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Recent Progress in Classical Fourier Analysis

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In what sense does $\int_{R^n} e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi$ converge to a given function f on R^n? How do properties such as the size and smoothness of f influence the behavior of its Fourier transform $\hat{f}$? These simple questions lie at the heart of much of classical analysis. Their deeper study leads naturally to certain basic auxiliary operators defined on functions on R^n; and Fourier analysts seek to understand these operators and their generalizations, and to apply them to various branches of analysis. In this paper I shall describe some basic results and applications of Fourier analysis and speculate briefly on the future. I have left out many topics of great importance, and emphasized merely those subjects I know something about.

Let me begin by sketching the state of the art as of about 1950. At that time, the field was well developed only in the one-dimensional case. Since it had long been known that the Fourier series of a continuous function on $[0, 2\pi]$ need not converge at every point, Lebesgue measure (and in particular L^p) was clearly recognized as a basic tool. The Plancherel theorem $\int_0^{2\pi} |f(x)|^2\, dx = 2\pi \sum_{-\infty}^{\infty} |a_k|^2$ with $f(x) \sim \sum_{-\infty}^{\infty} a_k e^{ikx}$ gave a complete characterization of L^2 functions in terms of their Fourier coefficients and established norm convergence of Fourier series. However, the study of L^p $(p \neq 2)$ was known to be much harder. As an indication of the difficulty of the problems of L^p, take a function $f(x) \sim \sum_{-\infty}^{\infty} a_k e^{ikx}$ belonging to L^p $(p < 2)$ but not to L^2, and modify its Fourier series by writing $g(x) \sim \sum_{-\infty}^{\infty} \pm a_k e^{ikx}$ with each $\pm$ sign picked independently by flipping a coin. Then with probability one, g does not belong to L^p (or even to L^1) but is merely a distribution with nasty singularities. Consequently, the assertion $f \sim \sum_{-\infty}^{\infty} a_k e^{ikx} \in L^p$ depends not only on the sizes $|a_k|$ of the Fourier coefficients, but also on subtle relationships among the phases $\arg(a_k)$.

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Despite the difficulty of the problem, a fair amount was known by 1940 about the relationship between the size of a function and the nature of its Fourier series, thanks to pioneering efforts by Hardy and Littlewood, M. Riesz, Paley, Zygmund, Marcinkiewicz and others. A result typical of the deepest work is as follows (see [95]):

THEOREM 1 (LITTLEWOOD-PALEY). *Let $\{S_k\}_{k=-\infty}^{+\infty}$ be a sequence of $\pm$ signs which stays constant on each dyadic block. (A dyadic block is an interval of the form $[2^N, 2^{N+1})$ or $(-2^{N+1}, -2^N]$; the collection of all dyadic blocks will be denoted by Δ.) Then if $f(x) \sim \sum_{\infty}^{-\infty} a_k e^{ikx}$ belongs to L^p $(1 < p < \infty)$, it follows that $\sum_{-\infty}^{\infty} S_k a_k e^{ikx}$ also belongs to L^p.*

Thus, although the phases $\arg(a_k)$ play a decisive role in determining the size of $\sum_{-\infty}^{\infty} a_k e^{ikx}$, only the relationship of $\arg(a_k)$ to relatively "nearby" $\arg(a_{k'})$ really matters.

Although the original techniques used to prove this and related theorems are very complicated, the underlying strategy is simple. The starting point is to rewrite Dirichlet's formula for the Nth partial sum of a Fourier series as

$$\begin{aligned} S_N f(x) &= e^{-iNx} \int_{R^1} e^{iN(x-y)} f(x-y) \frac{dy}{y} - e^{iNx} \int_{R^1} e^{-iN(x-y)} (x-y) \frac{dy}{y} \\ &= e^{-iNx} H(e^{iNy} f(y)) - e^{+iNx} H(e^{-iNy} f(y)) \end{aligned}$$

with $Hf(x) \equiv \int_{R^1} (f(x-y)/y)\, dy$, the integral being interpreted in the principal-value sense. (Hf is called the Hilbert transform of f.) This is a bold step, since for $C_0^\infty(R^1)$ (say), the integral in Dirichlet's formula converges absolutely, while that defining the Hilbert transform does not.

Now the Hilbert transform also arises in complex analysis, for if $F = u + iv$ is a well-behaved analytic function on the upper half-plane R^2_+, then on the boundary R^1, v is the Hilbert transform of u. Therefore we may hope to prove theorems on the Hilbert transform and related operators via complex analysis (e.g., Cauchy's theorem, Jensen's formula and Blaschke products, conformal mapping) and then translate the results into information on Fourier series. To illustrate the "complex method", let us prove a simple case of M. Riesz's famous theorem that the Fourier series of an L^p function on $[0, 2\pi]$ converges in norm $(1 < p < \infty)$. This comes down to proving that the Hilbert transform is bounded on $L^p(R^1)$, and we give the argument for the easiest nontrivial case $p = 4$. Given a well-behaved analytic function $F = u + iv$ on R^2_+, we have to show that $\int_{R^1} v^4\, dx \leqq C \int_{R^1} u^4\, dx$ with C independent of F. However, Cauchy's theorem for $F^4 = u^4 + 4iu^3v - 6u^2v^2 - 4iuv^3 + v^4$ yields $\int_{R^1} F^4\, dx = 0$ so that $0 = \int_{R^1} \mathrm{Re}(F^4)\, dx = \int_{R^1} (u^4 - 6u^2 v^2 + v^4)\, dx$. Hence $\int_{R^1} v^4\, dx \leqq 6 \int_{R^1} u^2v^2\, dx \leqq 6 \left(\int_{R^1} u^4\, dx\right)^{1/2} \left(\int_{R^1} v^4\, dx\right)^{1/2}$ by Cauchy-Schwarz. Dividing both sides by $\left(\int_{R^1} v^4 dx\right)^{1/2}$ and squaring gives the desired inequality $\int_{R^1} v^4\, dx \leqq 36 \int_{R^1} u^4 dx$. The general case ($p \neq 4$) is similar, though not so easy.[1]

[1]See the ingenious paper of S. Pichorides [72] for the exact norm of the Hilbert transform on L^p and other related constants.

Now I can give a vague idea of the proof of the Littlewood-Paley theorem. The idea is to relate an auxiliary operator S arising from complex analysis with an operator G arising from Fourier series. Specifically, given $f \sim \sum_k a_k e^{ikx}$ on $[0, 2\pi]$ (say $a_0 = 0$), we break up the Fourier series into dyadic blocks

$$f \sim \sum_k a_k e^{ikx} = \sum_{I \in \Delta} \Big(\sum_{k \in I} a_k e^{ikx} \Big) \equiv \sum_{I \in \Delta} f_I(x)$$

and define $G(f)$ as $G(f)(x) = (\sum_{I \in \Delta} |f_I(x)|^2)^{1/2}$. The function $S(f)$ is defined in terms of the Poisson integral $u(r, \theta)$ of f by the equation

$$S(f)(x) = \Big(\iint_{(r,\theta) \in \Gamma(x)} |\nabla u(r, \theta)|^2 \, r \, dr \, d\theta \Big)^{1/2}$$

where $\Gamma(x)$ is the Stoltz domain $\{(r, \theta) | \, |x - \theta| < 1 - r < \frac{1}{2}\}$ in the unit disc. $S^2(f)$ has a natural interpretation as the area of the image of $\Gamma(x)$ under the analytic function $u + iv$ whose real part is u. For our purposes, the basic facts concerning S and G are:

(a) $\|S(f)\|_p \sim \|f\|_p \ (1 < p < \infty)$. In other words, $\|S(f)\|_p / \|f\|_p$ is bounded above and below. This can be proved by complex methods. Note that already (a) contains the L^p-boundedness of the Hilbert transform, since for $F = u + iv$ analytic we have $|\nabla u| = |\nabla v|$ by the Cauchy-Riemann equations, and hence $S(u) = S(v)$.

(b) $\|S(f)\|_p \sim \|G(f)\|_p \ (1 < p < \infty)$. Limitations of space prevent even a vague description of the proof, but the basic tool here is the L^p-boundedness of the Hilbert transform acting on functions which take their values in a Hilbert space.

Once we know (a) and (b), the Littlewood-Paley theorem follows at once, since evidently $f = \sum_{I \in \Delta} f_I$ and $g = \sum_{I \in \Delta} \pm f_I$ always have the same G-function. An extensive discussion of the Littlewood-Paley theorem and of complex methods in general may be found in Zygmund **[95]**. It must be admitted that the ingenious complex-variable proofs of classical Fourier analysis leave the researcher in the unhappy position of accepting the main theorems of the subject without any real intuitive explanation of why they are true.

Now I want to speak of the profound changes which took place in classical Fourier analysis, starting with the fundamental paper of Calderón and Zygmund **[17]** in 1952.[2] We shall be concerned here with efforts to generalize the basic operators, especially the Hilbert transform, from R^1 to R^n. These generalizations are anything but routine, because Blaschke products do not generalize to functions of several complex variables, and consequently (for this and other reasons) the whole complex method has to be abandoned and the results reproved by real-variable techniques. Moreover, the real-variable methods and the n-variable analogues of the Hilbert transform, S-function, etc., play an important role in partial differential equations, several complex variables, probability and potential theory, and will probably continue to find further applications as time goes on.

The operators. Let us begin with the Laplace equation $\Delta u = f$ in R^n $(n > 2)$

[2] In retrospect we can see many of the ideas anticipated in the work of Titchmarsh, Besicovitch, and Marcinkiewicz. (See **[95]**.)

which one solves with the standard Newtonian potential

$$u(x) = c_n \int_{R^n} \frac{f(y)\,dy}{|x-y|^{n-2}}. \tag{1}$$

If f belongs to some function space (L^p, Lip(α), $C(R^n)$, etc.) does it follow that the second derivatives of u all belong to the same function space? Differentiating the right-hand side of (1) (carefully) under the integral sign, we obtain for the second derivatives of u the formula

$$u_{jk}(x) = \frac{\partial^2 u(x)}{\partial x_j \partial x_k} = c_{jk} f(x) + \int_{R^n} \frac{\Omega_{jk}(x-y)}{|x-y|^n} f(y)\,dy, \tag{2}$$

where Ω_{jk} is homogeneous of degree zero, and smooth away from the origin. Note that the integral in (2) diverges absolutely, but at least for "nice" functions f we may define that integral as

$$\lim_{\varepsilon\to 0+} \int_{|x-y|>\varepsilon} \frac{\Omega_{jk}(x-y)}{|x-y|^n} f(y)\,dy,$$

and the limit exists by virtue of the essential cancellation $\int_{S^{n-1}} \Omega_{jk}(y)\,dy = 0$. In general, a *singular integral operator* is defined on functions on R^n by

$$Tf(x) = \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} \frac{\Omega(x-y)}{|x-y|^n} f(y)\,dy, \tag{3}$$

where Ω is reasonably smooth and homogeneous of degree zero, and $\int_{S^{n-1}} \Omega(y)dy = 0$. For example, if we set $\Omega(y) = \mathrm{sgn}(y)$ on R^1, then (3) becomes $Tf(x) = \int_{R^1} (f(y)\,dy/(x-y))$, i.e., T is the Hilbert transform. Thus regularity properties of solutions to the Laplace equation come down to boundedness on various function spaces of a few specific singular integral operators; that is, certain n-variable generalizations of the Hilbert transform.

More generally, the theory of singular integral operators plays an essential role in a host of problems of partial differential equations. To see why, start with a pure mth order differential operator

$$L = \sum_{|\alpha|=m} a_\alpha(x) \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n},$$

and write

$$L = \left(\sum_{|\alpha|=m} a_\alpha(x) R_1^{\alpha_1} \cdots R_n^{\alpha_n}\right) \cdot (-\Delta)^{-m/2},$$

where $R_j = (\partial/\partial x_j)(-\Delta)^{-1/2}$. Now R_j is called the jth Riesz transform, and is given as a singular integral operator by the formula

$$R_j f(x) = c \int_{R^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\,dy.^{3}$$

(Note that in one dimension, the single Riesz transform is just the Hilbert trans-

[3] See Horváth [52] and Stein [85].

form.) Therefore, L factors as $L = T(-\Delta)^{m/2}$, where T is a *variable-coefficient singular integral operator*, i.e., an operator of the form

$$Tf(x) = c(x)f(x) + \int_{R^n} \frac{\Omega(x, (x-y)/|x-y|)}{|x-y|} f(y)\, dy, \tag{4}$$

with $c(\cdot) \in C^\infty(R^n)$, $\Omega \in C(R^n \times S^{n-1})$, and $\int_{S^{n-1}} \Omega(x, \omega)\, d\omega = 0$ for all x. In other words, modulo the factor $(-\Delta)^{m/2}$ a partial differential operator is merely a special type of singular integral operator.

As a substitute for the Fourier transform, we associate to the operator T of (4) its *symbol* $\sigma(T)$ defined by

$$\sigma(x, \xi) = c(x) + \int_{R^n} \frac{\Omega(x, \omega/|\omega|)}{|\omega|^n} e^{i\xi\cdot\omega}\, d\omega. \tag{5}$$

Clearly, $\sigma(x, \xi)$ is homogeneous of degree zero in ξ and smooth on $R^n \times (R^n \backslash 0)$. In the special case $T = (\sum_{|\alpha|=m} a_\alpha(x)(\partial/\partial x)^\alpha)(-\Delta)^{-m/2}$ the symbol is just $\sigma(x, \xi) = \sum_{|\alpha|=m} a_\alpha(x)(i\xi)^\alpha/|\xi|^m$. Moreover,

(6) Every smooth homogeneous $\sigma(x, \xi)$ on R^{2n} arises as the symbol of a unique singular integral operator, which we denote by $\sigma(x, D)$.

(7) The class of all symbols forms an algebra of functions. The mapping $\sigma(x, \xi) \to \sigma(x, D)$ is an approximate homomorphism from functions to operators. That is, $\sigma_1(x, D) \circ \sigma_2(x, D) = (\sigma_1 \cdot \sigma_2)(x, D) +$ a "negligible" error.

(8) The adjoint of $\sigma(x, D)$ is given approximately by the complex-conjugate symbol: $(\sigma(x, D))^* = \bar{\sigma}(x, D) +$ a "negligible" error.

By virtue of (6)—(8) we may construct useful operators merely by making elementary manipulations with symbols. For instance, an elliptic singular integral operator $\sigma(x, D)$ (i.e., an operator with nonvanishing symbol) evidently has an approximate inverse—we simply take $(1/\sigma)(x, D)$—and the standard interior regularity results on elliptic partial differential equations follow easily from these observations.

So far we have described the theory as it first appeared in the pioneering work of Calderón **[12]** on uniqueness of solutions to Cauchy problems. (Calderón used singular integrals to diagonalize a matrix of differential operators. See also earlier work of Giraud **[43]** and Mihlin **[66]**.) Nowadays it is more common to work with the closely related theory of pseudodifferential operators, invented by Kohn and Nirenberg **[60]** and developed by Seeley **[75]**, Hörmander **[48]**, **[49]**, Calderón and Vaillancourt **[16]** and others. To arrive at the notion of pseudodifferential operators[4] one uses (5) and the Fourier inversion formula in (4) to obtain

$$Tf(x) = \int_{R^n} e^{ix\cdot\xi}\sigma(x, \xi)\hat{f}(\xi)\, d\xi. \tag{9}$$

Now we take (9) as the definition of $\sigma(x, D)$, broaden the class of symbols to include all functions satisfying suitable estimates, say

$$|(\partial/\partial x)^\alpha (\partial/\partial\xi)^\beta \sigma| \leq C_{\alpha\beta}|\xi|^{-|\beta|} \quad \text{for all } \alpha, \beta, \tag{10}$$

[4]Actually Kohn and Nirenberg were led to pseudodifferential operators by their work on the $\bar{\partial}$-Neumann problem of several complex variables.

and prove refinements of (7) and (8) directly from (9). Pseudodifferential operators have the advantage of making it relatively easy to refine (7) and (8) to "Leibniz' rules"

(11) $a(x, D)\circ b(x, D) = (a\circ b)(x, D)$ with $(a\circ b)(x, \xi) \sim \sum_{\alpha} (1/\alpha) [\partial/\partial\xi]^{\alpha} a\cdot[i^{-1}\partial/\partial x]^{\alpha} b$

and

(12) $$(a(x, D))^* = a\sharp(x, D) \text{ with } a\sharp \sim \sum_{\alpha} \frac{1}{\alpha!}\left[\frac{1}{i}\frac{\partial}{\partial x}\right]^{\alpha}\left[\frac{\partial}{\partial\xi}\right]^{\alpha}\bar{a}.$$

Later on, we shall see problems in which singular integrals have advantages over pseudodifferential operators. However, for many purposes the two theories are equivalent.

The applications of pseudodifferential operators to index problems in topology and geometry are so well known that it is enough for me to pay them lip service. But I would like to take a few paragraphs to explain two recent developments in partial differential equations in which pseudodifferential operators and singular integrals played a crucial rôle. Both developments have their roots in a basic phenomenon of several complex variables, namely that the restriction of an analytic function F to a hypersurface $V \subseteq C^n$ satisfies a system of partial differential equations. To see this, we start with the n Cauchy-Riemann equations $\partial F/\partial\bar{z}_j = 0$ in C^n. From the restriction of F to the hypersurface V, we know only the $2n - 1$ tangential derivatives of F, and thus we must solve one of the Cauchy-Riemann equations for the remaining (normal) derivative. Consequently, the restriction of F to V must satisfy $n - 1$ first-order partial differential equations, called the *tangential Cauchy-Riemann equations* on V.

Our first topic in partial differential equations arises from the case $V =$ the unit sphere in C^2, where are we dealing with one equation in one unknown. In a suitable coordinate system on the sphere, that equation takes the form

$$[\partial/\partial t + i(\partial/\partial x + t\partial/\partial y)]F = 0.$$

Therefore it is natural to try to "correct" functions which are "close to to analytic" by solving

(13) $$[\partial/\partial t + i(\partial/\partial x + t\partial/\partial y)]u = f,$$

with $f \in C^\infty$ (say). Such "correction" procedures are common practice in complex variables. Thus, the discovery, by H. Lewy in 1957 **[63]** that equation (13) cannot be solved, even if we require $f \in C^\infty$ and demand only that u be a distribution defined in some neighborhood of a point, came as a great shock to researchers in partial differential equations. Prior to Lewy's discovery, it was universally assumed that all nondegenerate linear partial differential equations (and certainly those arising from "real life") could be solved. After Lewy's paper, intensive research began on the problem of deciding which equations admit local solutions. At the moment, systematic results are available only for equations of principal type, i.e., roughly equations in which all lower-order terms may be regarded as trivial perturbations of the highest-order terms. These include the Laplace and wave equations, but not the heat equation or the Schrödinger equation. For equations $\sum_{|\alpha|=m} a_\alpha(x)(i^{-1}\partial/\partial x)^\alpha u = f$ of principal type, Nirenberg and Trèves [70], [71] formulated the following condition and amassed overwhelming evidence to show that it is necessary and sufficient for existence of local solutions:

(P) Let $a(x, \xi)$ and $b(x, \xi)$ be the real and imaginary parts of $\sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$. Then for any point $(x_0, \xi_0) \in R^n \times (R^n \backslash 0)$ with $a(x_0, \xi_0) = b(x_0, \xi_0) = 0$, the function b has constant sign when restricted to the "bicharacteristic curve" $(x(t), \xi(t))$ obtained by solving the ordinary differential equations $\dot{x}_j = \partial a/\partial \xi_j$, $\dot{\xi}_j = -\partial a/\partial x_j$, $(x(0), \xi(0)) = (x_0, \xi_0)$.

In fact condition (P) is now known to imply local solvability (see Beals and Fefferman [4], [5] as well as Hörmander [50], Egorov [26], [27], and Trèves [92], [93]). There is no space here to discuss the ideas in any detail. Let me just mention two of the main techniques, namely the use of canonical transformations in (x, ξ)-space to "straighten out" the zero sets of symbols of pseudodifferential operators via conjugation with Fourier integral operators (discussion of which would take us too far afield), and "microlocalization", i.e., the use of suitable partitions of unity $1 = \sum_j \phi_j(x, \xi)$ in (x, ξ)-space to define approximate projection operators $\phi_j(x, D)$ and thus split $L^2(R^n)$ into a big direct sum of subspaces $H_j =$ image of $\phi_j(x, D)$. By microlocalizing, we hope to split up one hard problem into many easy ones, and then patch the easy results together. In patching together, one has to use a calculus of pseudodifferential operators with "exotic" symbols Ω satisfying merely

$$|(\partial/\partial x)^\alpha (\partial/\partial \xi)^\beta \sigma| \leqq C_{\alpha\beta} |\xi|^{|\alpha|/2 - |\beta|/2}$$

instead of the usual estimates (10). We shall say more about exotic symbols later on.

Now let us return to the tangential Cauchy-Riemann equations on the sphere $S^{2n-1} \subseteq C^n$, and this time suppose $n > 2$. A linear fractional transformation maps the sphere to the hypersurface $H = \{(z', z^n) \in C^{n-1} \times C^1 \,|\, \mathrm{Re}(z^n) = |z'|^2\}$, which has the structure of a nilpotent Lie group under the multiplication law $(z', z^n) \cdot (w', w^n) = (z' + w', z^n + w^n + 2z' \cdot \bar{w}')$. By analogy with the R^n theory sketched above, one expects that very sharp results on existence and regularity of solutions of the tangential Cauchy-Riemann equations on H can be proved by using "singular integrals" of the form $Tf(x) = \int_H K(xy^{-1}) f(y)\, dy$, where K has appropriate properties of cancellation and homogeneity with respect to the natural "dilations" $\delta \circ (z', z^n) = (\delta z', \delta^2 z^n)$ on H. Moreover, once the results are known for H, one can build a "variable-coefficient" theory of "singular integrals" on (say) the boundary of a strongly pseudoconvex domain in C^n, by osculating the domain with biholomorphic images of H. Thus, a natural analogue of singular integrals provides a powerful machine to study the tangential Cauchy-Riemann equations. (Note that we cannot use the pseudodifferential operators viewpoint here, because the non-abelian Fourier transform on H is [so far] too cumbersome even to deal with the constant-coefficient case.) The ideas explained here come from Folland and Stein [41], although singular integrals on nilpotent Lie groups have already appeared in Knapp and Stein [59] in connection with irreducibility of the principal series. See also Folland and Kohn [40] for the initial work of Kohn on tangential Cauchy-Riemann equations, as well as Folland [39] and Stein [87].[5]

I have attempted to show by a few examples how n-dimensional analogues of the Hilbert transform enter naturally into various branches of analysis. Let us now review some techniques which have been used to study such operators, and then see what insights we can gain into the Fourier transform in R^n.

The techniques. The first step in analyzing operators that generalize the Hilbert transform is to prove L^2-boundedness. Fortunately, this is often an easy consequence of the Plancherel theorem, as in the case of a constant-coefficient singular integral operator

$$Tf(x) = \int_{R^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)\, dy$$

where one has $(\hat{T}f)(\xi) = \sigma(\xi)\hat{f}(\xi)$ with $\sigma \in L^\infty$. The S-function falls into this category —it is not hard to show that $\|S(f)\|_2 = (\text{const})\|f\|_2$. However, when an operator cannot be diagonalized by the Fourier transform or its variants, there are remarkably few L^2-techniques available to deal with it. Sometimes in a lucky case we may be able to reduce matters back to constant-coefficient questions. For instance, let

$$Tf(x) = \int_{R^2} \frac{\Omega(x, (x-y)/|x-y|)}{|x-y|^2} f(y)\, dy$$

be a variable-coefficient singular integral operator on R^2. For each fixed x we expand $\Omega(x, \cdot)$ in a Fourier series on the unit circle, obtaining $\Omega(x, \omega) = \sum_{k=-\infty}^{\infty} c_k(x)\Omega_k(\omega)$ with $\Omega_k(\omega) = e^{ik\theta}$ for $\omega = (r, \theta)$, and $c_0(x) \equiv 0$. Now our operator T may be expanded in a series of constant-coefficient operators $Tf(x) = \sum_{k=-\infty}^{\infty} c_k(x) T_k f(x)$, with

$$T_k f(x) = \int_{R^2} \frac{\Omega_k(x-y)}{|x-y|^2} f(y)\, dy.$$

Since $\Omega(x, \omega) \in C^\infty$, it follows that $|c_k(x)| \leqq C/(k^2+1)$ (say); moreover, the T_k $(k \neq 0)$ are uniformly bounded on L^2, as one sees from Plancherel. Therefore,

$$\|Tf\|_2 \leqq \sum_{k\neq 0} \frac{C}{k^2+1} \|T_k f\|_2 \leqq C\|f\|_2,$$

and our L^2-result is proved. In R^n $(n > 2)$ the same trick works, with Fourier series replaced by spherical harmonics.

A promising idea which has begun to find applications recently is Cotlar's lemma on "almost orthogonal operators".

LEMMA. *Suppose that the operators $T_1, T_2, \cdots, T_N$ on a Hilbert space H satisfy the "orthogonality conditions"*

[5]Compare with the theory of "parabolic" singular integrals devised by Jones [58], Fabes and Rivière [28], Lizorkin [64], Krée [62] and others; and in connection with parabolic singular integrals, see the recent striking results of Negel, Rivière and Wainger [69].

(14) $$\| T_i^* T_j \| \leqq C(i-j),$$
(15) $$\| T_i T_j^* \| \leqq C(i-j),$$

where $\| \cdot \|$ *denotes the operator norm, and* $\sum_{k=-\infty}^{\infty} (C(k))^{1/2} \leqq A$. *Then* $\| \sum_{k=1}^{N} T_k \| \leqq A$.

The simplest special case says merely that a direct sum $\sum_i \oplus T_i : \sum_i \oplus H_i \to \sum_i \oplus H_i$ of operators $T_i : H_i \to H_i$ has norm $\sup_i \| T_i \|$. The lemma was first given by Cotlar **[24]** in the case of commuting operators, and then extended by Knapp and Stein **[59]** to the general case. See also Calderón and Vaillancourt **[16]**.

The proof of Cotlar's lemma is so simple that I can give it here. We start with the formulas

$$\left\| \sum_{i=1}^{N} T_i \right\| = \left\| \left(\sum_{i=1}^{N} T_i \right) \left(\sum_{i=1}^{N} T_i^* \right) \right\|^{1/2} = \left\| \left[\left(\sum_{i=1}^{N} T_i \right) \left(\sum_{i=1}^{N} T_i^* \right) \right]^k \right\|^{1/2k},$$

which imply

$$\left\| \sum_{i=1}^{N} T_i \right\|^{2k} \leqq \sum_{i_1, i_2, \cdots, i_{2k}=1}^{N} \| T_{i_1} T_{i_2}^* T_{i_3} \cdots T_{i_{2k-1}} T_{i_{2k}}^* \|.$$

Hypotheses (14) and (15) show that each summand on the right is dominated both by

$$\begin{aligned} \Lambda &= \| T_{i_1} \| \cdot \| T_{i_2}^* T_{i_3} \| \cdot \cdots \cdot \| T_{i_{2k-2}}^* T_{i_{2k-1}} \| \cdot \| T_{i_{2k}}^* \| \\ &\leqq C^{1/2}(0) C(i_2 - i_3) C(i_4 - i_5) \cdots C(i_{2k-2} - i_{2k-1}) C^{1/2}(0) \end{aligned}$$

and by

$$P = \| T_{i_1} T_{i_2}^* \| \cdot \| T_{i_3} T_{i_4}^* \| \cdot \cdots \cdot \| T_{i_{2k-1}} T_{i_{2k}}^* \| \leqq C(i_1 - i_2) C(i_3 - i_4) \cdots C(i_{2k-1} - i_{2k})$$

and hence also by the geometric mean $\Lambda^{1/2} P^{1/2} \leqq C^{1/2}(0) C^{1/2}(i_1 - i_2) C^{1/2}(i_2 - i_3) \cdots C^{1/2}(i_{2k-1} - i_{2k})$. Consequently,

$$\begin{aligned} \left\| \sum_{i=1}^{N} T_i \right\|^{2k} &\leqq \sum_{i_1, \cdots, i_{2k}=1}^{N} C^{1/2}(0) C^{1/2}(i_1 - i_2) C^{1/2}(i_2 - i_3) \cdots C^{1/2}(i_{2k-1} - i_{2k}) \\ &\leqq C^{1/2}(0) N A^{2k-1}, \end{aligned}$$

so that $\| \sum_{i=1}^{N} T_i \| \leqq (C^{1/2}(0) N)^{1/2k} A^{(2k-1)/2k}$. Now just let k tend to infinity, and Cotlar's lemma is proved.

To see how Cotlar's lemma applies to the operators we have been discussing, let us reprove the L^2-boundedness of the Hilbert transform without using the Plancherel theorem. The idea is simply to write

$$Hf(x) = \int_{R^1} \frac{f(x-y)\,dy}{y} = \sum_{j=-\infty}^{\infty} \int_{2^j \leqq |y| < 2^{j+1}} \frac{f(x-y)\,dy}{y} \equiv \sum_{j=-\infty}^{\infty} H_j f(x).$$

Each H_j is a convolution operator whose convolution kernel

$$\begin{aligned} K_j(y) &= y^{-1} \quad \text{if } 2^j \leqq |y| < 2^{j+1}, \\ &= 0 \qquad \text{if not,} \end{aligned}$$

has L^1 norm dominated by a constant independent of j. Moreover, $H_i^* H_j = H_i H_j^*$ is the convolution operator with kernel $- K_i * K_j$, and elementary estimates using

$\int_{R^1} K_i(y)\,dy = \int_{R^1} K_j(y)\,dy = 0$ show that $\|K_i * K_j\|_1 \leqq C\cdot 2^{-|i-j|}$. Thus, $\|H_i^* H_j\| \leqq C\cdot 2^{-|i-j|}$ and $\|H_i H_j^*\| \leqq C^{-|i-j|}$, and the L^2-boundedness of $H = \sum_{i=-\infty}^{\infty} H_i$ is immediate from Cotlar's lemma.

Of course the L^2-boundedness of the Hilbert transform is nothing new. However, the proof sketched above applies also to the Knapp-Stein singular integrals on nilpotent groups—in fact it is the only method known to handle those operators, since as we pointed out earlier, the nonabelian Fourier transform does not help. Details are in **[59]**.

A second application of Cotlar's lemma is the theorem of Calderón and Vaillancourt **[16]** on L^2-boundedness of pseudodifferential operators with exotic symbols. (See also Hörmander **[49]** for earlier work on the subject, and Beals **[2]**, **[3]** for extensions and applications.) The basic special case of their result which one uses in microlocalization arguments for equations of principal type is the following.

THEOREM 2. *Assume that $\sigma(x, \xi)$ satisfies the estimates*

$$\left|(\partial/\partial x)^\alpha (\partial/\partial \xi)^\beta \sigma(x, \xi)\right| \leqq C_{\alpha\beta}(1 + |\xi|)^{|\alpha|/2-|\beta|/2}$$

for all multi-indices α, β. Then the corresponding pseduodifferential operator $\sigma(x, D)$ is bounded on L^2.

The main idea in the proof of the Calderón-Vaillancourt theorem is to apply Cotlar's lemma to the decomposition $\sigma(x, D) = \sum_{j=1}(\phi_j\sigma)(x, D)$, where $\sum_j \phi_j(x,\xi) = 1$ is a smooth partition of unity in (x, ξ)-space, constructed so that each ϕ_j is supported in a region of the form $\{(x,\xi)|\ |x - x_0| \leqq |\xi_0|^{-1/2}, |\xi - \xi_0| \leqq |\xi_0|^{+1/2}\}$.

When neither the Plancherel theorem nor Cotlar's lemma applies, L^2-boundedness of singular operators presents very hard problems, each of which must (so far) be dealt with on its own terms. I shall mention two outstanding L^2-results of the last decade, and say a few words about their proofs and implications.

Commutator integrals. Let $D \subseteq C^1$ be a domain bounded by a C^1 curve Γ. Just as in the case of the unit disc, there is a "Hilbert transform" T defined on functions on Γ which sends the real part $u|_\Gamma$ of an analytic function $F = u + iv$ to its imaginary part $v|_\Gamma$, and it is natural to ask whether T is bounded on $L^2(\Gamma)$ with respect to the arclength measure on Γ. This question is closely connected to the problem of understanding harmonic measure on Γ, i.e., the probability distribution of the place where a particle undergoing Brownian motion starting at a fixed point $P_0 \in D$ first hits Γ.

In effect, T is an integral operator on functions on R^1, given by the formula

$$Tf(x) = \int_{-\infty}^{\infty} \frac{f(y)\,dy}{(x - y) + i(A(x) - A(y))}$$

with $A \in C^1(R^1)$. Expanding the denominator of the integrand in a geometric series, we obtain T as an infinite sum of operators

$$T_k f(x) = \int_{-\infty}^{\infty} \frac{(A(x) - A(y))^k}{(x - y)^{k+1}} f(y)\,dy.$$

T_k is called the kth commutator integral corresponding to $A(x)$.

Commutator integrals also arise naturally when one tries to construct a calculus of singular integral operators to handle differential equations with nonsmooth coefficients. T_0 is just the Hilbert transform, but already the following two results are deep.

THEOREM 3. *Let A be a C^1 function on the line. Then*
(A) (*Calderón* **[14]**, 1965) *T_1 is bounded on L^2.*
(B) (*Coifman and Y. Meyer* 1974, *still unpublished*) *T_2 is bounded on L^2.*

See also Calixto Calderón **[18]**. To prove (A), Calderón used special contour integration arguments which unforunately do not apply to higher T_k's. Coifman and Meyer modified and built on Calderón's ideas to produce a far more flexible proof, which can probably be pushed further in the near future to cover all the T_k's and possibly T itself. We shall return to commutators in a moment.

Pointwise convergence of Fourier series. No discussion of Fourier analysis can be complete without mentioning the fundamental theorem of Carleson **[19]** to the effect that the Fourier series of an L^2 function on $[0, 2\pi]$ converges almost everywhere. Carleson's theorem provides the sharpest and most satisfactory answer to the historic problem of representation of an "arbitrary" function as the sum of a Fourier series. The result came as a surprise for several reasons. First of all, most specialists thought that pointwise convergence would turn out to be false even for continuous functions, the supporting evidence being an old example of Kolmogoroff (see **[95]**) of an L^1 function with everywhere divergent Fourier series, and the fact that for thirty years no one had succeeded in improving the classical result of Kolmogoroff-Seliverstoff-Plessner which said that the nth partial sum of an L^2 Fourier series is $o((\log n)^{1/2})$ almost everywhere. Moreover, it was widely assumed that some radical new techniques would be needed to crack the pointwise convergence problem, while Carleson succeeded by pushing the known techniques very far and very hard.

Unfortunately, Carleson's proof is so technical that it is impossible in so little space to give even the vaguest idea of its inner workings. I will only point out that the problem reduces immediately to showing that

$$f \to Mf(x) = \sup_n \left| \int_{R^1} \frac{e^{iny} f(y)\, dy}{x - y} \right|$$

is bounded on L^2, so that pointwise convergence is really a problem about the Hilbert transform. R. Hunt extended Carleson's result to L^p $(p > 1)$ in **[54]**, and his paper also gives the best presentation of Carleson's proof. P. Sjölin **[76]** proved the sharpest known result near L^1 (the Fourier series of f converges a.e. if $f \log^+ |f| \log^+ \log^+ |f| \in L^1$), and Sjölin **[77]**, Tevzadze **[90]**, and Fefferman **[30]**, **[31]** discovered some extensions to functions of n variables. See also the alternate proof of Carleson's theorem **[33]** (based partly on Cotlar's lemma) whose relationship with Carleson's proof is not well understood.

Both Carleson's convergence theorem and the Calderón-Coifman-Meyer results are stated purely in terms of L^2, but, at least as far as we know today, purely L^2 methods are not strong enough for the proofs. In fact, the known proofs of the pointwise convergence and commutator theorems in one form or another involve the full force of the "Calderón-Zygmund" machinery described below, whose usual purpose is to pass from L^2 to L^p. I am not the only analyst who suspects a strong hidden connection between commutators and pointwise convergence. In any event, our understanding of L^2 boundedness of variable-coefficient operators is still rudimentary.

The "Calderón-Zygmund" techniques used to prove L^p boundedness of singular integrals contain the deepest ideas of the theory. In the next two sections, I hope to convey more than a superficial notion of how the proofs go, even though this necessitates a more technical discussion than is customary in a survey article.[6] We begin with a seeming digression on a topic in real variables.

The maximal function. As preparation for the L^p-theory of singular integrals, we shall discuss the following basic result of Hardy and Littlewood **[44]** and Wiener **[94]**.

THEOREM 4 (THE MAXIMAL THEOREM). *Define the maximal function Mf of a locally integrable function f on R^n by the equation*

$$Mf(x) = \sup_{Q \ni x} |Q|^{-1} \int_Q |f(y)|\, dy.$$

(*Here Q denotes a cube in R^n with sides parallel to the coordinate axes.*) *Then we have the inequalities*

$$\text{(A)} \qquad \| Mf \|_p \leqq C_p \|f\|_p \; (1 < p \leqq \infty),$$

$$\text{(B)} \qquad |\{Mf > \alpha\}| \leqq C\|f\|_1/\alpha.$$

The technical-looking result (B) is the heart of the matter—it is the natural conjecture that comes to mind upon staring at the simple example $f = \delta^{-1} \chi_{[-\delta,\delta]}$ on the line. (In that case, $Mf(x) \sim (\delta + x)^{-1}$.)

The maximal theorem is really a sharp form of Lebesgue's theorem on differentiability of the integral. For, one knows trivially that $|Q|^{-1}\int_Q f(y)\,dy \to f(x)$ as Q shrinks to x, whenever f belongs to the dense subspace $C_0^\infty \subseteq L^1$. To pass from the dense subspace to all of L^1 one needs an a priori inequality, and part (B) of the maximal theorem exactly does the job.

One set of applications of the maximal theorem concerns stronger theorems than Lebesgue's on differentiation of multiple integrals. In the plane R^2, for example, let $\boldsymbol{R}_0, \boldsymbol{R}_1, \boldsymbol{R}_2$ be respectively the family of all squares, the family of all rectangles with sides parallel to the coordinate axes, and the family of all rectangles with arbitrary direction. The standard Lebesgue theorem in R^2 says that $|R|^{-1} \int_R f(y_1, y_2)\, dy_1\, dy_2 \to f(x_1, x_2)$ a. e. for $f \in L^1(R^2)$, when $R \in \boldsymbol{R}_0$ shrinks to (x_1, x_2). What happens if we allow R to belong to the larger familes $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$? The answer is contained in the following list of results:

[6]Much has been deleted from an original version of this article.

(16) $|R|^{-1}\int_R f(y_1,y_2)\,dy_1\,dy_2 \to f(x_1,x_2)$ a.e. as $R \in \boldsymbol{R}_1$ shrinks to (x_1, x_2), provided $f \in L^p(R^2)$ with $p > 1$.

(17) The result (16) may be sharpened—instead of $f \in L^p$ $(p > 1)$, it is enough to assume that $f \log^+|f|$ is integrable on R^2.

(18) However, there exist L^1 functions f for which $|R|^{-1}\int_R f(y_1, y_2)\,dy_1\,dy_2$ does not tend to a finite limit as $R \in \boldsymbol{R}_1$ shrinks to any point $(x_1, x_2) \in R^2$.

(19) The family $\boldsymbol{R}_2$ is even worse. Even for bounded functions f it may happen that $|R|^{-1}\int_R f(y_1, y_2)\,dy_1\,dy_2$ tends to $f(x_1, x_2)$ almost nowhere, as $R \in \boldsymbol{R}_2$ shrinks to (x_1, x_2).

The positive results (16) and (17) cannot be established by the usual textbook proof of Lebesgue's theorem, because the Vitali covering lemma is false if we use $\boldsymbol{R}_1$ in place of $\boldsymbol{R}_0$. However, with the aid of the maximal theorem (16) is a triviality. Since $|R|^{-1}\int_R f(y_1, y_2)\,dy_1\,dy_2 \to f(x_1, x_2)$ for f in the dense subspace $C_0^\infty \subseteq L^p$ $(1 < p < \infty)$, it is enough to prove the maximal inequality

$$\text{(A}^+\text{)}\qquad \begin{gathered}\|M^+f\|_p \leqq C_p\|f\|_p \ (1 < p < \infty) \text{ with}\\ M^+f(x_1, x_2) = \sup_{R \ni (x_1, x_2); R \in \boldsymbol{R}_1} |R|^{-1}\int_R |f(y_1, y_2)|\,dy_1\,dy_2,\end{gathered}$$

just as in the familiar case of Lebesgue's theorem. Now set

$$M_1f(x_1, x_2) = \sup_{Q \ni x_1; Q \subseteq R^1} |Q|^{-1}\int_Q |f(y_1, x_2)|\,dy_1$$

and

$$M_2f(x_1, x_2) = \sup_{Q \ni x_2; Q \subseteq R^1} |Q|^{-1}\int_Q |f(x_1, y_2)|\,dy_2.$$

The ordinary one-dimensional maximal theorem shows that M_1 and M_2 are bounded operators on L^p. On the other hand, it is trivial to show that $M^+f \leqq M_1(M_2f)$ pointwise, so that $\|M^+f\|_p \leqq \|M_1(M_2f)\|_p \leqq C_p\|M_2f\|_p \leqq C_p^2\|f\|_p$ and (A$^+$) is proved. Thus, the maximal theorem implies statement (16), the "strong differentiability" of the integral. The refined positive result (17) again follows from $M^+f \leqq M_1(M_2f)$, using a more detailed version of the maximal theorem. Limitations of space prevent adequate discussion of the negative results (18) and (19), but I want to point out that they are intimately connected with the failure of the conjectures

(B$^+$) $|\{M^+f > \alpha\}| \leqq C\|f\|_1/\alpha$, and

(A^{++}) $\|\sup_{R \ni (x_1, x_2); R \in \boldsymbol{R}_2} |R|^{-1}\int_R |f(y_1, y_2)|\,dy_1\,dy_2\|_p \leqq C_p\|f\|_p$.

In particular (19) and (A^{++}) are strongly related to the Kakeya needle problem. (See Busemann and Feller [**10**].)

Let us now try to understand why the maximal theorem is true. To simplify the discussion, I shall weaken the result slightly by restricting attention from all cubes to the special family of *dyadic* cubes. We start with the unit cube $Q_0 \subseteq R^n$, "bisect" Q_0 into 2^n subcubes of side $\frac{1}{2}$, "bisect" each of these cubes into 2^n subcubes with side $\frac{1}{4}$, "bisect" each of these cubes, etc., etc., and continue forever. The family $\mathscr{D}$ of all cubes so obtained is called the family of dyadic cubes. From now on, we shall look only at dyadic cubes—in particular we change the definition of the maximal function so that the "sup" is taken only over dyadic cubes. This restriction is not severe, for given any cube $Q \subseteq Q_0$ we can find a dyadic cube $\tilde{Q}$ of about the same size, at about the same place; so dyadic cubes are almost as "general" as arbitrary cubes. However, for dyadic cubes we have the very convenient observation

(20) Two dyadic cubes are always disjoint, unless one is contained in the other.

The easiest way to become convinced of the dyadic inequality (B) is to vent one's probabilistic intuition on the following game of chance, constructed from the set-up for the maximal theorem. Let $f \geqq 0$ be a fixed L^1 function on the unit cube Q_0. Our fortune at time $t = 0$ is $|Q_0|^{-1} \int_{Q_0} f(y)\, dy$, and we can either rest content or take a chance. If we decide to gamble, the dealer picks a cube Q_1 at random from among the 2^n dyadic subcubes of Q_0 of side $\frac{1}{2}$ (all possible Q_1's have equal probability), and our fortune at time $t = 1$ is $|Q_1|^{-1} \int_{Q_1} f(y)\, dy$. Again we may rest content or take a chance. If we again decide to gamble, the dealer picks a cube Q_2 at random from among the 2^n dyadic subcubes of Q_1 of side $\frac{1}{4}$ (all possible Q_2's have equal probability), and our fortune at time $t = 2$ is $|Q_2|^{-1} \int_{Q_2} f(y)\, dy$. The game continues in this way, either forever or until we decide to quit.

The most important feature of our game of chance is that it is absolutely fair (i.e., it is a "martingale"). More precisely, suppose we find ourselves at time $t = k$ at the cube Q_k so that our fortune is $|Q_k|^{-1} \int_{Q_k} f(y)\, dy$. If we gamble once more, we may win or lose money, but our average fortune at time $t = k + 1$ will be

$$\sum_{\substack{Q_{k+1} \subseteq Q_k; \\ \text{side}(Q_{k+1}) = 2^{-(k+1)}}} \frac{1}{2^n} \cdot \frac{1}{|Q_{k+1}|} \int_{Q_{k+1}} f(y)\, dy = \frac{1}{|Q_k|} \int_{Q_k} f(y)\, dy,$$

i.e., exactly the same as our present fortune. Thus, the game is fair.

Now consider the strategy "quit while you're ahead". We pick in advance a large number $\alpha > \int_{Q_0} f(y)\, dy$, and we stop playing the first time our fortune exceeds α—if our fortune never exceeds α, we keep playing forever. In the lucky case (one of our fortunes exceeds α), we shall have fortune at least α at the end of the game; and even in the unlucky case we shall have at least zero, since $f \geqq 0$. Therefore our average (or expected) fortune at the end of the game is at least $\alpha \times$ Probability of the lucky case $= \alpha \times$ Probability$\{\sup_k |Q_k|^{-1} \int_{Q_k} f(y)\, dy > \alpha\}$, and a few moments' thought shows that this is the same as $\alpha \cdot |\{Mf > \alpha\}|$. On the other hand, since the game is fair, our average fortune at the end of the game is merely our initial fortune $\int_{Q_0} f(y)\, dy$, no matter which clever strategy we use. Therefore, $\alpha \cdot |\{Mf > \alpha\}| \leqq \int_{Q_0} f(y)\, dy$, which is exactly the estimate (B). Part (A) of the maximal theorem follows from part (B) by a useful "interpolation" theorem which we state only in a basic special case. (For more general results, see Zygmund **[95]** and Hunt **[53]**.)

THEOREM 5 (MARCINKIEWICZ INTERPOLATION THEOREM). *Let T be a linear or sublinear operator defined on functions on some measure space, and suppose that $p_0 < p < p_1 \leqq \infty$. If T is bounded on L^{p_1}, and if the "weak-type (p_0, p_0) inequality" $|\{|Tf| > \alpha\}| \leqq C\|f\|_{p_0}^{p_0}/\alpha^{p_0}$ holds, then it follows that T is bounded on L^p.*

To deduce the maximal theorem, we take $p_0 = 1, p_1 = \infty$.

L^p-estimates for singular integrals. The techniques we have just discussed for the maximal function apply also to a wide class of singular integral operators. For simplicity, we will start with a constant-coefficient operator $T: f \to K * f$ on R^n, where K is a distribution locally integrable away from the origin. Thus, K might be x^{-1} on the line, or $\Omega(x)/|x|^n$ in R^n.

Our assumptions on K are

(21) T is bounded on $L^2(R^n)$, and

(22) $\int_{|x|>2|y|} |K(x) - K(x-y)|\, dx \leqq C < \infty$ for all $y \in R^n$.

Condition (22) is always satisfied if $|\text{grad } K(x)| \leqq C/|x|^{n+1}$, so (a) and (b) hold for all the usual singular integral operators.

THEOREM 6 (CALDERÓN-ZYGMUND INEQUALITY). *Let T be a convolution operator satisfying hypotheses* (21) *and* (22). *Then*

(A) *T is bounded on L^p* $(1 < p < \infty)$,

(B) $|\{|Tf| > \alpha\}| \leqq C\|f\|_1/\alpha$.

The proof of Theorem 6 is based on further careful study of the game of chance used to prove the maximal theorem. See Stein [**85**].

Although for simplicity we stated the Calderón-Zygmund inequality only for convolution operators, its proof applies to virtually all the variable-coefficient singular integral operators mentioned above. In particular, the following operators are bounded on L^p $(1 < p < \infty)$:

(A) A singular integral

$$Tf(x) = c(x)f(x) + \int_{R^n} \frac{\Omega(x, (x-y)/|x-y|)}{|x-y|^n} f(y)\, dy$$

with c and Ω as described above. (Actually, one can weaken considerably the assumptions on Ω.)

(B) A "classical" pseudodifferential operator

$$Tf(x) = \int_{R^n} e^{ix\cdot\xi}\sigma(x, \xi)\hat{f}(\xi)\, d\xi,$$

where $|(\partial/\partial x)^\alpha(\partial/\partial\xi)^\beta\sigma(x, \xi)| \leqq C_{\alpha\beta}|\xi|^{-|\beta|}$ for all multi-indices α, β.

(C) The commutator integrals

$$T_1f(x) = \int_{R^1} \frac{A(x) - A(y)}{(x-y)^2} f(y)\, dy, \quad T_2f(x) = \int_{R^1} \frac{(A(x) - A(y))^2}{(x-y)^3} f(y)\, dy$$

on R^1, with $A' \in L^\infty$.

(D) The Knapp-Stein singular integrals on nilpotent Lie groups. (See Korányi and Vági [**61**].)

Moreover, the Calderón-Zygmund inequality turns out to be exactly the right tool to prove the classical results of Fourier analysis on the S-function and the G-function, which we discussed briefly at the beginning of this paper in connection with complex methods. (See Stein [**81**], [**83**], J. Schwartz [**74**], Hörmander [**47**], Benedek, Calderón and Panzone [**6**].) Typical results are

(E) $\| S(f) \|_p \sim \|f\|_p \, (1 < p < \infty)$.

(F) Let $\psi_k(\xi) = \psi_0(2^{-k}\xi)$ on R^1, with ψ_0 a fixed smooth function supported in $\{\frac{1}{2} \leqq |\xi| \leqq 2\}$, chosen so that $\sum_{k=-\infty}^{\infty} |\psi_k(\xi)|^2 \equiv 1$. Define $\mathscr{G}(f)(x) = (\sum_{k=-\infty}^{\infty} |A_k f(x)|^2)^{1/2}$, where $(A_k f(\xi))^\wedge = \psi_k(\xi) \hat{f}(\xi)$. Then $\|G(f)\|_p \sim \|f\|_p$ $(1 < p < \infty)$.

(G) $\|G(f)\|_p \sim \|f\|_p \, (1 < p < \infty)$. Recall that (in effect) $G(f)(x) = (\sum_{k=-\infty}^{\infty} |B_k f(x)|^2)^{1/2}$, where $(B_k f(\xi))^\wedge = \chi_{\{2^k \leqq |\xi| < 2^{k+1}\}}(\xi) \cdot \hat{f}(\xi)$.

The main idea in proving (E), (F), (G) is to regard S, $\mathscr{G}$ and G as convolution operators mapping ordinary scalar-valued functions to functions with values in a Hilbert space, and then apply the Calderón-Zygmund inequality.

Actually, the connections between the maximal function, the Hilbert transform, and the S-function are now known to be far closer even than had been suggested by the Calderón-Zygmund inequality and its applications (A)—(G). The main ideas here were developed by Burkholder, Gundy and Silverstein **[8]**, **[9]** and Fefferman and Stein **[38]** in the context of the H^p spaces. The key to the new results is the game of chance introduced above in connection with the maximal function. We consider a fair game of chance (e.g., matching pennies) in which the gambler is allowed to vary the size of his bets depending on past history. (For example: Bet \$1.00 the first time. If you win, bet 2^{-k} dollars at time k ($k \geqq 2$); if you lose, bet 2^{+k} dollars at time k ($k \geqq 2$).) Then the following three events are equivalent, except on a set with probability zero. (See Burkholder and Gundy **[8]**.)

(a) The gambler's fortune remainds bounded as time tends to ∞.

(b) The gambler's fortune approaches a finite limit as time tends to ∞.

(c) The sum of the squares of the bets is finite.

The simplest special case is the old "three series" theorem, which says that a series $\sum_n \pm c_n$ with random $\pm$ signs converges with probability one if $\sum_n |c_n|^2 < \infty$ and diverges with probability one if $\sum_n |c_n|^2 = \infty$.

By analogy, one hopes that for an arbitrary harmonic function u on the upper half-plane (not necessarily a Poisson integral), the following conditions on a boundary point x are equivalent outside a set of measure zero:

(a′) u is nontangentially bounded at x, i.e., $\sup_{z \in \Gamma(x)} |u(z)| < \infty$.

(b′) u has a nontangential limit at x, i.e., $\lim_{z \to x;\, z \in \Gamma(x)} u(z)$ exists.

(c′) $S(u)(x) = (\iint_{\Gamma(x)} |\nabla u(z)|^2 \, dz \, d\bar{z})^{1/2} < \infty$.

See Privalov **[73]**, Marcinkiewicz and Zygmund **[65]**, and Spencer **[79]** for the case of the upper half-plane, and Calderón **[11]** and Stein **[82]** for extensions to harmonic functions of several variables. Note that since $S(u) \equiv S(v)$ for conjugate harmonic functions, the equivalence of (b′) and (c′) shows that u and v have nontangential limits at essentially the same set of boundary points. Thus, we obtain a "local" analogue of M. Riesz's theorem on the Hilbert transform.

So far, the analogy with gambling had done nothing but clarify the known results (a′) $\Leftrightarrow$ (b′) $\Leftrightarrow$ (c′) and the maximal theorem. However, further work of Burkholder, Gundy and Silverstein **[9]** and Fefferman and Stein **[38]** uses probabilistic methods in recasting the theory of H^p-spaces into a "Calderón-Zygmund" real-variable framework. Unfortunately, I have not the space here to say anything about H^p, and I must simply refer the interested reader to a relevant survey paper [1].

Up to now we have seen how singular integrals act on L^p $(1 < p < \infty)$ and on L^1. I want to close this section with a brief discussion of L^∞. Surprisingly, one can write down explicitly the essential characterizing property of the Hilbert transform of a bounded function. The basic example to keep in mind is $H(\mathrm{sgn}(x)) = (2/\pi)\log|x|^{-1}$.

THEOREM 7 (SPANNE [78], STEIN [84]). *Let $g \in L^\infty$ and let K be a convolution kernel satisfying the hypotheses of the Calderón-Zygmund inequality. Then $K * g$ is a function of bounded mean oscillation.*

A function $f \in L^1_{\mathrm{loc}}(R^n)$ is said to be of bounded mean oscillation (BMO) if it satisfies the condition

$$\sup_Q |Q|^{-1} \int_Q |f(x) - f_Q|\, dx < \infty, \quad \text{with } f_Q = |Q|^{-1} \int_Q f(y)\, dy. \tag{23}$$

Thus on R^1, $L^\infty \subseteq$ BMO, $|x|^{-\delta} \notin$ BMO, $\log|x|^{-1} \in$ BMO, but $\mathrm{sgn}(x)\log|x|^{-1} \notin$ BMO. Functions of bounded mean oscillation were introduced by John and Nirenberg [57], who proved the following result in connection with partial differential equations.

THEOREM 8. *The condition* (23) *is equivalent to the seemingly far stronger statement*

$$\sup_Q |Q|^{-1}\int_Q \exp(\lambda|f(x) - f_Q|)\, dx < \infty \quad \textit{for some } \lambda > 0. \tag{24}$$

In particular, functions of bounded mean oscillation are (locally) exponentially integrable.

The claim that (23) and (24) are the basic properties of $K * g$ with $g \in L^\infty$ is supported by the following converse result in the case of Riesz transforms:

THEOREM 9. *Every function f of bounded mean oscillation may be written in the form $f = g_0 + \sum_{j=1}^n R_j g_j$ with $g_0, g_1, \cdots, g_n \in L^\infty$.*

This is equivalent to the duality of H^1 and BMO [38]. In the one-dimensional case of the Hilbert transform H, we can say even more.

THEOREM 10. *A function $f \in L^1_{\mathrm{loc}}(R^1)$ may be written in the form $f = g_0 + Hg_1$ with $g_0 \in L^\infty$ and $\|g_1\|_\infty < 1$ if and only if* (24) *holds with $\lambda = \pi/2$.*

The proof of Theorem 10 is truly remarkable. One starts with the following question, which seemingly has nothing to do with bounded mean oscillation: Given a positive measure $d\mu = \omega(x)dx$ on R^1, is the Hilbert transform H a bounded operator on $L^p(d\mu)$? Clearly, various partial results could be proved without much trouble, but a complete solution seems too much to expect. However, at least for L^2, one has not merely one necessary and sufficient condition, but two.

THEOREM 11 (HELSON AND SZEGÖ [45]). *H is bounded on $L^2(d\mu)$ if and only if $\log \omega(x)$ may be written in the form $g_0 + Hg_1$, with $g_0 \in L^\infty$ and $\|g_1\|_\infty < \pi/2$.*

THEOREM 12 (HUNT, MUCKENHOUPT AND WHEEDEN [55]). *H is bounded on $L^p(d\mu)$ if and only if*

$$(\mathrm{A}_p) \qquad \sup_Q \left(|Q|^{-1} \int_Q \omega(x)\, dx\right)\left(|Q|^{-1} \int_Q \omega^{-1/(p-1)}\, dx\right)^{p-1} < \infty$$

holds.

The Helson-Szegö theorem is proved by a simple but ingenious application of the Hahn-Banach theorem, while the proof of the Hunt-Muckenhoupt-Wheeden theorem uses Calderón-Zygmund methods, and builds on Muckenhoupt's solution of the corresponding problem for the maximal function [68]. (See also Coifman and Fefferman [22].) Since the Helson-Szegö condition and (A_2) are necessary and sufficient conditions for the same thing, they must be equivalent. That is the proof of Theorem 10.

Various applications of BMO are presented in John [56], Moser [67], Fefferman and Stein [38], and [34].

Multiple Fourier transforms. After all the progress of Fourier analysis in the last twenty years, we still know almost nothing about the Fourier transform in R^n. We can use the techniques of singular integrals to prove theorems like the following (see [85]).

THEOREM 13 (LITTLEWOOD-PALEY THEOREM IN R^n). *Let $f \sim \sum_{k \in Z^n} a_k e^{ik\cdot x}$ be the multiple Fourier series of a function $f \in L^p([0, 2\pi]^n)$ $(1 < p < \infty)$, and let $\{S_k\}_{k \in Z^n}$ be a sequence of $\pm$ signs. Suppose that $\{S_k\}$ is constant on each parallelopiped of the form $I_1 \times I_2 \times \cdots \times I_n$, where each I_j is a dyadic block (see Theorem 1). Then $Tf \sim \sum_k S_k a_k e^{ik\cdot x}$ also belongs to L^p, and $\|Tf\|_p \leqq C_p \|f\|_p$.*

But in many respects, R^n is fundamentally different from R^1, so that merely proving R^n analogues of R^1-theorems misses a great deal. For example, given $f \in L^p(R^n)$ with $1 < p < 2$, what can we say about the size of the Fourier transform $\hat{f}$? The familiar Hausdorff-Young theorem $\|\hat{f}\|_{p'} \leqq \|f\|_p$ $(1/p' + 1/p = 1)$ is virtually all we can say in R^1.[7] (There are further results, but they are in the nature of refinements.) Already in R^2, however, we can go much further. Here is an elementary "restriction theorem" to drive home the point.

THEOREM 14 [29]. *For $f \in L^p(R^2) \cap L^1(R^2)$ $(1 \leqq p < 4/3)$ we have a priori inequality*

$$(25) \qquad \|\hat{f}\|_{L^1(S^1)} \leqq C_p \|f\|_{L^p(R^2)}$$

where S^1 denotes the unit circle.

It follows that $\hat{f}|_{S^1}$ is well defined for $f \in L^p$ $(p < 4/3)$ even though in principle the Fourier transform is defined only up to sets of measure zero.[8] The correspond-

[7]However, recent work of Babenko and Beckner shows that the norm of the Fourier transform as an operator from L^p to $L^{p'}$ is strictly less than one and can be computed. See Stein's lecture in [1].

[8]Actually, the sharp estimate is $\|\hat{f}\|_{L^{p'/3}(S^1)} \leqq C_p \|f\|_{L^p(R^2)}$ for $p < 4/3$. The example $f = \hat{\chi}_B$ with $B =$ unit disc ($f \in L^p$ for $p > 4/3$) shows that we cannot expect to define $\hat{f}|_{S^1}$ for $f \in L^p$ $(p > 4/3)$.

ing statement for a straight line (replacing S^1) is utter nonsense. The first theorem of this kind is due to Stein (see [29]).

The proof of the restriction theorem takes only a paragraph. We have to show that the operator $T : f \to \hat{f}|_{S^1}$ is bounded from $L^p(R^2)$ to $L^1(S^1)$; to do so, we prove that the adjoint T^* maps $L^\infty(S^1)$ to $L^{p'}(R^2)$ for $p' > 4$. This comes down to showing that

$$\|(f d\theta)^\wedge\|^2_{L^{p'}(R^2)} \in C_p \|f\|_{L^\infty(S^1)},$$

where $d\theta$ denotes uniform measure on the circle. Now we write

$$\begin{aligned}\|(f d\theta)^\wedge\|^2_{L^{p'}(R^2)} &= \|((f d\theta)^\wedge)^2\|_{L^{p'/2}(R^2)} \\ &= \|((f d\theta) * (f d\theta))^\wedge\|_{L^{p'/2}(R)^2} \leqq \|(f d\theta) * (f d\theta)\|_{L^r(R^2)}\end{aligned}$$

with $1/r + 1/(p'/2) = 1$ (the last step follows from Hausdorff-Young, since $1 \leqq r < 2$ for $p' > 4$), and the obvious pointwise inequality $|(f d\theta) * (f d\theta)| \leqq \|f\|^2_\infty \cdot (d\theta * d\theta)$ yields $\|f d\theta\|^2_{L^{p'}(R^2)} \leqq \|f\|^2_{L^\infty(S^1)} \cdot \|(d\theta * d\theta)\|_{L^r(R^2)}$. Thus, our restriction theorem comes down to checking that $d\theta * d\theta \in L^r(R^2)$ for $r < 2$. We omit the details, but we note that it is here that the difference between circles and straight lines shows up in the proof. A closely related idea appears in Zygmund [97].

In some ways, the Fourier transform is more intractable in R^n than in R^1. For instance, for many problems on partial sums of multiple Fourier series, the natural analogue of the Hilbert transform is an operator T_0 defined on $L^2(R^n)$ by $(T_0 f)^\wedge(\xi) = \chi_B(\xi)\hat{f}(\xi)$, where χ_B is the characteristic function of the unit ball. T_0 behaves far worse than the usual singular integrals, for its convolution kernel looks like $e^{i|x|}/x^{(n+1)/2}$ at infinity, compared to which $\Omega(x)/|x|^n$ is very tame. As a "Hilbert transform", T_0 is intimately connected to a certain maximal function, but it is not the usual maximal function. Rather (in R^2, say) the right maximal function is $M_2 f(x) = \sup_{R \ni x} |R|^{-1} \int_R |f(y)|\, dy$, where R is a rectangle of arbitrary size, shape, and direction. We have already noted that M_2 is not bounded on L^p $(p < \infty)$, by virtue of the Besicovitch-Perron constructions for the Kakeya needle problem, and consequently T_0 is unbounded on L^p $(p \neq 2)$. (See [32], [46].) Thus, a basic analogue of the Hilbert transform is a "bad" operator, and so, in dealing with multiple Fourier series, we expect trouble.

This is not to imply that nothing positive can be said about T_0. We define the Bochner-Riesz operators T_δ $(\delta > 0)$ on L^2 by

$$(T_\delta f)^\wedge(\xi) = (1 - |\xi|^2)^\delta \chi_B(\xi)\hat{f}(\xi);$$

T_δ is related to T_0 just as Cesaró summation of Fourier series on $[0, 2\pi]$ is related to ordinary convergence (see Bochner [7]). By analogy between the Bochner-Riesz operators and restriction theorems on Fourier transforms, Carleson and Sjölin [21] proved the following result in the two-dimensional case. (See also [35] and Hörmander [51].)

THEOREM 15. *T_δ $(\delta > 0)$ is bounded on $L^p(R^2)$ for* $4/3 \leqq p \leqq 4$.

The result is essentially sharp (Herz [46]).

A. Cordoba **[23]** has recently shown that the Carleson-Sjölin theorem can be related to a positive result for a maximal function closely connected to M_2. In fact, setting $M^N f(x) = \sup_{R \ni x} |R|^{-1} \int_R |f(y)|\, dy$ where R is a rectangle of arbitrary direction with (Longer side of R)/(Shorter side of R) $< N$, we have:

THEOREM 16 (CORDOBA MAXIMAL THEOREM). $\|M^N f\|_2 \leqq C(\log N)^3 \|f\|_2$.

The three basic Theorems 14, 15 and 16 suggest a program to force us to come to grips with some genuinely n-dimensional Fourier analysis. First of all, the known results should be extended from the two-dimensional case (where they are really too easy) to R^n. The natural conjectures are

(26) $\|\hat{f}\|_{L^{2n/(n+1)}(S^{n-1})} \leqq C_p \|f\|_{L^p(R^n)}$ if $1 \leqq p < 2n/(n+1)$.

(27) T_δ is bounded on $L^p(R^n)$ if $|1/p - 1/2| < (\delta + \frac{1}{2})/n$ and $\delta > 0$.

(28) Let $M^N f(x) = \sup_{R \ni x} |R|^{-1} \int_R |f(y)|\, dy$ where R is any rectangular parallelopiped of arbitrary direction, and sides $\delta_1 \times \delta_1 \times \cdots \times \delta_1 \times \delta_2$ with $1 \leqq \delta_2/\delta_1 \leqq N$. Then

$$\|M^N f\|_{L^2(R^n)} \leqq C(\log N)^A \|f\|_{L^2(R^n)}.$$

So far, the best partial result known is a clever theorem of P. Tomas **[91]**:

THEOREM 17. *The following inequalities hold.*

$$\|\hat{f}\|_{L^2(S^{n-1})} \leqq C\|f\|_{L^{[(2n+2)/(n+3)]^{-1}}(R^n)}, \tag{29}$$

and

$$\|T_\delta f\|_{L^p(R^n)} \leqq C\|f\|_{L^p(R^n)} \tag{30}$$

for $|1/p - 1/2| < (\delta + \frac{1}{2})/n$ *and* $\delta > (n-1)/(2n+2) + \varepsilon$ (*cf.* **[29]** *and* **[35]**).

See Carleson and Sjölin **[21]** for the three-dimensional case.[9]

It seems that we are still far from complete solutions. Even after our conjectures have been settled, we shall only have barely started to grasp the real situation. It is as if we had just proved Cesaró summability of Fourier series on $[0, 2\pi]$ but still knew nothing about the Hilbert transform. One natural problem is to relate the geometry of the maximal function M_2 to the behavior of the "Hilbert transform" T_0 in R^n.[10] The only result known in this direction is Cordoba's Theorem 16. We still know so little that we cannot answer intelligently the question "How big is the Fourier transform of a function in $L^p(R^2)$?" Perhaps $\{|\hat{f}| > \alpha\}$ for large α can be covered efficiently by rectangles (of no fixed direction). If true, this would explain why $\hat{f}$ can be restricted to circles but not to straight lines, for a circle is harder to cover by thin rectangles than a straight line. Coverings by rectangles play a major role in the study of T_0, where the "Kakeya" set of Besicovitch exerts an influence all out of proportion to its small area. A recent counterexample of Carleson **[20]** to various conjectures on the polydisc related to Theorems 9 and 10 has a similar flavor. Perhaps in dealing with the Fourier transform in R^n, we must abandon our fixation on Lebesgue measure, and search for new quantities (defined

[9] E. M. Stein has modified Tomas' argument to handle $\varepsilon = 0$ in (29) and (30).

[10] There is also an analogue of the S-function for T_0, which we have not mentioned.

possibly in terms of coverings by thin rectangles) to express the size or importance of a set of points. This is easier said than done, but we have seen evidence suggesting that it is forced on us by the phenomena we seek to understand. I do not know where—if anywhere—these ideas lead.

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