CONVERGENCE AND SUMMABILITY OF FOURIER SERIES

LENNART CARLESON

Let me first state quite explicitly that I do not intend to give in this lecture any survey of the very large field covered by the title. There is also no need for this since the Congress was presented such a survey quite recently. I rather want to present my personal interests which are concentrated on the almost everywhere behaviour of the partial sums. Also the subject of summability will only be touched upon.

1. Background

For a very long time, the outstanding result in the area of almost everywhere convergence has been the following result of Kolmogorov-Seliverstov-Plessner: *if for* $\lambda_n = \log n$

$$\sum_1^\infty (a_n^2 + b_n^2)\lambda_n < \infty, \tag{1.1}$$

then

$$s_n(x) = \frac{a_0}{2} + \sum_1^n (a_\nu \cos \nu x + b_\nu \sin \nu x) \tag{1.2}$$

converges a.e. The outstanding question was whether $\log n$ is a relevant sequence or not.

It has been known that conditions of the type (1.1) are related to capacities with respect to a kernel

$$K(x) \sim \sum \frac{\cos nx}{\lambda_n} \tag{1.3}$$

(Beurling [1]: $\lambda_n = n$, $K(x) \sim \log \frac{1}{|x|}$; Salem-Zygmund [4]: $\lambda_n = n^\alpha$, $K(x) \sim |x|^{\alpha-1}$). However, what they really prove is that the capacity of the divergence set vanishes for

$$K^*(x) = \frac{1}{|x|}\int_0^{|x|} K(t)\,dt$$

(see Temko [5]). When $\lambda_n = (\log n)^\beta$,

$$K(x) \sim |x|^{-1}\left(\log \frac{1}{|x|}\right)^{-1-\beta}$$

while

$$K^*(x) \sim |x|^{-1}\left(\log\frac{1}{|x|}\right)^{-\beta}.$$

Since $K^*(x) \in L^1$ only if $\beta > 1$, the result is meaningless unless $\beta > 1$ which could be considered as an indication of the relevance of the Kolmogorov factor $\log n$.

There is, however, a strong objection to this way of arguing: nothing better was known for summability (Abel or (C, 1), for example) either. But already the Fatou theorem here shows that we have a.e. summability for $f \in L^2$, $\lambda_n = 1$. We would then be faced with a possible interval $0 \leqslant \beta < 1$ with no distinction between the sizes of the exceptional sets for summability. This was of course, most unlikely and I recently proved [2] that the set where the Hardy-Littlewood maximal function

$$f^*(x) = \sup_t \int_x^{x+t} f(u)\,du \tag{1.4}$$

is infinite has K- (not only K^*-) capacity zero. But then, why could not the same be true for convergence?

The other aspect on the background is also quite subjective but it seems to me quite possibly to be of central importance. It depends on the following trivial observation.

If

$$\bar{\varphi}_j(x) = e^{2\pi i 2^j x}, \quad j = 0,\ 1,\ 2,\ \ldots,$$

and

$$n = \sum \varepsilon_j 2^j, \quad \varepsilon_j = 0,\ 1,$$

then

$$e^{2\pi i n x} = \bar{\psi}_n(x) = \prod \bar{\varphi}_j(x)^{\varepsilon_j}.$$

This means that the exponentials e^{inx}, $0 < n < 2^N$, are completely known by the knowledge of N functions.

To emphasize the point of view, let us replace $\bar{\varphi}_j(x)$ by the Rademacher functions $\varphi_j(x)$:

$$\varphi_j(x) = \operatorname{sign}(\operatorname{Im}(\bar{\varphi}_j(x)))$$

and $\psi_n(x)$ by the corresponding product—the Walsh functions.

φ_j and ψ_n can for j, $n \leqslant 2^N$ be represented by two matrices M_R and M_W respectively of —1's and +1's where each column corresponds to a function and each row to a certain set. The number of columns are 2^N while the number of rows are 2^{2^N} and 2^N respectively. Let $M = (\varepsilon_{ij})$ be any such matrix. The divergence problem concerns the existence of a_j, $\sum_{j=1}^{2^N} a_j^2 = 1$, such that

$$S_i^* = \max_{k<2^N} \sum_{j=1}^{k} a_j \varepsilon_{ij}$$

is large for a large proportion of the possible i's. This is, of course, the more difficult to arrange the larger the number of possible i's are. This is in good correspondence with the fact that $\Sigma\, a_j\, \varphi_j\, (x)$ converges a.e. It should also be observed that if M is to correspond to an orthogonal system then the number of rows must be $\geqslant 2^N$. The Walsh system (and by analogy, the trigonometrical system) is therefore particularly advantageous to obtain divergence on sets of positive measure.

The following result is now quite surprising:

Given $\delta > 0$ there is a constant C so that a random square matrix $M = (\varepsilon_{ij})$, $1 \leqslant i,j \leqslant 2^N$, where $\varepsilon_{ij} = \pm 1$, with probability $>1 - \delta$, has the property that for any $\{a_j\}$ and any $\lambda > 0$

$S_i^* > \lambda$ only for at most $C \cdot \lambda^{-2} \cdot 2^N$ indices i.

This means that—from this point of view—the existence of a divergent L^2-Walsh-series has probability zero.

If we introduce the possibility of changing the orders of the terms, i.e. permuting the columns of the matrix, the problem changes and the following may be true:

Given any square ± 1-matrix M there is a permutation of the columns so that for the new matrix M^* there exists a_j with S_i^* large for all i.

That purely combinatorial result would give the construction of a rearranged divergent L^2-Walsh-series and could most likely be used for a corresponding construction for the Fourier system. This line of work seems to me most promising and possibly the Kolmogorov $\log n$-factor could find its proper place here.

2. Two recent results

In a recent paper [3] I have proved the following result :

If $f \in L^2$ then $s_n\,(x)$ converges a.e.

If $\int |f|\, (\log |f|)^{1+\delta}\, dx < \infty$, then $s_n\,(x) = o\,(\log \log n)$ a.e.

I should like to try to give an idea of the method used to obtain these results.

We consider $f \in L^2\,(-\pi, \pi)$ and the dyadic intervals ω obtained by successive bisections of $(-2\pi, 2\pi)$. $\omega_0^* = (-2\pi, 2\pi)$ and generally ω^* is two neighbouring ω's of equal lengths. The first observation is that $s_{n_0}(x)$, $n_0 < 2^N$, behaves as

$$(2.1) \qquad \int_{\omega_0^*} \frac{e^{-in_0t} f(t)}{x-t}\, dt = I(p_0^*;\, x), \quad p_0^* = (n_0,\, \omega_0^*).$$

This is essentially a conjugate function which makes it possible to use a well-known theory, in particular that of maximal Hilbert transforms.

We now consider a suitable decomposition $\Omega(p_0^*)$ of ω_0^* into subintervals ω and find for a certain $\omega_1^* \subset \omega_0^*$,

$$I(p_0^*;\, x) = I(p_1^*;\, x) + R(p_0^*;\, x).$$

$R(p_0^*;\, x)$ is a remainder term and can be estimated outside a certain exceptional set $E(p_0^*)$ by means of *weak* norms such as

$$\| f;\, p_0^* \|^2 = \sum_{\nu=-\infty}^{\infty} \frac{1}{1+\nu^2} \left| \int_{\omega_0^*} e^{-in_0t - i\frac{\nu}{3}t} f(t)\, dt \right|^2 .$$

The estimate is such that e.g. $|R(p^*;\, x)| <$ const $\| p^*;\, f \|^\eta \cdot \lambda$ outside a set $E(p^*)$ of measure $< e^{-\|p^*;\, f\|^{\eta-1}\lambda}$.

We next observe that $I(p_1^*;\, x)$ is of the type as $I(p_0^*;\, x)$ after a change of scale except that n_0 is then not an integer. However, the change by moving n_0 to the closest multiple of $|\omega_0^*| / |\omega_1^*|$ can also be estimated by $\| f;\, p_0^* \|$ and can be incorporated in R. We repeat the construction and find

$$(2.2) \qquad I(p_0^*;\, x) = I(p_k^*;\, x) + \sum_{0}^{k-1} R(p_j^*;\, x)$$

where we have stopped when $|\omega_k^*| \leqslant 2\pi \cdot 2^{-N}$ in which case $n_k = 0$. Therefore there is no mentioning of n_0 in the main term and this term is easily estimated.

Now the larger we allow $R(p^*;\, x)$ to be, the smaller we make the total exceptional set

$$(2.3) \qquad E = \bigcup_S E(p^*),$$

where the union comprises the set S of all p^* used in the different steps. On the other hand, if the R's are large, we get a bad estimate of $I(p_0^*;\, x)$ in the formula (2.2).

The log log n-result is obtained as follows. We let S be the set of *all* p^*'s whether used or not. For the estimation we use simply the Hausdorff-Young inequality corresponding to the integrability assumption, which gives for $\eta = \frac{\delta}{1+\delta}$,

$$\sum_S e^{-\|f;\, p^*\|^{-1+\eta} \cdot \log N} \cdot |\omega^*| < \varepsilon, \quad N \text{ large}.$$

The factor $\lambda = \log N$ ($n_0 \leqslant 2^N$) that is introduced here to give the small total measure ε enters as a factor of the R's and gives the estimate $\log N$, i.e. $\log \log n_0$.

To get the L^2-result the set S has to be controlled. This is quite involved and I here only want to say that the starting point for this is the modified partial sums

$$S_a(x) = \sum_{|c_\nu| \geqslant a} c_\nu e^{i\nu x}, \quad c_\nu = \frac{1}{2\pi} \int f e^{-i\nu x} \, dx, \tag{2.4}$$

which give the best L^2-approximations using the least number of terms. These sums have, to my knowledge, not been studied and ought to be important for many problems.

3. Some open problems

(1) There is a classic example by Kolmogorov of an L^1-function whose Fourier series diverges everywhere. If one studies this example, one can quite easily see that for any $\varepsilon(n) \to 0$, there exists $f \in L^1$ with

$$s_n(x) \neq O(\varepsilon(n) \log \log n) \text{ a. e.}$$

This means that for a certain integrability between L and $L(\log L)^{1+\delta}$, $\delta > 0$, $\log \log n$ is the relevant order. The methods used above do not work, because the Hausdorff-Young inequality fails.

(2) By the result for L^2, the possible divergence sets are completely described for all classes between L^2 and C. Kahane and Katznelson have namely recently proved that for any set E of measure zero, there is a continuous function whose Fourier series diverges on E.

For L^p, $1 < p < 2$, the problem remains open. I can prove that $s_n(x) = o(\log \log \log n)$ a.e. This result obviously very strongly suggests that we actually have convergence a.e.

The L^2-proof fails for L^p because I cannot handle $S_a(x)$ in (2.4). $S_a(x)$ is an example of a function φ_a operating on L^p as a Banach algebra,

$$\varphi_a(z) = \begin{cases} z, & |z| \geqslant a \\ 0, & |z| < a. \end{cases}$$

However, it can be proved that φ_a is *not* a uniformly bounded operator as $a \to 0$, so some smoothing must be done. Also a complete solution of this problem would not immediately solve the convergence problem, but I think that important work in the general area of operators and multipliers that depend on the function could be done.

(3) In connection with $S_a(x)$ it is natural to ask for pointwise convergence. There is rather strong evidence that this fails in L^2.

(4) The study of the maximal partial sum

$$S^*(x) = \sup_n |s_n(x)| \tag{3.1}$$

is of very great interest. By a well-known theorem by Calderon, we have a weak (L^2, L^2) result, but is it true that

$$\|S^*(x)\|_q \leqslant C_q \|f\|_q, \quad 2 \leqslant q < \infty \text{ or possibly } 1 < q < \infty$$

and that

$$\text{meas}\{S^* > \lambda\} \leqslant Ce^{-c\lambda}, \quad |f| \leqslant 1?$$

The convergence proof is, in principle, constructive but it is not clear that it can give so strong results.

(5) In connection with the problem discussed in the first part, one should now be able to prove that we have convergence (not only summability) outside a set of K-capacity zero.

(6) By standard methods the convergence result for L^2 can be used to prove the pointwise convergence of Fourier integrals of functions in L^2 and of expansions of regular eigen-functions. Nothing follows, however, for several variables or for other systems such as the closely related Walsh system. As a last indication of how subtle these questions are, let me mention that there is a function $f \in L^\infty$ so that

$$\sup_n \left| \int \frac{f(t)\,\psi_n(t)}{x-t}\,dt \right| = +\infty \text{ a.e.} \tag{3.2}$$

The difference between this expression and the Dirichlet formula, for which (3.2) does not hold, is only that sin nt has been replaced in (3.2) by the function $\psi_n(t) = \text{sign}(\sin nt)$.

University of Uppsala,
Uppsala, Sweden

REFERENCES

[1] Beurling A., Ensembles exceptionnels, *Acta Math.*, **72** (1939), 1-13.
[2] Carleson L., Maximal functions and capacities, *Ann. Inst. Fourier* **15** (1965), 59-64.
[3] Carleson L., On convergence and growth of partial sums of Fourier series, *Acta Math.*, **116** (1966), 135-157.
[4] Salem R., Zygmund A., Capacity of sets and Fourier series, *Trans Amer. Math. Soc.*, **59** (1946), 23-41.
[5] Темко К. В., Выпуклая емкость и ряды Фурье, *ДАН СССР*, **11(** (1956), 943-944.