SOME ASPECTS OF THE THEORY OF ALMOST PERIODIC FUNCTIONS

Børge Jessen

1. In this address, I shall endeavour to give an account of some of the work that has been done in Copenhagen, in recent years, on almost periodic functions.

Let me first recall the main points of the theory of almost periodic functions of a real variable as developed by Bohr [1]) in 1924—25. We are concerned with complex functions $F(t) = U(t) + iV(t)$ of an unrestricted real variable t. In order to be almost periodic, the function must be continuous, and for every $\varepsilon > 0$ it must have translation numbers τ, i.e., numbers for which $| F(t + \tau) - F(t) | \leqq \varepsilon$ for all t, and not too few — more precisely, there must be a length $l(\varepsilon)$ such that every interval of this length on the real axis contains one of these numbers τ.

Every almost periodic function is bounded. The sum or product of two almost periodic functions and the limit of a uniformly convergent sequence of almost periodic functions are again almost periodic. Every almost periodic function possesses a mean value, $\lim\limits_{T\to\infty} \dfrac{1}{T}\displaystyle\int_0^T F(t)dt$. For a given almost periodic function, the mean value $\lim\limits_{T\to\infty} \dfrac{1}{T}\displaystyle\int_0^T F(t)e^{-i\lambda t}dt$, where λ is real, differs from 0 for at most an enumerable set of values of λ. Let these values be denoted by λ_n and put $\lim\limits_{T\to\infty} \dfrac{1}{T}\displaystyle\int_0^T F(t)e^{-i\lambda_n t}dt = a_n$. The trigonometric series

$$F(t) \sim \sum a_n e^{i\lambda_n t}$$

is called the Fourier series of $F(t)$ and the numbers λ_n the Fourier exponents of $F(t)$. For a continuous periodic function with period 2π, these exponents λ_n are integers and the series is the usual Fourier series. The main result of the theory is the approximation theorem according to which the class of almost periodic functions is identical with the class of those functions which can be approximated uniformly by trigonometric polynomials

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[1]) Acta Math. 45 (1924), 29—127, 46 (1925), 101—214; *Coll. Math. Works* II, C3, C7.

$$P(t) = c_1 e^{i\lambda_1 t} + \ldots + c_N e^{i\lambda_N t}$$

with arbitrary real exponents λ_n and arbitrary complex coefficients c_n. In other words, a function $F(t)$ is almost periodic if and only if to each ε there corresponds a trigonometric polynomial $P(t)$ such that $| F(t) - P(t) | \leqq \varepsilon$ for all t. As shown by Bochner [2]), such approximating trigonometric polynomials can be obtained from the Fourier series by a suitable summation method.

The crucial point of Bohr's definition of almost periodicity is the existence of the length $l(\varepsilon)$. We express the existence of this length by saying that, for each ε, the translation numbers form a relatively dense set on the real axis. It was therefore quite surprising when it turned out, as a consequence of a very beautiful direct proof of the approximation theorem due to Bogolyubov [3]), that the relative density condition can be replaced by a weaker condition. It suffices, indeed, to assume that for each ε there is a sequence of translation numbers τ_n such that $| \tau_n - \tau_m | \geqq \alpha > 0$ for all n and m, and $\tau_n = O(n)$. On closer examination, it turned out that actually the difference is not at all deep, since, as shown by Følner [4]), it can be proved easily that the existence of translation numbers satisfying Bogolyubov's condition implies the existence of a relatively dense set of translation numbers. The last paper of Bohr on almost periodic functions [5]) deals with the problem of whether or not Bogolyubov's condition can be replaced by a still weaker one. He proved that this is not the case — more precisely, that if, instead of $\tau_n = O(n)$, we assume $\tau_n = O(\psi(n))$, where $\psi(n)$ is any function which goes to infinity more rapidly than n, then we obtain a class of functions which is properly larger than the class of almost periodic functions.

Let us define the uniform norm of a function $F(t)$ by

$$|| F ||_U = \sup_{-\infty < t < \infty} | F(t) |.$$

It is finite for the bounded functions. By taking $|| F - G ||_U$ as distance, we obtain a metric space. The content of the approximation theorem is then that the class of almost periodic functions is the closure, with respect to this metric, of the class of trigonometric polynomials; in symbols,

$$\{\text{a.p.}\} = \text{Cl}_U \{\text{trig. pol.}\}.$$

2. Shortly after the appearance of the theory of almost periodic functions, a number of generalizations were introduced and discussed. We shall not go

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[2]) Math. Ann. 96 (1927), 119—147.

[3]) Ann. Chaire Phys. Math. Kiev 4 (1939), 195—205.

[4]) Mat. Tidsskr. B 1944, 24—27.

[5]) J. Anal. Math. 1 (1951), 11—27; *Coll. Math. Works* III, C 54.

into the definitions of these generalizations here [6]) but merely indicate how the various classes of generalized almost periodic functions are characterized through approximation by trigonometric polynomials. Instead of continuous functions, let us consider functions which are measurable in the Lebesgue sense. The first generalization was made by Stepanov. By the Stepanov norm of order $p \geqq 1$ of a measurable function $F(t)$, we mean

$$\| F \|_{S^p} = [\sup_{-\infty<t<\infty} \int_t^{t+1} |F(u)|^p du]^{1/p}.$$

The class of Stepanov almost periodic functions of order p is then the closure with respect to the distance $\| F - G \|_{S^p}$, of the class of trigonometric polynomials; in symbols,

$$\{S^p\text{a.p.}\} = \mathrm{Cl}_{S^p}\,\{\text{trig. pol.}\}.$$

This generalization was also made by Wiener. Another generalization, which leads to a much wider class of functions, was given by Besicovitch who considered the norm

$$\| F \|_{B^p} = [\limsup_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |F(t)|^p dt]^{1/p}.$$

The class of Besicovitch almost periodic functions of order p is then the closure, with respect to the distance $\| F - G \|_{B^p}$, of the class of trigonometric polynomials; in symbols

$$\{B^p \text{ a.p.}\} = \mathrm{Cl}_{B^p}\,\{\text{trig. pol.}\}.$$

There is also an intermediate generalization introduced by Weyl based on a norm $\| F \|_{W^p}$. In all these cases, we have Fourier series $F(t) \backsim \sum a_n e^{i\Lambda_n t}$ just as in the original case.

A number of important questions concerning these generalizations were settled by Ursell [7]). Among other things, he proved that, whereas the Stepanov and Besicovitch classes are complete metric spaces, the Weyl classes are not. A very thorough examination of the relations between the different classes was made by Bohr and Følner [8]) in 1944 and continued by Følner [9]). It would lead us too far afield to go into the very detailed description they obtained.

In the case $p = 2$, we have the Parseval equality

[6]) See, for example, the comprehensive treatment by Besicovitch and Bohr, Acta Math. 57 (1931), 203—292; Bohr, *Coll. Math. Works* II, C 27; or Besicovitch, *Almost periodic functions*, Cambridge, 1932.

[7]) Proc. London Math. Soc. (2) 32 (1931), 402—440.

[8]) Acta Math. 76 (1944), 31—155; Bohr, *Coll. Math. Works* III, C 47.

[9]) Thesis, Copenhagen, 1944.

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T |F(t)|^2 dt = \Sigma\, |a_n|^2$$

in all three cases. Besicovitch's generalization is interesting because, for this generalization, the analogue of the Riesz-Fischer theorem is also valid, that is to say, any trigonometric series $\Sigma\, a_n e^{i\lambda_n t}$ with $\Sigma\, |a_n|^2$ finite is the Fourier series of a B^2 almost periodic function. This is not the case for Stepanov or Weyl functions.

One can, however, look at the question of generalization of the theorems on Fourier series of periodic functions in another way, and I would like to comment on the generalization of the Riesz-Fischer theorem by way of example. In many respects, the Stepanov generalization is the most satisfactory generalization, and one would therefore like to have analogues of the Riesz-Fischer theorem for Stepanov functions. Now, the Lebesgue class L^2 of periodic functions with period 2π can also be characterized as the class of those S^2 almost periodic functions for which the Fourier exponents λ_n are integers. Thus the Riesz-Fischer theorem holds for these functions. One can now ask: For which sequences of exponents λ_n does the Riesz-Fischer theorem hold for S^2 almost periodic functions having these exponents. This question has a very simple answer. As pointed out by Tornehave [10]), it follows from an old result of Wiener and a method of proof due to Stepanov that a sequence of exponents has the property in question if and only if there exists a constant k such that no interval of length 1 contains more than k exponents. Presumably, further interesting results could be obtained along this line.

3. The generalization of the theory of almost periodic functions which is of greatest interest for the original theory is, of course, von Neumann's generalization of the theory to functions on groups [11]). It would fall entirely outside the scope of this lecture to go into this theory which recently has received a beautiful exposition in Maak's book [12]). I would merely like to mention that the theory of almost periodic functions on groups may sometimes be put to use in the ordinary theory in cases where one would expect a direct analytical approach to be simpler. In this connection, we may mention a very simple treatment of Besicovitch almost periodic functions recently given by Doss [13]), and also the theory of linear functionals on spaces of generalized almost periodic functions. This

[10]) Math. Scand. 2 (1954), 237—242. The result is a corollary of Wiener's result and of its counterpart proved by Tornehave in this paper.

[11]) Trans. Amer. Math. Soc. 36 (1934), 445—492.

[12]) *Fastperiodische Funktionen*, Berlin/Göttingen/Heidelberg, 1950.

[13]) Bull. Sci. Math. (2) 77 (1953), 186—194.

latter theory was first considered by Doss [14]), and, recently, Følner [15]) has obtained further results generalizing the classical result of F. Riesz on the L^p spaces.

4. The remainder of this lecture will be devoted to the subject of mean motions of almost periodic functions and related topics. For a moment, let us forget about almost periodic functions and go back to a classical investigation by Lagrange [16]) from 1782 on the perturbations of the large planets. To indicate the elliptic orbit of a planet, we use certain parameters called the elements of the orbit. Owing to the influence of the other planets, the elliptic orbit of a planet changes slowly in the course of time. These parameters are therefore functions of the time t. Among these parameters is the longitude of the perihelion, i.e., a certain angular variable indicating the direction to the perihelion.

As shown by Lagrange, this angle is, in the first approximation, determined as the argument of a certain trigonometric polynomial

$$F(t) = a_1 e^{i\lambda_1 t} + \ldots + a_N e^{i\lambda_N t}$$

with complex coefficients a_n and real exponents λ_n. Thus this polynomial is a sum of vectors each having a constant length and turning with a constant angular velocity. The number of terms N equals the number of planets. Thus the study of the variation of the longitude of the perihelion leads to a study of the variation of the argument $\arg F(t)$ of a trigonometric polynomial.

These polynomials were calculated by Lagrange for the different planets, and it turned out that, in most cases, the polynomial contains a preponderant term, i.e., a term whose absolute value exceeds the sum of the absolute values of the remaining terms. Suppose, for example, that the first term is preponderant, that is, $|a_1| > |a_2| + \ldots + |a_N|$. Then $F(t)$ does not assume arbitrarily small values, and it is easy to see that the argument of $F(t)$ will differ by less than $\frac{1}{2}\pi$ from the argument of the first term. Consequently we have, for a continuous branch of the argument, the formula

$$\arg F(t) = \lambda_1 t + O(1).$$

Thus the argument is, in this case, the sum of a secular term $\lambda_1 t$ and a bounded remainder.

5. Lagrange formulated the problem of investigating the variation of the argument in the case when there is no preponderant term. In this case, it may happen that $F(t)$ comes arbitrarily near to 0 or even assumes the value 0. The

[14]) Amer. J. Math. 72 (1950), 81—92.

[15]) Dan. Mat. Fys. Medd. 29, no. 1 (1954), 1—27.

[16]) Nouv. Mém. Acad. Berlin 1781—82, Oeuvres 5, 123—344.

line joining 0 and $F(t)$ will however vary continuously, even if $F(t)$ becomes 0, provided, that at 0, we replace the line by the tangent (which exists since the curve is analytic). It will however change its positive direction when passing through a zero of odd order. In order to be able to speak of a continuous branch of the argument, we must therefore consider the argument mod π and not as usual mod 2π. In the case of a planet, this means that instead of the perihelion itself we must consider the line of apsides.

6. After attempts by various astronomers, the first non-trivial case, $N = 3$, was completely treated by Bohl [17]) in 1909. He showed that, in this case, we always have

$$\arg F(t) = ct + o(t);$$

but the constant c is generally not. as before, one of the exponents, and the remainder is generally not bounded. This result is equivalent to the existence of the limit

$$c = \lim_{T \to \infty} \frac{\arg F(T) - \arg F(0)}{T}.$$

The constant c is called the *mean motion* of $F(t)$.

Bohl's proof depends on diophantine approximations. It is in this connection that the important notion of equidistribution mod 1 occurs for the first time. Bohl's investigation has since been extended to very general cases, notably by Weyl [18]) by means of his general theorem on equidistribution mod 1.

7. After the creation of the theory of almost periodic functions, it was natural to extend Lagrange's problem to this wider class of functions. It was Wintner who first called attention to the connection between almost periodic functions and astronomical problems. It was conjectured by Wintner and proved by Bohr [19]) in 1930 that if $F(t) \backsim \sum a_n e^{i\lambda_n t}$ is an almost periodic function which does not come arbitrarily near to 0, i.e., $|F(t)| \geqq k > 0$, then

$$\arg F(t) = ct + O(1),$$

even if there is no preponderant term in the Fourier series. The mean motion c need not be one of the Fourier exponents.

8. Before continuing the discussion of Lagrange's problem, I would like to digress slightly by mentioning certain results which are connected with Bohr's theorem. If the almost periodic function $F(t)$, or, as we may also say, the almost

[17]) J. Reine Angew. Math. 135 (1909), 189—283.

[18]) Enseignem. Math. 16 (1914), 455—467, Math. Ann. 77 (1916), 313—352, Amer. J. Math. 60 (1938), 889—896, 61 (1939), 143—148.

periodic movement $F(t)$ in the plane, does not come near to two points a and b, I could prove [20]) that the mean motions c_a and c_b of $F(t) - a$ and $F(t) - b$ have a rational ratio. This result is contained in a general theorem of Fenchel and myself [21]) to the effect that an almost periodic movement $F(t)$ in a closed domain D bounded by a finite number of circles is homotopic in D to a periodic movement $G(t)$. This means that there exists a family of almost periodic movements $H_\theta(t)$ in D depending uniformly continuously on the parameter θ for $0 \leqq \theta \leqq 1$ and such that $H_0(t) = F(t)$ and $H_1(t) = G(t)$. This follows from a similar result on almost periodic movements on closed surfaces with negative Euler characteristic. A movement $f(t)$ on such a surface is called almost periodic if, for each $\varepsilon > 0$, it has a relatively dense set of translation numbers τ, i.e., numbers for which dist $[f(t+\tau), f(t)] \leqq \varepsilon$ for all t. The theorem is to the effect that such a movement is homotopic, in the sense described before, to a periodic movement on the surface. The proof depends on the simple fact that a commutative subgroup of the fundamental group of the surface is cyclic. The theorem is not true for movements on a torus.

These investigations have been continued, in various ways, by Tornehave. Recently Tornehave [22]) has discussed almost periodic movements in arbitrary metric spaces. He proves, among other things, that the theorem just mentioned also fails for movements on a sphere, a case which Fenchel and I had not decided.

9. After this digression, let us return to the discussion of Lagrange's problem. If the almost periodic function $F(t)$ comes arbitrarily near to 0, the variation of its argument may be very complicated, and if the function has zeros, it may even be impossible to choose the argument as a continuous function of t. In this case, the study of the variation of the argument seems to be of interest only when the function $F(t)$ is obtained by considering an *analytic* almost periodic function on a vertical line. Let me briefly recall the main points of the theory of analytic almost periodic functions as developed by Bohr [23]) in 1926.

We are concerned, in this theory, with functions $f(s) = f(\sigma + it)$ of a complex variable $s = \sigma + it$ in a vertical strip $\alpha < \sigma < \beta$. In order to be almost periodic, the function must be regular, and, for every $\varepsilon > 0$ and every reduced strip $\alpha_1 < \sigma < \beta_1$, it must have a relatively dense set of translation

[19]) Dan. Mat. Fys. Medd. 10, no. 10 (1930), 5—11, Comment. Math. Helv. 4 (1932), 51—64; *Coll. Math. Works* II, C 24, C 29.

[20]) Math. Ann. 111 (1935), 355—363.

[21]) Dan. Mat. Fys. Medd. 13, no. 6 (1935), 1—28.

[22]) Dan. Mat. Fys. Medd. 28, no. 13 (1954), 1—42.

[23]) Acta Math. 47 (1926), 237—281; *Coll. Math. Works* II, C 12.

numbers τ, i.e., real numbers for which $| f(s+i\tau) - f(s) | \leqq \varepsilon$ for all s in the reduced strip. On every vertical line of the strip, i.e., for every fixed σ, the function is an almost periodic function of t. The Fourier series of these functions are obtainable from a certain exponential series $f(s) \sim \sum a_n e^{\lambda_n s}$ with complex coefficients a_n and real exponents λ_n by replacing s by $\sigma + it$. This series is called the Dirichlet series of $f(s)$ and the numbers λ_n the Dirichlet exponents of $f(s)$. The main result is the approximation theorem according to which a function is almost periodic in a strip if and only if it can be approximated uniformly in every reduced strip by exponential polynomials $p(s) = c_1 e^{\lambda_1 s} + \ldots + c_N e^{\lambda_N s}$ with complex coefficients c_n and real exponents λ_n, i.e., if to each ε and each reduced strip, there corresponds a polynomial $p(s)$ such that $| f(s) - p(s) | \leqq \varepsilon$ for all s in the reduced strip.

10. It is evident that for an analytic function the variation of the argument on vertical lines must be closely related to the distribution of the zeros of the function in vertical strips. This establishes a connection between Lagrange's problem and problems concerning the distribution of the values of analytic almost periodic functions. Such investigations were carried out by Bohr, partly in collaboration with other authors, especially in the case of the Riemann zeta function $\zeta(s) = \sum n^{-s}$. These investigations, which began about 1910, are closely related to the method of Bohl and Weyl. Historically they are at the origin of the theory of almost periodic functions [24].

It was natural to try to generalize these investigations to arbitrary analytic almost periodic functions $f(s)$. This problem was studied by myself [25] in 1933 and by Hartman [26]. The main results are as follows. For every σ between α and β, the mean value

$$\varphi(\sigma) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \log | f(\sigma + it) | \, dt$$

exists, and $\varphi(\sigma)$ is a continuous, convex function in the interval $\alpha < \sigma < \beta$. It has therefore a derivative $\varphi'(\sigma)$ at all points of the interval with the exception of at most an enumerable set. If $\varphi(\sigma)$ is differentiable at the point σ and if $\arg f(\sigma + it)$ denotes a continuous branch of the argument on the corresponding vertical line, then the mean motion $\lim\limits_{T\to\infty} \dfrac{\arg f(\sigma + iT) - \arg f(\sigma)}{T}$

[24]) See, for example, the comprehensive exposition by Bohr and the author, Acta Math. 50 (1930), 1—35, 54 (1932), 1—55; Bohr, *Coll. Math. Works* I, B 23, B 24.

[25]) Math. Ann. 108 (1933), 485—516. Certain results had been obtained previously by Favard, *Leçons sur les fonctions presque périodiques*, Paris, 1933, 129—140.

[26]) Trans. Amer. Math. Soc. 46 (1939), 64—81.

exists and is determined by the formula

$$\lim_{T\to\infty} \frac{\arg f(\sigma+iT) - \arg f(\sigma)}{T} = \varphi'(\sigma).$$

Moreover, if $\varphi(\sigma)$ is differentiable at the points σ_1 and σ_2, then the zeros of $f(s)$ in the strip $\sigma_1 < \sigma < \sigma_2$ have a relative frequency, i.e., if $N(T)$ denotes the number of zeros in the strip $\sigma_1 < \sigma < \sigma_2$ lying between the lines $t = 0$ and $t = T$, then the limit $\lim\limits_{T\to\infty} \frac{N(T)}{T}$ exists and its value is determined by the formula

$$\lim_{T\to\infty} \frac{N(T)}{T} = \frac{\varphi'(\sigma_2) - \varphi'(\sigma_1)}{2\pi}.$$

For a periodic function, this is merely another form of the classical formula of Jensen. The function $\varphi(\sigma)$ is called the Jensen function of $f(s)$.

Regarding the proof of these results, I shall restrict myself to the following remarks. If we differentiate the expression for $\varphi(\sigma)$ formally and interchange the differentiation and the formation of the mean value, we obtain

$$\varphi'(\sigma) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{d}{d\sigma} \log | f(\sigma + it) | \, dt.$$

Hence by the Cauchy-Riemann equations

$$\varphi'(\sigma) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{d}{dt} \arg f(\sigma + it) dt,$$

which is the formula for the mean motion. The formula for the relative frequency of zeros is obtained by expressing the number of zeros in the rectangle $\sigma_1 < \sigma < \sigma_2$, $0 < t < T$ by means of the variation of the argument along the contour and passing to the limit. The non-negativity of the frequency accounts for the convexity of $\varphi(\sigma)$. — The actual proof follows these lines but is complicated by the fact that $\varphi(\sigma)$ need not be differentiable.

11. By a combination of the preceding method and the method of Bohl and Weyl I obtained [27]) in 1938 a complete solution of Lagrange's problem. It turned out that every trigonometric polynomial possesses a mean motion. Subsequently, the Jensen function and its connection with mean motions and distribution of zeros has been the object of a detailed and systematic study by Tornehave and myself [28]) from 1945.

The principal result, so far as Lagrange's problem is concerned, is that, for

[27]) C. R. Acad. Sci. Paris 207 (1938), 1081—1084.

[28]) Acta Math. 77 (1945), 137—279.

an exponential polynomial $f(s) = a_1 e^{\lambda_1 s} + \ldots + a_N e^{\lambda_N s}$, the mean motion exists on every vertical line and is determined by

$$\lim_{T\to\infty} \frac{\arg f(\sigma - iT) - \arg f(\sigma)}{T} = \frac{\varphi'(\sigma - 0) + \varphi'(\sigma + 0)}{2}.$$

On the imaginary axis, we obtain the trigonometric polynomial

$$f(it) = F(t) = a_1 e^{i\lambda_1 t} + \ldots + a_N e^{i\lambda_N t},$$

which therefore possesses a mean motion.

Another important class of analytic almost periodic functions for which the last result is valid is the class of functions represented by a Dirichlet series of ordinary type $f(s) = \Sigma a_n n^{-s}$ in its half plane of uniform convergence.

In addition to these results dealing with special classes of almost periodic functions, the investigation of Tornehave and myself contains a detailed discussion of the variation of the argument on vertical lines in cases where the mean motion does not exist and a determination of all convex functions that can occur as the Jensen function of an analytic almost periodic function.

Further results have been obtained by Tornehave [29]) who, among other things, has studied the Jensen function for analytic functions of several variables.

12. As mentioned earlier, these investigations have their origin in the investigations of Bohr on the distribution of the values of the Riemann zeta function. It was natural to reconsider this question using the Jensen function. This was done by Miss Borchsenius and myself [30]) in 1948. The study of the distribution of the a-points of $\zeta(s)$, i.e., the zeros of $\zeta(s) - a$, by this method, leads to refinements of the results of Bohr. The results deal not only with the half plane $\sigma > 1$ where the function is almost periodic, but, as did the old results, with the part of the critical strip lying to the right of the critical line, i.e., the strip $\frac{1}{2} < \sigma \leqq 1$ where the function is almost periodic only in the Besicovitch sense. The treatment depends on the standard statistical method of characteristic functions, which previously had been applied by Wintner and myself [31]) to the study of the distribution of the values of the zeta function on vertical lines.

[29]) Thesis, Copenhagen, 1944.

[30]) Acta Math. 80 (1948), 97—166.

[31]) Trans. Amer. Math. Soc. 38 (1935), 48—88.