bounded by curves with piecewise continuous geodesic curvature.

4.2. The method of cutting and gluing is as old as geometry itself. The ancient proofs of Pythagoras's theorem as well as many other ancient proofs in elementary geometry consist just in cutting certain figures into suitable pieces and rearranging, or, let me say, gluing those pieces together so as to make the statement obvious. We apply just the same idea to our far more general and abstract figures.

4.3. In order to show that we do not only have general definitions let me mention a few results which were obtained by means of our methods and which were new for regular surfaces as well. I formulate these results in ordinary terms of differential geometry in order to avoid further definitions of certain concepts of the general theory.

(1) Consider surfaces S in a space of constant curvature K_0. Let them be homeomorphic to a circle and have prescribed the perimeter p and the positive part of their relative curvature, i.e.

$$\int_{K>K_0} K\,dS,$$

S being the area and K the Gaussian curvature. We put the question: what is the upper bound for the area of such surfaces? The answer is that it is the area of the circular cone with the same prescribed data (provided such a cone does exist, which is ensured by a simple condition).

(2) Consider the same surfaces in the same space and suppose they have non-positive relative curvature, i.e. $K \leqslant K_0$, so that for any domain, $\omega \leqslant K_0 S$. We ask, once again, about the maximum of the area. The answer is that the maximum is attained by the surfaces isometric to a circle with the same perimeter. It is worth mentioning that Reshetnak succeeded in proving that such an isometric inequality for any small circle on a surface is not only necessary but also sufficient for the Gaussian curvature of the surface to be $\leqslant K_0$.

(3) Consider now a curve L on a surface S homeomorphic to a circle. Let ω^+ be the positive part of the curvature of S, and s and τ the length and the integral of the absolute value of the geodesic curvature of the line L. The following results hold.

(3*a*) Let $\omega^+ + \tau < \pi$ and the distance between the ends of L be r. Then

$$s \leqslant \frac{r}{\cos\frac{1}{2}(\omega^+ + \tau)},$$

and the estimation is sharp. The equality is attained in the case of an isosceles triangle, r being its base and s the sum of two other sides.

(3*b*) Under the condition $\omega^+ + \tau < 2\pi$, the curve L either has no multiple points and it is necessarily so provided $\omega^+ + \tau < \pi$, or may be divided into two branches without such points. In the latter case it consists of a loop (i.e. a curve without multiple points and with coincident ends) and of one or two branches each of which has neither multiple points nor common points with the domain bounded by the loop.

(3*c*) Under the same condition $\omega^+ + \tau < 2\pi$ the length of the curve does not exceed a certain constant which depends on $\omega^+ + \tau$ and the size (the diameter, or the perimeter) of the surface. In particular, p being the perimeter of the surface,

$$s \leqslant \frac{p}{1+\cos\frac{1}{2}(\omega^{+}+\tau)}, \quad \text{if} \quad \omega^{+}+\tau \leqslant \pi,$$

and

$$s \leqslant \frac{p}{\sin\frac{1}{2}(\omega^{+}+\tau)}, \quad \text{if} \quad \pi < \omega^{+}+\tau < 2\pi.$$

These estimations are sharp.

The above theorems are cited as examples of numerous concrete results that are obtained in our theory. I could give many other examples, some similar to the above and some of quite different type, but there is not time.

5. The surfaces which are M.B.C.

5.1. The question arises as to what are the surfaces, in ordinary Euclidean space E^3, which, from the intrinsic point of view, are M.B.C., and whether or not it is possible to embed any M.B.C., at least locally, in E^3. According to the beautiful Nash–Kuiper theorem, any Riemannian manifold allows of such an embedding, and even in the large provided it is orientable and compact. Any M.B.C. can be approximated by Riemannian manifolds, and this makes it obvious enough that one can extend the Nash–Kuiper result to an M.B.C. But the Nash–Kuiper embedding is too arbitrary, and it does not ensure that deep connection between the external and the intrinsic properties of the surface which is characteristic for more regular embeddings of Riemannian manifolds. The most fundamental of these connections being Gauss's theorem, we define a Gaussian embedding as one which preserves the validity of Gauss's theorem in at least a suitably generalized form. Thus our problem consists in finding Gaussian embeddings.

5.2. But in order to approach the solution of the problem it is necessary, at first, to determine and to study the classes of surfaces among which the realization of a given M.B.C. may be sought. Hence, our first problem is to determine and to study sufficiently general surfaces which are M.B.C. from the viewpoint of their intrinsic metric and which allow of a generalization of Gauss's theorem. After that the second problem is to prove, if possible, that an M.B.C., possibly subject to certain additional conditions, can be realized in E^3 as a surface of the given class. The third problem consists of a deeper study of the dependence of the external properties of the surface upon the intrinsic ones. In particular there is the question concerning the dependence of the degree of regularity of the surface upon that of its intrinsic metric. The special importance of the last problem for the general theory consists in the fact that its positive solution makes one sure that the solutions of problems of embedding and bending of surfaces obtained in the scope of general theory give the solution of corresponding problems in terms of differential geometry provided the surfaces have sufficiently regular intrinsic metric.

5.3. The surfaces R.D.C. (representable by differences of convex functions). As far as our first problem is concerned, the following results have been obtained. I studied the surfaces which, at least locally, allow of a representation by an explicit equation of the form $z = f(x, y)$, where x, y, z are Cartesian co-ordinates and f is the difference of two convex functions. In short these are R.D.C. surfaces. Any polyhedron which allows of a local representation by the equation $z = f(x, y)$, any convex surface, and any surface with first derivatives subject to the Lipschitz condition are R.D.C. surfaces. It is not difficult to verify that an R.D.C. surface can be approximated by regular surfaces with uniformly bounded absolute curvatures. Therefore, our convergence theorem for M.B.C. being applied, we see that these surfaces are M.B.C.

To establish the connection between the intrinsic and the external properties of these surfaces was not so easy. A generalized Gauss's theorem exists, but the exact definitions of the spherical image and of the external curvature require some consideration because of the absence of either tangent or supporting plane at an arbitrary point. I will not give such details here.

The connection between the intrinsic and the external geometry is not exhausted by Gauss's theorem. For instance, we have an intrinsic concept of an angle and of a direction of a curve. For R.D.C. surfaces they prove to be equivalent to the corresponding external concepts. The existence of an intrinsic direction of a curve at its initial point O proves to be equivalent to the existence of the ordinary tangent line, and the angle between two curves proves to be equal to the angle between their tangents measured on the tangent cone of the surface at the point O.

5.4. Surfaces with generalized second derivatives. Backelman has studied the surfaces which allow of a parametric representation $x(u, v)$, $y(u, v)$, $z(u, v)$ with continuous first derivatives and with generalized second derivatives summable by squares. He succeeded in extending to such surfaces all basic results of ordinary differential geometry provided the second derivatives which occur in its formulae are understood as generalized ones.

5.5. Smooth S.B.C. Backelman's surfaces are not, in general, R.D.C. Still they are included in a class of surfaces studied by Pogorelov. These are smooth surfaces with bounded external curvature. The exact definition is the following. The surface S is supposed to have at each point a tangent plane which continuously depends upon the point. Therefore the spherical image is defined. Let now $F_1, \ldots, F_n$ be closed sets on S, pairwise without common points, and let $\sigma(F_i)$ be the areas of their spherical images. The condition of the boundedness of the external curvature requires that $\Sigma\sigma(F_i)$ be uniformly bounded for all such systems of closed sets F_i.

First Pogorelov proves that his surfaces are M.B.C. Now as far as Gauss's theorem is concerned, there is no difficulty in defining absolute external curvature as $\sup \Sigma\sigma(F_i)$. But it proved to be far more difficult to define curvature with suitable sign and to prove Gauss's theorem for it. Pogorelov's considerations are rather subtle. First he divides the points of the surface into two classes: the ordinary points and the non-ordinary ones, the first being characterized by the following property. An ordinary point A has such a neighbourhood U that no point $X \in U$ has the same spherical image as A. It is proved that the non-ordinary points are in a certain sense negligible.

The ordinary points are classified according to their indices, i.e. according to the number and the sense of the circuits of the spherical image of a closed simple curve surrounding the point. The sign of the index is ascribed to the point, and the positive part of the curvature is defined as the area of the spherical image of the set of all positive points, provided the multiplicity of the spherical image is taken into consideration. The negative part of the curvature is defined similarly. Then Pogorelov proves that these set-functions are equal to the corresponding intrinsic magnitudes.

Some additional results are worth mentioning. The surfaces with everywhere positive curvature are convex, and the surfaces whose positive and negative curvature both vanish are developable. If a surface has a locally one-to-one spherical image then it is convex, provided the spherical map preserves orientation, otherwise it is of negative curvature.

5.6. Backelman has noted lately that Pogorelov's proof of the theorem that his surfaces are M.B.C. does not make use of the smoothness of the surfaces and therefore allows of an immediate generalization to the surfaces with the following property.

Consider a surface S, and divide it into small pieces S_i. Consider the spherical image of those points of an S_i at which S_i has a supporting plane, with the exception of the points which lie upon the boundary of S_i. Let $\sigma^+(S_i)$ be the area of this spherical image. The condition imposed upon the surface is: the sums $\Sigma\sigma^+(S_i)$ are required to be uniformly bounded for all subdivisions of S into pieces S_i. It is obvious that in this construction we just catch the positive part of the external curvature of the surface. Hence the surfaces subject to the above condition may be characterized as the surfaces of bounded positive external curvature, quite analogously to our definition of M.B.C. where the condition of boundedness is imposed just upon the positive part of the curvature.

The simple repetition of Pogorelov's proof for smooth surfaces—for Pogorelov just makes use of the above construction—leads to the result: a surface with bounded positive part of the external curvature is an M.B.C. But up to now this is essentially all that is known about such surfaces. In particular we have neither a proof of Gauss's theorem nor even the definition of curvature for such surfaces. Their study is our next problem.

I believe that these surfaces form that general class of surfaces among which we have to expect the local realization of abstract M.B.C. They include all the above classes of surfaces, such as, for instance, the R.D.C. surfaces.

6. Embedding problems

6.1. As far as the embedding problems are concerned we have no general results except those for convex surfaces. First of all there is my old theorem on the embedding of manifolds into a space $R^3_{K_0}$ of constant curvature K_0. In the case of a compact manifold it reads as follows. Let M be an M.B.C. homeomorphic to a sphere and let its curvature ω for any domain G be subject to the condition $\omega(G) \geqslant K_0 S(G)$, S being the area. Then there exists in $R^3_{K_0}$ a convex surface isometric to M. According to Pogorelov's theorem this surface is unique up to a motion or reflection.

6.2. The problem of the regularity of the embedding provided the metric of M is representable by means of a line-element ds^2 has been solved to the following extent. Suppose the coefficients of the line-element have first derivatives subject to the Lipschitz condition. Then the surface S is smooth and realizes not only the metric but the line-element ds^2 itself. That is (in the case of Euclidean space) there exists a parametrization u, v of M such that the vector-function $\mathbf{x}(u, v)$ representing the surface S satisfies the equation $d\mathbf{x}^2 = ds^2$. In other words, the embedding solves not only the generalized problem but also the classical problem itself.

The same property of regularity takes place in the small, i.e. for a convex surface realizing any domain in an M.B.C., provided the curvature is subject to the inequalities:

$$C > \frac{\omega(G)}{S(G)} > K_0+\epsilon \quad (\epsilon = \text{const} > 0).$$

6.3. Pogorelov established stronger regularity of a convex surface with prescribed line-element, provided the latter is at least five times differentiable. Pogorelov's theorem reads as follows: If the line-element of a convex surface S in $R^3_{K_0}$ is $k+1$ times ($k \geqslant 4$) differentiable (analytic) and has curvature $K > K_0$ everywhere, then S is k times differentiable (analytic). The result is valid for any convex surface in a space.

6.4. Notwithstanding their strength, these results seem to me far from being sufficient. If a line-element is $k+1$ times differentiable the surface proves to be k times differentiable (or we have somewhat stronger results if a Lipschitz condition is implied). But in my theorem above $k = 1$ and in Pogorelov's theorem $k \geqslant 4$. Meanwhile, the ordinary formulae of differential geometry imply second derivatives, or third ones if we write the Peterson–Codazzi equations not in an integral but in their ordinary form. Therefore the most interesting and important problem consists in finding the minimal conditions which ensure two times or three times differentiability of the surface. This problem, however, still remains unsolved.

6.5. I would like to mention that Pogorelov succeeded recently in proving theorems on the embedding of a manifold into a three-dimensional Riemannian space (the curvature of the manifold being sufficiently great). The theorem of regularity and that of the uniqueness of the embedding are also established. A short exposition was published in *Vestnik* of Leningrad University, 1957, N 7, and the full details are given in a booklet published later by Kharkov University Press.

7. The surfaces with parallel translation of vectors

7.1. As was mentioned at the beginning of my lecture there is a different approach to the theory of general surfaces based upon the concept of parallel translation of a vector. This idea has been recently realized by Borisov.

The definition of parallel translation was given by Levi-Civita. Consider a smooth surface S and a line $L \subset S$ joining the points A, B. Take a vector $\mathbf{a}$ tangent to S at the point A. Take, now, successive points $A = X_0, X_1, \ldots, X_n = B \in L$. If we project the vector $\mathbf{a}$ onto the tangent plane P_1 at the point X_1, we get a vector $\mathbf{a}_1$ at X_1; then project this vector onto the tangent plane P_2 at the point X_2 and so on. At the end we get a vector $\mathbf{a}_n$ at the point B. Suppose that the vector $\mathbf{a}_n$ tends to a certain limit $\mathbf{b}$ provided the points X_i are taken to be more and more dense upon the line L. Then we say that $\mathbf{b}$ is the result of parallel translation of $\mathbf{a}$ along the curve L.

This is the external definition of parallel translation by means of projections. A somewhat different definition may be given when, instead of projecting a vector from one tangent plane P_i onto another one P_{i+1}, we revolve the first plane P_i around the line of intersection of P_i and P_{i+1} and transfer in this way a vector given in P_i into the plane P_{i+1}. This operation being applied, we have parallel translation by the development of tangent planes.

Both definitions are equivalent for regular curves on regular surfaces, but, in general, they are not. Borisov has proved the following simple theorem which seems to be the more interesting because of the simplicity of its result. The lengths of vectors $\mathbf{a}_n$ converge to a certain limit, provided we use the projection, if and only if the curve has the following property. The sum of the squares of the angles between the normals to the surface at the successive points $X_1, \ldots, X_n$ tends to zero when the points are distributed more and more densely on the curve. If the curve has such a property the parallel translation along it is unique for any vector, i.e. not only the lengths of the vectors $\mathbf{a}_n$, but the vectors themselves with their directions have a definite limit. Under the same condition for the curve the parallel translation by developing of tangent planes is unique also, and gives the same result. But the converse statement is not true, which shows, in particular, that the two definitions of parallel translations are not equivalent.

In the following we understand by a parallel translation the operation defined by means of projections.

7.2. Borisov proves, further, that the uniform convergence of the parallel translation on a compact set of rectifiable curves (on a given surface) is equivalent to the condition that $\theta^2/\rho \to 0$ uniformly as $\rho \to 0$,

ρ being the distance between any two points and θ the angle between the normals at them.

Borisov studies the surfaces subject to this condition. First of all he proves that the parallel translation has an intrinsic meaning. Its intrinsic definition may be given for Borisov's surfaces as follows. A vector **a** is translated along a geodesic, i.e. the shortest line, if the angle between the vector and the line remains unchanged. The translation along a curve is defined by means of parallel translation along inscribed geodesic broken lines with the natural passage to the limit. Borisov proves that this intrinsic parallel translation is defined for any rectifiable curve and is equivalent to the external parallel translation as it has been defined above.

It is necessary to note that this is not so very simple, for we must have, at first, an intrinsic co-ordinate-free, purely metric definition of a vector in the surface and of the angle between the vector and the shortest line. These definitions are based upon a concept of angle which is somewhat more general than that used in the theory of M.B.C. Two curves are said to have the same direction at a point O if the angle between them is equal to zero. The concept of a direction being thus introduced, we have immediately the concept of a vector.

We shall not mention Borisov's other results with the exception of the Gauss–Bonnet theorem. Borisov proves that for any domain G with rectifiable boundary L homeomorphic to a circle and having spherical image within a hemisphere the Gauss–Bonnet formula holds. The rotation of a vector under parallel translation along L is equal to the area of the spherical image of G defined as a certain curvilinear integral along the spherical image of L. This result seems to me the more interesting in that Borisov's surfaces are not, generally speaking, manifolds of bounded curvature, so that their curvature is not a totally additive set-function.