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Critical Points of Smooth Functions*

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The *critical points* of a smooth function are the points where the differential vanishes. A critical point is *nondegenerate* if the second differential is a nondegenerate quadratic form. In some neighbourhood of a nondegenerate critical point the function can be represented in the Morse normal form

$$f = \pm x_1^2 \pm \cdots \pm x_n^2 + \text{const}$$

using suitable local coordinates.

Every degenerate critical point bifurcates into some nondegenerate points after an arbitrarily small deformation ("morsification"). So generically, functions have no degenerate critical points.

Degenerate critical points appear naturally when the function depends upon parameters. For example, the function $f(x) = x^3 - tx$ has a degenerate critical point for the value $t = 0$ of the parameter. Every family of functions close enough to this one-parameter family has a similar degenerate critical point for some small value of the parameter.

When the parameters are few, only the simplest degeneracies appear generically, and one can explicitly list them, giving normal forms for functions and families. When the number of parameters increases, more complicated degeneracies appear, and their classification seems hopeless. In recent years it has been found, however, that at least the initial part of the hierarchy of singularities is remarkably simple, as is described below.

Families of functions appear in all branches of analysis and mathematical physics. In this report three applications will be discussed: Lagrange singularities (or caustics), Legendre singularities (or wavefronts), and oscillating integrals (or stationary phase method).

*Delivered by E. Brieskorn.

Unexpectedly enough, the classification of the simplest singularities in all these problems turns out to be related to the Lie, Coxeter, and Weyl groups A_k, D_k, E_k, to Artin's and Brieskorn's braid groups, and to the classification of the platonics in Euclidean three space.

The occurrence of the diagrams A, D, E and of Coxeter groups in such different situations as the simple Lie algebra theory, the classification of simple categories of linear spaces (Gabriel, Gel'fand-Ponomarev, Roiter-Nasarova), the Kodaira classification of elliptic curves degenerations, the theory of platonics, and the theory of simple singularities gives an impression of a wonderful chain of coincidences of the results of independent classifications (certain relations between some of them being known, others suspected). As we will now see, the classification of more complex singularities provides new wonderful coincidences, where Lobatchevski triangles and automorphic functions take part.

1. Classification of critical points. Let f be a germ of a holomorphic function at a critical point O. The *multiplicity* (or the *Milnor number*) μ of the critical point is defined as the number of nondegenerate critical points to which O bifurcates after a morsification.

Two germs of functions are *equivalent* if one of them can be transformed into the other by a local diffeomorphism of the domain space. The jet (the Taylor polynomial) of a function at O is *sufficient* if it determines the germ up to equivalence.

Every germ of a smooth function at a critical point of finite multiplicity is equivalent to a germ of a polynomial (namely, of its Taylor polynomial), and its jet of sufficiently high order is sufficient (see Tougeron **[76]**, M. Artin **[14]**, Mather **[53]**, and also **[3]**, for four different proofs).

So, the classification problem for critical points with finite μ is reduced to a sequence of algebraic problems dealing with linear actions of Lie groups on finite dimensional spaces of jets. The first steps in solving these algebraic problems were taken by Thom **[70]**, Mather **[53]**, and Siersma **[66]**.

The classification of the first degeneracies is discrete, but the further types of critical points depend upon parameters (moduli). One finds that it is the classification of singularities with a small number of moduli that is nice while the classification of classes with small μ or small codimension is not.

Let us call *modality* (or number of moduli) of a point $x \in X$ under the action of a Lie group G on X the minimum number m such that some neighbourhood of x is covered by a finite number of m-parameter families of orbits of the group G. The point x is called *simple* if its modality is 0, that is, if some neighbourhood of x intersects only a finite number of orbits.

The *modality of the germ* of a function at a critical point is the modality of its sufficient jet in the space of jets of functions with critical point O and critical value 0.

Two germs of holomorphic functions with different numbers of arguments are called *stably equivalent* if they become equivalent after the direct addition of a nondegenerate quadratic form of the suitable number of variables.

THEOREM 1 (SEE [6]). *Up to stable equivalence, simple germs of holomorphic functions are exactly the following germs*:

$$A_k : f(x) = x^{k+1}, \qquad D_k : f(x, y) = x^2y + y^{k-1},$$
$$E_6 : f(x, y) = x^3 + y^4, \qquad E_7 : f(x, y) = x^3 + xy^3,$$
$$E_8 : f(x, y) = x^3 + y^5.$$

(See Figure 1.)

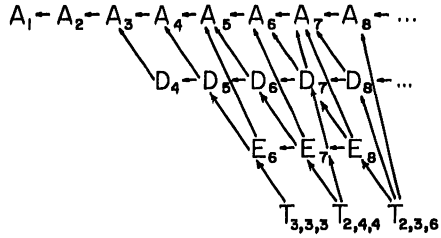


FIGURE 1. All the adherences of simple and parabolic singularities.

THEOREM 2 (SEE [7]). *Unimodular germs* (*that is, germs of modality* $m = 1$) *of holomorphic functions belong* (*up to stable equivalence*) *either to the following series of one-parameter families of functions*:

$$T_{p,q,r} : f(x, y, z) = axyz + x^p + y^q + z^r, \qquad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1, a \neq 0,$$

or to one of the following three families:

$$T_{3,3,3} : f(x, y, z) = x^3 + y^3 + z^3 + axyz, \qquad a^3 + 27 \neq 0,$$
$$T_{2,4,4} : f(x, y, z) = x^4 + y^4 + z^2 + ax^2y^2, \qquad a^2 \neq 4,$$
$$T_{2,3,6} : f(x, y, z) = x^3 + y^6 + z^2 + ax^2y^2, \qquad 4a^3 + 27 \neq 0,$$

or to one of the fourteen "exceptional" one-parameter families, given by the table below (*whose columns* 3—7 *will be explained later*).

There also exist tables of normal forms for all functions of two variables with nontrivial 3-jets [9] or nontrivial 4-jets, and tables of real normal forms.

THEOREM 3 (SEE [6], [7]). *The set of all nonsimple germs of functions of* $n \geqq 3$ *variables has codimension* 6, *and the set of germs with modality* $m > 1$ *has codimension* 10 *in the space of all germs of functions with critical value* 0.

Therefore every s-parameter family of functions can be made generic by a small variation, so that all germs of functions for all values of parameters will be stably equivalent to the germs of Theorem 1 (+ const) if $s < 6$, or to the germs of Theorems 1 and 2, if $s < 10$.

2. Factor singularities. The group $SU(2)$ acts linearly on C^2. Discrete subgroups of $SU(2)$ are known as binary groups of a polygon, a dihedron, a tetrahedron, a cube, or of an icosahedron (because they define the corresponding subgroups of the

TABLE

Notation	Normal form	Weights	Coxeter number	Dolgatchev numbers	Gabrielov numbers	Dual class
Q_{10}	$x^2z+y^3+z^4+ayz^3$	8 9 6	−24	2 3 9	3 3 4	K_{14}
Q_{11}	$x^2z+y^3+yz^3+az^5$	7 6 4	−18	2 4 7	3 3 5	Z_{13}
Q_{12}	$x^2z+y^3+z^5+ayz^4$	6 5 3	−15	3 3 6	3 3 6	Q_{12}
S_{11}	$x^2z+yz^2+y^4+ay^3z$	6 5 4	−16	2 5 6	3 4 4	W_{13}
S_{12}	$x^2z+yz^2+xy^3+ay^5$	5 4 3	−13	3 4 5	3 4 5	S_{12}
U_{12}	$x^3+y^3+z^4+axyz^2$	4 4 3	−12	4 4 4	4 4 4	U_{12}
Z_{11}	$x^3y+y^5+z^2+axy^4$	15 8 6	−30	2 3 8	2 4 5	K_{13}
Z_{12}	$x^3y+xy^4+z^2+ay^6$	11 6 4	−22	2 4 6	2 4 6	Z_{12}
Z_{13}	$x^3y+y^6+z^2+axy^5$	9 5 3	−18	3 3 5	2 4 7	Q_{11}
W_{12}	$x^4+y^5+z^2+ax^2y^3$	10 5 4	−20	2 5 5	2 5 5	W_{12}
W_{13}	$x^4+xy^4+z^2+ay^6$	8 4 3	−16	3 4 4	2 5 6	S_{11}
K_{12}	$x^3+y^7+z^2+axy^5$	21 14 6	−42	2 3 7	2 3 7	K_{12}
K_{13}	$x^3+xy^5+z^2+ay^8$	15 10 4	−30	2 4 5	2 3 8	Z_{11}
K_{14}	$x^3+y^8+z^2+axy^6$	12 8 3	−24	3 3 4	2 3 9	Q_{10}

rotation group of the sphere CP^1 after factoring $SU(2)$ by its center $\pm E$).

The quotient space C^2/Γ, for a binary group Γ, is an algebraic surface with one singular point.

The algebra of polynomials in two variables invariant under Γ possesses three generators. The relation (syzygy) between these generators is exactly the equation of the quotient variety C^2/Γ, embedded in C^3. The following theorem has been known since the time of H. A. Schwarz.

THEOREM 4 (SEE [43], [41], [56], [18]). *All the surfaces C^2/Γ for binary groups Γ have simple singularities of types A_k (for polygons), D_k (for dihedrons), E_6 (for the tetrahedron), E_7 (for the cube), or E_8 (for the icosahedron).*

Now let us consider the group $SU(1,1)$ of 2×2 matrices with determinant one preserving the quadratic form $|Z_1|^2 - |Z_2|^2$. This group acts on the set P of vectors in C^2 with positive value on this form. A discrete group of motions of the Lobatchevski plane with compact fundamental domain defines a "binary subgroup" $\Gamma \subset SU(1, 1)$ and an algebraic surface $M = (P/\Gamma) \cup O$ with singular point O. The coordinate ring of M is isomorphic to the ring of integer Γ_0-automorphic forms.

Let Δ be a Lobatchevski triangle with angles π/p, π/q, π/r. The reflections in its sides define a discrete group, and motions form a subgroup of index two in it. Thus for every such triangle Δ there is a binary group of the triangle in $SU(1,1)$.

The study of the 14 exceptional singularities led I. V. Dolgatchev to the following result.

THEOREM 5 (SEE DOLGATCHEV [27]). *There exist exactly 14 Lobatchevski triangles for which the surfaces $M = (P/\Gamma) \cup O$, Γ the binary group of the triangle, allow embeddings in C^3 (in other words, for exactly 14 triangles the algebra of integer automorphic forms allows three generators). These 14 quotient surfaces have at O exactly (the 14) exceptional quasi-homogeneous unimodular singularities (see Theorem 2 above). The values of p, q, r are given in the column under "Dolgatchev numbers" in the table.*

The binary group for E_8 is $\mathrm{PSL}(2, F_5)$; and for K_{12} it is $\mathrm{PSL}(2, F_7)$ (Klein [43]). This example was the starting point of Dolgatchev's work.

3. Quadratic forms of singularities. To each isolated critical point of a holomorphic function f in n variables one can associate a manifold V with boundary ∂V. V is the *local nonsingular level manifold* of real dimension $2n - 2$. Let us consider a small ball with its centre at the critical point. Then V is the part of a level set $f^{-1}(z)$ inside the ball (for a z sufficiently close to the critical value) (Figure 2).

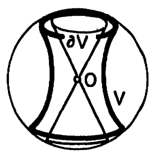


FIGURE 2. The local nonsingular level manifold.

[The boundary ∂V provides standard examples in differential topology, e.g., for the simple critical point E_8 in five variables, ∂V is one of the exotic 7-dimensional spheres of Milnor, which is homeomorphic but not diffemorphic to S^7. By attaching a disc to ∂V for E_8 in seven variables, one obtains a nonsmoothable 12-manifold (see Hirzebruch [42], Brieskorn [19], Milnor [56], Kuiper [44]).]

Milnor has proved that the local level manifold V^{2n-2} is homotopically equivalent to a bouquet of μ spheres S^{n-1}, so $H_{n-1}(V, Z) = Z^\mu$ (Milnor [56], Brieskorn [24]). The intersection index defines on H_{n-1} an integral bilinear form.

The *quadratic form of a singularity* is the self-intersection form on the homology of the nonsingular level manifold of a function in $n \equiv 3 \bmod 4$ variables, stably equivalent to the given function. [It is convenient to add squares to the function to obtain a symmetric intersection form. The effect of adding squares (or other direct summands) is described by the Sebastiani-Thom theorem [64]; see also [32].]

A singularity of a hypersurface is called *elliptic* (resp. *parabolic* or *hyperbolic*), if its quadratic form is negative definite (resp. nonpositive, or has 1 positive square).

Elliptic singularities have been classified by Tjurina [73].

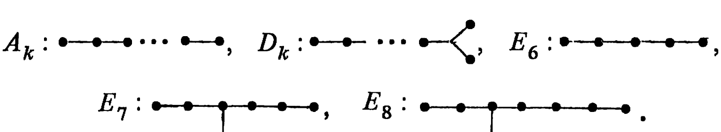
THEOREM 6 (SEE [73], [6], [71]). *Elliptic singularities of hypersurfaces are exactly the simple singularities A, D, E of Theorem* 1. *The parabolic singularities are exactly $T_{3,3,3}$, $T_{2,4,4}$ and $T_{2,3,6}$ of Theorem* 2. *The hyperbolic singularities are exactly $T_{p,q,r}$, with $1/p + 1/q + 1/r < 1$.*

The assertion on parabolic singularities has been formulated as a conjecture by Milnor, inspired by Wagreich's work [80].

It is convenient to describe quadratic forms of singularities using Dynkin (or Coxeter) diagrams. Such a diagram is a graph, whose points correspond to "vanishing cycles" (basis vectors with square -2 in H_{n-1}). Two points are connected by k lines if the scalar product of corresponding vectors is equal to k, e.g. ·—· is a diagram for $-2x_1^2 + 2x_1x_2 - 2x_2^2$.

Very effective methods for determining diagrams of singularities have been elaborated by A. M. Gabrielov [32], [33] and S. M. Guseĭn-Zade [37], [38]. The method of the latter gives the diagrams for all functions in two variables directly from the picture of level curves of a convenient real morsification. Recently A'Campo has independently rediscovered the Guseĭn-Zade method.

The quadratic forms of simple singularities A, D, E are given by standard diagrams (Hirzebruch [42]):



Gabrielov [32], [33] has found the quadratic forms for all unimodular singularities. Let τ_{p_1, p_2, p_3} denote the quadratic form, defined by a diagram having the shape of a letter T, with p_1, p_2, p_3, points on its three closed segments (e.g., $E_7 = \tau_{2,3,4}$).

THEOREM 7 (SEE [7], [32], [33]). *The quadratic form of every hyperbolic (parabolic) singularity $T_{p,q,r}$ is a direct sum $\tau_{p,q,r} \oplus 0$ (where 0 is a 0-form in one variable). The quadratic forms of the 14 exceptional singularities are of the form $\tau_{p,q,r} \oplus \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, where the 14 triples (p,q,r) are given by the column "Gabrielov numbers" of the table above.*

4. The strange duality. The comparison of Dolgatchev and Gabrielov numbers of the 14 exceptional singularities leads to the following.

THEOREM 8. *Gabrielov numbers of every exceptional singularity are Dolgatchev numbers of another one; the Gabrielov numbers of the latter are the Dolgatchev numbers of the former.*

So there is an involution which transposes the eight singularities $Q_{10} \leftrightarrow K_{14}$, $Q_{11} \leftrightarrow Z_{13}$, $Z_{11} \leftrightarrow K_{13}$, $S_{11} \leftrightarrow W_{13}$ and leaves invariant all the six other (having multiplicity $\mu = 12$) (Figure 3).

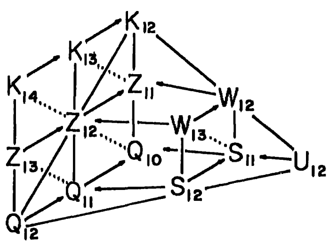


FIGURE 3. The pyramid of the 14 exceptional singularities.

There is no evident relation between singularities dual to each other (or between their Lobatchevski triangles, or quadratic forms), nor between Gabrielov and Dolgatchev numbers of the same singularity.

When Dolgatchev first reported his theorem, D. B. Fuks remarked that the sum of the multiplicity μ with the three Dolgatchev numbers is 24 for all the exceptional singularities but one (where Dolgatchev made some mistake).

This remark of Fuks joined to Theorem 8 implies that the sum of all the six Dolgatchev and Gabrielov numbers is 24 for any of the 14 exceptional singularities. One can also see that dual singularities are exactly the singularities with the same Coxeter number (defined below). There is no explanation for all these empirical facts. Singularity theory is, in its present state, an experimental science.

5. Versal deformation and the level bifurcation set. The *deformations* of a function f are the germs at O of smooth mappings from finite-dimensional "base spaces" to the function space which map O to f. A deformation is called *versal* if this mapping is transversal (in an understandable sense) to the orbit of f under the action of the pseudo-group of diffeomorphisms of the argument space. If the dimension of the base space has the minimal possible value (equal to the codimension of the orbit), the deformation is called *miniversal.*

One can define versal deformations for germs of functions. A germ of a function at a critical point of finite multiplicity μ allows a μ-parameter miniversal deformation; all other deformations are equivalent to deformations induced from this one by mappings of base spaces (for the proof see Tjurina **[74]**, Mather **[53]**, Latour **[46]**, Zakaljukin **[84]**).

The *local algebra* of the germ of a function f at a point O is the factor algebra of the algebra of (formal or convergent) series at O by the ideal generated by the partial derivatives of f. The dimension of this algebra as a module over the constant functions ($\boldsymbol{C}$ or $\boldsymbol{R}$) is exactly the multiplicity μ of f (see Palamodov **[59]**).

One can choose as a miniversal deformation of the germ of f at O the deformation $\lambda \to f + \lambda_1 e_1 + \cdots + \lambda_\mu e_\mu$ where the functions e_i define the generators of the local algebra as a module over the constants.

Let us fix some miniversal deformation of the germ of f at the origin. The *level bifurcation* set for f is the germ at O of the hypersurface in the base space, formed by all the values of the parameter λ such that 0 is a critical value for the corresponding function near the origin.

The complement to this bifurcation set is the base space of the fibration, whose typical fibre is a nonsingular level set of f. The action of the fundamental group of this complement on the homology of the fibre is called the *monodromy* of the singularity, and its image is called *monodromy group*.

THEOREM 9 (SEE **[6]**). *The complement of the level bifurcation set of a simple singularity is a $K(\pi, 1)$ space, where π is the corresponding braid group (defined by E. Artin for the case A, and by E. Brieskorn in the general case; see* **[21]**, **[22]**, **[23]**). *This complement is the space of regular orbits of the action of the corresponding Coxeter group on the complexification of the Euclidean space $\boldsymbol{R}^\mu$. The monodromy group of a simple singularity is the natural representation of the braid group on the Coxeter group.*

In case E the proof uses one theorem of Deligne **[26]** and one of Brieskorn **[20]**.

The topology of the complement to the bifurcation subsets seems to be very interesting, and might bring some algebraic structure to the amorphous set of singularity classes. The few results known on the homology (see **[4]**, **[22]**) show promising relations to loop spaces of the sphere (G. Segal **[65]**, Fuks **[30]**, **[31]**); there exist also some relations to pseudo-isotopies (Cerf **[25]**, Thom **[71]**) and to the algebraic K-theory (Volodin **[79]**, Wagoner **[81]**, Hatcher **[39]**).

Returning to the level bifurcation set for a function at a critical point of multiplicity μ, let us consider a straight line $\boldsymbol{C}^1$ near the origin of the base space $\boldsymbol{C}^\mu$ of the miniversal deformation. A generic $\boldsymbol{C}^1$ intersects the bifurcation set at μ different points near O. Let us fix such a $\boldsymbol{C}^1$ and call these μ points *distinguished points*.

Fix a base point in $\boldsymbol{C}^1$ (different from the μ distinguished points), and let V be the fibre of our fibration over the base point (V is the nonsingular local level manifold). Let us choose μ *distinguished paths*, coming from the base point to the μ distinguished points and having no intersections outside the base point. The fibre over a point near the distinguished one has a *vanishing cycle* of Picard-Lefshetz (that is, an embedded sphere S^{n-1} which generates the homology of the local level manifold at the distinguished point) (Figure 4).

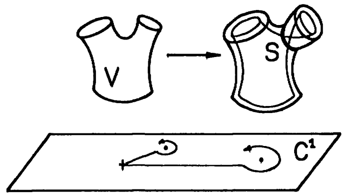


FIGURE 4. A distinguisted vanishing cycle.

Returning to the base point along the distinguished path, one defines a *distinguished vanishing cycle* in $H_{n-1}(V)$. The μ distinguished cycles thus obtained form the *distinguished basis* of H_{n-1} (Lamotke [45], Gabrielov [32]). The fundamental group of the complement to the level bifurcation set is generated by the μ *distinguished loops* on C^1; one obtains these loops from the distinguished paths, turning around the distinguished points (the fundamental group theorem of Zariski [85], see also [77], [48]).

Now let us suppose that n is odd (n is the number of variables). In this case the action of every distinguished generator of the fundamental group on the homology of V is the reflection in the orthogonal complement to the distinguished vanishing cycle (the Picard-Lefshetz theorem).

So, to calculate the monodromy group of a singularity it is sufficient to find the Dynkin diagram *for the base formed by the* μ *distinguished vanished cycles.* The first important examples of this were given by Pham (see Pham [60], Brieskorn [19]): The Pham basis is in fact a distinguished one. Articles of Gabrielov [32], [33] and of Guseĭn-Zade [37], [38] include many examples of such diagrams (e.g., [33] includes all the unimodular singularities and [38] all the singularities stably equivalent to functions of two variables).

The Dynkin diagram for a distinguished basis is always connected (Lazzeri [47], see also [34]). It follows that the critical point cannot bifurcate if the critical value does not.

The *classical monodromy* of a function germ f is the action on $H_{n-1}(V)$ of the product of all distinguished generators. This operator is related to the asymptotics of different integrals containing f, and it is important to calculate it (see, e.g., Milnor and Orlik [57], Brieskorn [24], A'Campo [1], Malgrange [51]). If the diagram for a distinguished basis is known, the calculation of the classical monodromy is reduced to a multiplication of matrices.

For simple singularities, the classical monodromy operator is the Coxeter element of the Coxeter group. Its order N is the Coxeter number $N(A_k) = k + 1$, $N(D_k) = 2k - 2$, $N(E_6) = 12$, $N(E_7) = 18$, $N(E_8) = 30$.

A'Campo [2] has proved that the classical monodromy operator for degenerate singularities is never the identity.

6. The function bifurcation set. Let $\mathfrak{m}^2$ be the space of germs of functions with critical value 0 of the critical point $O \in C^n$. The group of germs of diffeomorphisms of C^n preserving O acts on $\mathfrak{m}^2$. A germ T of a manifold of minimal dimension, transversal to the orbit of f in $\mathfrak{m}^2$, has dimension $\mu - 1$. One can consider the embedding of T as a $(\mu - 1)$-parameter deformation of the germ f. This deformation, as any other deformation, is induced from the miniversal deformation by some mapping of the base spaces $\tau: T \to C^\mu$.

The level bifurcation set Σ is the image of T under τ. Σ is irreducible and has a nonsingular normalisation T (see Teissier **[68]**, Gabrielov **[34]**).

Let $\mathfrak{m}$ be the space of germs at O of functions with value 0 at O (O not necessarily being a critical point). The deformation inside this class will be called *restricted deformations*. A miniversal restricted deformation has $\mu - 1$ parameters. We obtain an *extended miniversal deformation* with μ parameters from the restricted one by adding an arbitrary constant at the μth parameter.

Let us fix the representative of a miniversal restricted deformation of a germ f. One calls the points in the base space $C^{\mu-1}$ for which the associated function has less than μ different critical values near O the *function bifurcation points*. The set of all such points is the *function bifuracation subset* for f; this is a hypersurface Δ in $C^{\mu-1}$ (more precisely, we will consider the germ of Δ at O) (Figure 5).

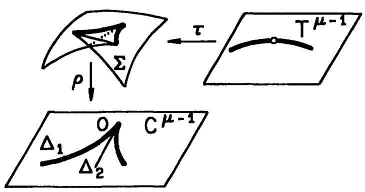


FIGURE 5. The level bifurcation set Σ and the function bifurcation set Δ for A_3.

THEOREM 10 (SEE **[67]**, **[34]**). *The restriction to the level bifurcation set Σ of the natural mapping $\rho: C^\mu \to C^{\mu-1}$ from the base space of the extended miniversal deformation to that of the restricted one defines a μ-fold covering over the complement to Δ in $C^{\mu-1}$ (in some neighbourhood of O). The group of this covering is the whole symmetric group, S_μ.*

THEOREM 11 (LOOIJENGA **[50]**, LIASCHKO **[9]**). *For simple germs of functions the complement of the function bifurcation set (in some neighbourhood of O in $C^{\mu-1}$) is a $K(\pi, 1)$ space, where π is a subgroup of finite index $\nu = \mu! N^\mu W^{-1}$ (N = Coxeter number, W = order of the Weyl group) in the Artin braid group with μ strings.*

The function bifurcation set Δ is the union of two hypersurfaces Δ_1 and Δ_2; Δ_1 corresponds to functions having degenerate critical points, and Δ_2 to functions having coincident critical values.

The smooth mapping $\rho \circ \tau: T^{\mu-1} \to C^{\mu-1}$ from the transversal space to the base space of the restricted deformation has Δ_1 as the critical value set and defines a μ-fold covering over the complement to Δ_1.

The hypersurface Δ_1 is called the *caustic*, and Δ_2 the *cut locus* (or the Maxwell stratum).

7. Lagrange singularities and caustic classification. One can see caustics on a wall lit up by sun rays reflected by some curved surface (e.g., by the inside surface of a cup). Moving the cup, one can see that generic caustics allow only standard singularities, while more complicated singularities bifurcate into standard ones under small perturbation.

The study of caustics is a part of the theory of *Lagrange singularities* (see [6] and articles of J. Guckenheimer [36], A. Weinstein [82], Hörmander [40]) similar to the usual theory of singularities of smooth mappings of Whitney, Thom, and Mather ([83], [69], [53]).

The *symplectic structure* on a manifold M^{2n} is a closed 2-form ω, nondegenerate at every point of M.

A *Lagrange submanifold* of a symplectic manifold (M^{2n}, ω) is a submanifold of the greatest possible dimension where ω vanishes (that is, of dimension n). The fibration p: $M^{2n} \to B^n$ is a *Lagrange fibration* if all its fibres are Lagrange submanifolds. A typical example is the cotangent fibration $T^*B \to B$ (the "phase space" of classical mechanics).

Let $i : L \to M$ be the embedding from a Largrange submanifold to the total space of a Lagrange fibration $p : M \to B$. Then $p \circ i : L \to B$ is called a *Lagrange mapping*, and one calls its set of critical value *caustics*.

A *Lagrange equivalence* is a mapping between two Lagrange fibrations respecting the symplectic structure. Two Lagrange mappings are *equivalent* iff there exists a Lagrange equivalence which maps the first of the corresponding Lagrange submanifolds on the second. Caustics of equivalent mappings are diffeomorphic.

A Lagrange mapping is *stable* at a point O iff every Lagrange mapping, close enough to the given one, has, at some point near O, a germ equivalent to the given germ at O.

The germ of a Lagrange mapping is *simple* iff all nearby germs belong to a finite number of equivalence classes. A simple germ can be nonstable and a stable germ can be nonsimple.

THEOREM 12 (SEE [6]). *Simple stable germs of Lagrange mappings are classified by the A, D, and E singularities. Iff $n \leqq 5$ every Lagrange mapping of L^n can be approximated by a mapping whose germ at every point is stable and simple.*

We give below an explicit description of Lagrange germs of the types A, D, and E, listing coordinate normal forms. It follows from the list, for example, that generic caustics in three-space have besides normal crossings only Lagrange singularities A_3 (cuspidal edges), discrete point singularities A_4 (swallow tails) and $D_4^{\pm}$ (points of contact of three cuspidal edges, two of which may be complex) (Figure 6).

8. Legendre singularities and classification of wavefronts. To obtain an example of a wavefront one can start from an ellipse, construct inside normals, and choose

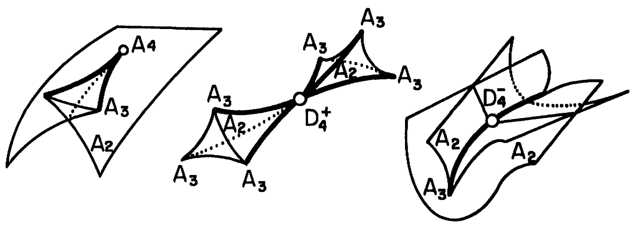


FIGURE 6. Generic singularities of the caustics in the 3-space.

points at the distance t from the ellipse points on the normals (Figure 7). The curve so obtained may have singularities which cannot be removed by a small variation of the initial ellipse. The study of front singularities is a part of the theory of Legendre singularities (see **[10]**, **[11]**; the name comes from classical "Legendre transformation", which provides typical examples of Legendre singularities).

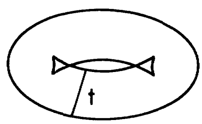


FIGURE 7. The singularities of a wavefront.

The theory of Legendre singularities is parallel to the theory of Lagrange singularities, with the following differences: One has to replace the symplectic structure with the contact one, the affine structure with the projective one, gradients with Legendre transformations, functions with hypersurfaces, and so on.

The parallelism between the two theories is the real origin of the Hamilton "optical-mechanical analogy".

The *contact structure* on a manifold M^{2n+1} is a field of tangent hyperplanes (called *contact planes*), verifying the "maximum nonintegrability" condition (if α is the 1-form defining contact planes, $\alpha \wedge (d\alpha)^n$ is nondegenerate). Standard examples of contact manifolds are the total space of the projective cotangent bundle PT^*V^{n+1} and the manifold of 1-jets of functions $J^1(W^n, \boldsymbol{R})$ with their natural contact structures (defined by the integrability conditions).

The *Legendre submanifold* of a contact manifold M^{2n+1} is an integral submanifold of maximal dimension (that is, of dimension n). The fibration $p : M^{2n+1} \to B^{n+1}$ is a *Legendre fibration* if all its fibres are Legendre submanifolds.

A typical example is the projective cotangent fibration $p: PT^*B \to B$. All Legendre fibrations of the same dimension are locally equivalent (locally = near every point of the total space). Fibres of a Legendre fibration locally have the structure of a projective space defined intrinsically by the Legendre fibration.

Let $i : L^n \to M^{2n+1}$ be an embedding of a Legendre submanifold in the total space of a Legendre fibration $p : M^{2n+1} \to B^{n+1}$. The mapping $p \circ i : L^n \to B^{n+1}$ is then the *Legendre mapping*, and its image the *front*.

Legendre equivalence, stability, and simplicity of germs are defined in the same way as in the Lagrange case.

THEOREM 13 (SEE [10], [11]). *Simple stable germs of Legendre mappings are classified by the A, D, and E singularities. Iff $n \leqq 5$, every Legendre mapping of L^n can be approximated by a mapping whose germs at all points are stable and simple.*

We give the list of explicit normal forms for simple stable Legendre germs in the next section. It follows from the list, e.g., that generic fronts in three-space have (besides normal crossings) only Legendre singularities A_2 (cuspidal edges) and A_3 (swallow tails). The singularity of the moving front slips along the caustic, and at some discrete moment may change its shape under some standard "catastrophe" of the types A_4 or $D_4^{\pm}$ (compare the pictures in the book of Thom [70]) (Figure 8).

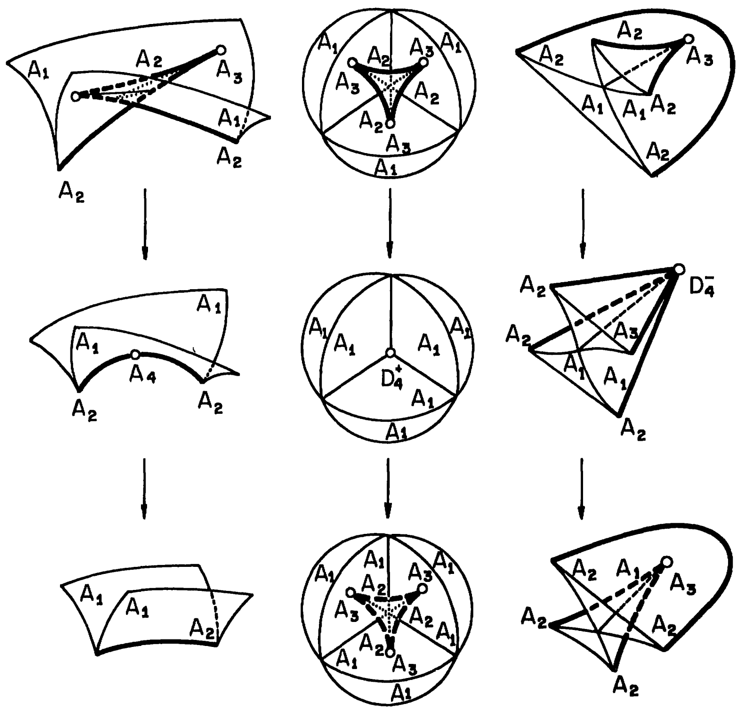


FIGURE 8. The modifications of the wavefronts near the catastrophes A_4 and $D_4^{\pm}$.

There exists a *symplectisation functor* associating to a contact M^{2n+1} a symplectic E^{2n+2}. However, the symplectisations of generic Legendre singularities are very special (conical) Lagrange singularities. The right way to deal with the Legendre case is rather the *contactisation functor*, associating to a symplectic M^{2n} a contact E^{2n+1} (defined, in fact, either for germs or for symplectic structures defining an integer class in $H_2(M^{2n})$).

9. Normal forms for caustics and fronts. I shall use here the old-fashioned coordinate notations: Let $F(x, \lambda)$ be a deformation of a function $f(x)$ in k variables, $x \in \boldsymbol{R}^k$ and one parameter, $\lambda \in \boldsymbol{R}^l$. Let $n = k + l$, and let us consider a symplectic space $\boldsymbol{R}^{2n}$ with coordinates $x \in \boldsymbol{R}^k, y \in \boldsymbol{R}^{k*}, \lambda \in \boldsymbol{R}^l, \kappa \in \boldsymbol{R}^{l*}$, with a symplectic structure $\omega = dx \wedge dy + d\kappa \wedge d\lambda$ and with a Lagrange fibration structure $(x, y, \lambda, \kappa) \mapsto (y, \lambda)$. The equations

$$(1) \qquad y = \partial F/\partial x, \qquad \kappa = -\partial F/\partial \lambda$$

define a Lagrange submanifold, and we denote by $\mathscr{L}$ the Lagrange mapping so obtained.

Let us construct two more families of functions in the variable x

$$\Phi(x; \lambda, y, z) = F(x, \lambda) - z - xy$$

(parameters are $\lambda \in \boldsymbol{R}^l$, $y \in \boldsymbol{R}^{k*}$, $z \in \boldsymbol{R}$);

$$G(x; a, \lambda) = F(a + x, \lambda) - F(a, \lambda) - xF'_a(a, \lambda)$$

(parameters are $a \in \boldsymbol{R}^k$, $\lambda \in \boldsymbol{R}^l$). Let $G(x, O, O)$ be $g(x)$.

THEOREM 14. *The following conditions are equivalent*:

(i) *The germ of $\mathscr{L}$ at the point $x = 0$, $\lambda = 0$ is Lagrange stable.*

(ii) *The deformation G is transversal to the orbit of g in $\mathfrak{m}^2$.*

If $f \in \mathfrak{m}^2$, each of the conditions (i), (ii) *is equivalent to:*

(iii) *The deformation Φ is a versal deformation of f at O.*

THEOREM 15 (SEE [6]). *Simple stable germs of Lagrange mappings are equivalent to the germs $\mathscr{L}$ defined by* (1), *where F is a deformation of a simple germ of f such that the deformation Φ is versal.*

For example, if $f = x^4$ (the A_3 case), one can choose $F = x^4 + \lambda x^2$ (the complete list of F for all the cases A, D, E can be found in [6]).

Turning now to the Legendre case, let us extend $\boldsymbol{R}^{2n}$ to $\boldsymbol{R}^{2n+1} = \boldsymbol{R}^{2n} \times \boldsymbol{R}^1$ and let z be the coordinate in $\boldsymbol{R}^1$. Let us define the contact structure of $\boldsymbol{R}^{2n+1}$ by the form $\alpha = xdy + \kappa d\lambda + dz$ and the Legendre fibration by $(x, y, \lambda, \kappa; z) \mapsto (y, \lambda; z)$.

The equations

$$(2) \qquad y = \partial F/\partial x, \qquad \kappa = -\partial F/\partial \lambda, \qquad z = F - x\partial F/\partial x$$

define a Legendre submanifold, and we denote by $\mathscr{L}'$ the Legendre mapping so obtained.

[Every Lagrange (resp. Legendre) submanifold or mapping in the standard coordinate symplectic (contact) manifold is locally defined by at least one of the 2^n formulae (1) (resp. (2)), corresponding to the 2^n choices of a coordinate "x-subspace" $\boldsymbol{R}^k \subset \boldsymbol{R}^n$.]

THEOREM 16 (SEE [10], [11]). *Simple stable germs of Legendre mappings are equivalent to the germs $\mathscr{L}$, defined by* (2), *F being the same as in the previous theorem.*

Besides the argument change group, there are the multiplications by the group of nonvanishing functions acting on the function space. The direct product of those two groups acts on the function space too. One calls the deformation of a function f *versal for levels*, if it is transversal to the orbit of this group.

As a versal for level deformation of a germ of f at O one can choose the deformation $\lambda \mapsto f + \lambda_1 e_1 + \cdots + \lambda_r e_r$, the e_s defining generators over $\boldsymbol{R}$ of the factor algebra of power series at O by the ideal $(f, \partial f/\partial x)$.

The product of the group of multiplications by nonvanishing germs at O with the group of diffeomorphisms leaving O fixed acts on $\mathfrak{m}^2$.

THEOREM 17. *The following conditions are equivalent:*

(i′) *The germ of* $\mathscr{L}'$ *at the point* $x = 0$, $\lambda = 0$ *is Legendre stable.*

(ii′) *The deformation* G *is transversal to the orbit of* g *under the action of the product group on* $\mathfrak{m}^2$.

If $f \in \mathfrak{m}^2$, *each of the conditions* (i′), (ii′) *is equivalent to*:

(iii′) *The deformation* Φ *is versal for levels.*

Comparing these results with those of previous sections, we can formulate

THEOREM 18. *The mapping* τ *from the transversal space* $T^{\mu-1}$ *to the base space of the miniversal deformation is a Legendre mapping, the level bifurcation set being its front. The mapping* $\rho\circ\tau$ *from the transversal space* T *to the restricted miniversal deformation space is a Lagrange mapping, the function bifurcation set being its caustic.*

[The above theorems may become more clear if we introduce the germ of the restricted critical set C of the deformation $F(x, \cdot)$ defined as $C = \{(x, \lambda) : \partial F/\partial\lambda = 0, F(x, \lambda) = 0\}$.

If the deformation F is miniversal, C is a germ of a smooth $(\mu - 1)$-manifold. The canonical projection $(x, \lambda) \mapsto \lambda$ defines a mapping $\pi : C \to \Sigma$. The coordinate system defines a diffeomorphism $j : C \to T$ to the transversal to the orbit (j is defined by the translations of the critical points to the origin). The diagram

$$\begin{tikzcd} T \arrow[dr, "\tau"'] & & C \arrow[ll, "j"'] \arrow[dl, "\pi"] \\ & \Sigma & \end{tikzcd}$$

commutes; therefore π as τ, normalizes Σ; $\rho\circ\pi$ has the properties of $\rho\circ\tau$ and so on.]

10. Oscillating integrals. The study of the intensity of light near the caustic leads to the problem of asymptotics for an "oscillating integral"

$$I(h, \lambda) = \int e^{iF(x, \lambda)/h} \phi \, dx, \qquad x \in \boldsymbol{R}^k, \lambda \in \boldsymbol{R}^l,$$

depending on a parameter λ, for $h \to 0$. Here the parameter λ represents the point of observation, ϕ has compact support, F is a real smooth "phase function", and h defines the wave length.

Of course, one meets such integrals in all branches of mathematics and physics —e.g., in number theory and P.D.E. theory (see **[78]**, **[54]**, **[40]**).

If the light is intense enough to destroy the medium, the destruction will begin at the singular points of the caustics, where I is maximal. Thus arises the problem of defining asymptotics for $h \to 0$ of the maximum of I in λ, which can be met for generic F. The classification of simple singularities was found as a byproduct when this problem, communicated to me by Maslov [55], was being solved for $l = 3$ (see [5]).

The stationary phase principle is the assertion that the main part of the oscillating integral is given by the integration over the neighbourhoods of the critical points of F (for fixed λ). For a generic function all these points are nondegenerate, and the integral decreases for $h \to 0$ as $h^{k/2}$ (Fresnel [29]). However, degenerate critical points appear for some "caustic value" of the parameter λ even for a generic $F(x, \lambda)$. So at some points λ the integral decreases more slowly (as $h^{(k/2)-\beta}$).

The number β so defined is called the *degree of singularity* [5] of the corresponding critical point.

To be more precise, let us consider a critical point of finite multiplicity μ. The integral I allows then an asymptotic expansion

$$I \sim \sum_{\alpha, \kappa} C_{\alpha, \kappa} h^{(k/2)-\alpha} \ln^{\kappa} h,$$

where $0 \leqq \kappa \leqq k - 1$ and α belongs to the union of a finite number of rational arithmetical progressions (see I. Bernstein and S. Gel′fand [16], Atiyah [15], I. Bernštein [17], B. Malgrange [51] and [52]). Now β is the minimum of α such that there exists $C_{\alpha, \kappa} \neq 0$ for some ϕ with arbitrarily small support containing the critical point.

THEOREM 19 (SEE [5], [6], [7]). *For simple critical points, $\beta = 1/2 - 1/N$, where N is the Coxeter number. For parabolic and hyperbolic singularities $\beta = 1/2$.*

Probably, for all other critical points, $\beta > 1/2$. We define the *Coxeter number N of* any *singularity* by the formula $\beta = 1/2 - 1/N$, where β is the degree of singularity.

THEOREM 20 (SEE [7]). *The maximum of the degrees of singularities inevitable in generic families of functions in $k \geqq 3$ variables depending on $l \leqq 10$ parameters is $\beta = 1/2 - 1/N$, where N is given by the table*

l	0	1	2	3	4	5	6	7	8	9	$10, k=3$	$11, k=3$	$10, k>3$
N	$+2$	$+3$	$+4$	$+6$	$+8$	$+12$	∞	∞	-24	-16	-12	-8	-6

All the numbers $\beta_l(k)$ are rational, and do not depend on k when k is large enough; β_l, the limit for $k \to \infty$, increases probably as $\sqrt{2l}/6$ with l.

Probably, β is semicontinuous and even more, for every λ near λ_0,

$$|I(h, \lambda)| \leqq C(\varepsilon, \phi) h^{(k/2)-\beta(\lambda_0)-\varepsilon} \quad \text{for all } \varepsilon > 0.$$

Such an "uniform estimation" has been proved by I. M. Vinogradov [78] for the singularities of the type A and by Duistermaat [28] for all simple and parabolic singularities.

11. Semi-quasi-homogeneous functions and the Newton diagram. The first proofs of the classification theorems [6], [7] need long calculations, which can be replaced with some geometrical arguments, based upon the Newton diagrams.

A function $f(x_1, \cdots, x_n)$ is quasi-homogeneous of degree d with weights $(\alpha_1, \cdots, \alpha_n)$, if $f(t^{\alpha_1}x_1, \cdots, t^{\alpha_n}x_n) = t^d f(x_1, \cdots, x_n)$ identically in $t \in \boldsymbol{C}^*$. Here $0 \leqq \alpha_i \leqq \frac{1}{2}$ are rational numbers (see Saito [62], Milnor and Orlik [57], Orlik and Wagreich [58], Saito [63]).

The function f is semi-quasi-homogeneous, if $f = f_0 + f'$, where f_0 is quasi-homogeneous of degree 1, and has an isolated critical point at o, while the degrees of all the monomials of f' are higher than 1.

THEOREM 21 (SEE [8]). *Every semi-quasi-homogeneous function is equivalent to a "normal form" $f \sim f_0 + c_1e_1 + \cdots + c_re_r$ where c_s are numbers and the monomials e_s are the elements of a monomial basis of the local algebra of f_0 at O, whose degrees are more than* 1.

The Newton diagram of $f(x_1, \cdots, x_n)$ is a convex polyhedron in $\boldsymbol{R}^n$ constructed from the exponents of the monomials having nonzero coefficients in the Taylor series; it contains a lot of information on the singularity, but I shall formulate here only one result of A. G. Kushnirenko.

Let us suppose the the Newton diagram contains points on all coordinate axes (that is not a restriction, see the theorem of Tougeron [76]).

THEOREM 22 (KUSHNIRENKO). *Let us denote by V the volume of the positive orthant of $\boldsymbol{R}^n$ under the Newton diagram, by V_i the $(n-1)$-dimensional volume under the diagram on the ith coordinate hyperplane, by V_{ij} the $(n-2)$-dimensional volume on the coordinate plane orthogonal to the ith and the jth coordinate lines, and so on.*

Then for all functions f having a given Newton diagram

$$\mu(f) \geqq n!V - (n-1)! \sum V_i + (n-2)! \sum V_{ij} - \cdots \pm 1,$$

and for almost all functions f having this diagram, the equality holds.

For instance, for almost all functions in two variables with a given Newton diagram, we have $\mu = 2S - a - b + 1$, where S is the area under the diagram, a and b the coordinates of the diagram points on the axis (Figure 9).

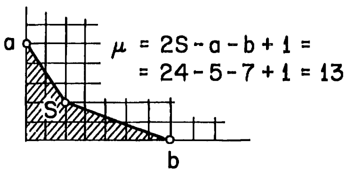


FIGURE 9. The calculation of the Milnor number.

12. Concluding remarks. It is not known whether the μ = const stratum (that is, the subvariety of the versal deformation base space, formed by points corresponding to functions with a critical point of multiplicity μ) is smooth. It was proved by Le Dung Trang and Ramanujan [49] that, for $n \neq 3$, neither the topology of the singular level set nor the topology of the Milnor fibration change along μ = const stratum. Probably, neither the topology of the function nor the β changes (for n = 3, as for other n).

The topology of bifurcation sets may change (Pham [60]). The dimension of the μ = const stratum is semicontinuous and so equal to the modality m of the critical point (Gabrielov [34]). Using some results of Teissier [68], Kushnirenko and Gabrielov [35] were able to prove that the modality of semihomogeneuos singularities is equal to the number of generators of a monomial basis of the local algebra above and on the Newton diagram.

The same is probably true for all semi-quasi-homogeneous singularities. The modality m of functions of two variables is probably equal to the number of integer points between the coordinate rays passing through the point (2, 2) and the Newton diagram (for almost all functions with a given diagram, see [8]).

In this article I did not even mention many important sides of the theory of critical points of functions, especially the algebraic ones (see, e.g., [65]). I like to stress the importance and power of transcendental, topological methods, based upon the study of the hierarchy of singularities (first for the cases of small codimension), of the adherence of different classes of singularities to others, upon semicontinuity and general position arguments, arguments which go back to the bifurcation theory of Poincaré (see [12]) and were formalised by Thom's transversality theorems. G. M. Tjurina was the first to apply these ideas to the study of singular points of hypersurfaces (see [13], [73], [74], [75].)

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