HOMOLOGY AND HOMOTOPY THEORY

HOMOTOPY AND HOMOLOGY

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The concept of homotopy is a mathematical formulation of the intuitive idea of a continuous transition between two geometrical configurations. The concept of homology gives a mathematical precision to the intuitive idea of a curve bounding an "area" or a surface bounding a "volume."

1. The first step toward connecting these two basic concepts of topology was taken by L. E. J. Brouwer in 1912 by demonstrating that two continuous mappings of a two-dimensional sphere into itself can be continuously deformed into each other if and only if they have the same degree (that is, if they are equivalent from the point of view of homology theory).

After having generalized Brouwer's result to an arbitrary number of dimensions, H. Hopf undertook a systematic study of the problem of classifying the continuous mappings of a polytope P into a polytope Q. Each mapping f induces homomorphisms of homology groups of P into the corresponding groups of Q. Two mappings f and g are said to belong to the same *homology class* if they induce identical homomorphisms of homology groups (for all dimensions and all coefficient domains). The mappings f and g are said to belong to the same *homotopy class* if they can be embedded into a common one-parameter continuous family of mappings. The homotopy class of a mapping determines its homology class, but not conversely, as shown by the example of the mappings of the sphere S_3 into S_2 which all belong to the same homology class although there is an infinite number of homotopy classes. The question arises: under what special conditions the homotopy classification of the mappings of P into Q coincides with their homology classification. The classical result of Hopf states that this is the case if P is a polytope of dimension n and Q the n-dimensional sphere S_n. Using cohomology groups instead of homology groups, H. Whitney gave the following elegant formulation to Hopf's theorem. Homotopy classes of mappings of an n-dimensional polytope P into the sphere S_n are in one to one correspondence with the elements of the n-dimensional cohomology group of P with integers as coefficients.

2. In 1934–1935 the author developed the concept and theory of higher dimensional homotopy groups. Given an arcwise connected topological space Y, the n-dimensional homotopy group $\pi_n(Y)$ is defined as follows: Let an arbitrary point $y_0 \in Y$ be singled out once and for all as the "reference point," and let also a fixed point x_0 be selected on the fixed n-sphere S_n. An element of $\pi_n(Y)$ is determined by a continuous mapping of S_n into Y, satisfying the condition $(x_0) = y_0$. Two mappings f and g determine the same element of $\pi_n(Y)$ if and only if they can be continuously deformed into each other, in such a fashion that the image of x_0 remains at y_0 during the entire process of deformation (this condition can be dispensed with if Y is simply connected). In this case every continuous mapping of S_n into Y determines uniquely an element of $\pi_n(Y)$. The group composition law is defined in a fairly obvious way by identifying two n-spheres tangent at x_0 with the two hemispheres of a single sphere whose equator has been shrunk into x_0 .

An alternative way to introduce homotopy groups is to consider the topological space F of all continuous mappings of S_{n-1} into Y satisfying the condition $f(x_0) = y_0$, where x_0 is a fixed point of S_{n-1} . Although this functional space is, generally speaking, disconnected, it can be shown that its arcwise connected components have the same homotopy type (see below) and consequently have isomorphic fundamental groups. We can thus speak about the fundamental group of the functional space F, and this group turns out to be isomorphic to the n-dimensional homotopy group $\pi_n(Y)$ in the sense of the previous definition. The group $\pi_1(Y)$ is, of course, the fundamental group of Y. A simple geometric argument shows that for $n > 1$ the groups $\pi_n(Y)$ are *abelian*. In contrast with homology groups the homotopy groups of an n-dimensional space may be non-trival even in dimensions higher than n. For instance, $\pi_3(S_2)$ is an infinite cyclic group (Hopf's theorem) and for $n > 2$ the group $\pi_{n+1}(S_n)$ is of order 2 (Freudenthal-Pontrjagin theorem).

To determine the homotopy groups of a given space is, generally speaking, an extremely difficult problem (even for finite polytopes) which so far has been solved only in a few special cases. In this respect there is a significant difference between homotopy and homology. When a polytope P is broken up in two sub-polytopes Q and R, there is a relatively simple relation (Meyer Vietoris theorem restated recently in terms of the so-called exact sequences) between the homology invariants of the polytopes P, Q, R and the intersection $Q \cap R$. No analogous relation exists for homotopy groups. This is tied up to the fact that a continuous image of the n-sphere in P cannot be decomposed into "small" spherical images, the way a simplicial chain can be decomposed into "small parts." Therefore the basic process of homology theory consisting in decomposing a space into smaller pieces with simpler homology structure has no counterpart in homotopy theory. The difficulty is illustrated by the fact that even in the case of a space P represented as the union of two subspaces Q and R with only one point in common, there is no simple relation between higher dimensional homotopy groups of P, Q, and R.

3. In certain "elementary" cases, homotopy groups can be reduced to homology groups. Let Y be an arcwise connected space and let $H_n(Y)$ be the n-dimensional homology group of Y based on singular chains, with integers as coefficients. A continuous image of S_n in Y can be regarded as a singular n-cycle. Since two homotopic spherical images determine homologous singular cycles, one obtains a "natural" homomorphism of $\pi_n(Y)$ into $H_n(Y)$. The fundamental *equivalence theorem* states:

If $n \geqq 2$ and the homotopy groups $\pi_i(Y)$ are trivial for $i < n$, the nth homotopy group $\pi_n(Y)$ is isomorphic to the nth homology group $H_n(Y)$ under the natural homomorphism.

For example $\pi_i(S_n)$ is trivial for $i < n$ and hence $\pi_n(S_n)$ is infinite cyclic. Arcwise connected spaces whose homotopy groups in dimensions less than or equal to n vanish are called *n-connected.* This property is equivalent to the condition that every continuous image of an arbitrary n-dimensional polytope in Y be homotopic to a single point. An obvious corollary to the equivalence theorem states that Y is n-connected if and only if the groups $\pi_1(Y)$, $H_2(Y)$, $\cdots$, $H_n(Y)$ are trivial. It follows that a polytope can be shrunk to a point in itself if and only if it is simply connected (= 1-connected) and has vanishing homology groups in all dimensions.

4. The equivalence theorem just stated can be formulated in the following way: If the arcwise connected space Y is $(n - 1)$-connected $(n \geqq 2)$, the homotopy classes of mappings of S_n into Y coincide with their homology classes. Comparing this result to Hopf's theorem mentioned above we find that the assertions in both theorems are of the same type. Hopf's theorem and the equivalence theorem are both contained in the following more general theorem:

If Y is an $(n - 1)$-connected space $(n \geqq 2)$ and P an n-dimensional polytope, the homotopy classification of P into Y agrees with their homology classification. More refined results in this direction can be obtained by using the concept of a *homotopy obstruction* developed by S. Eilenberg (implicitly this idea was used for the first time by H. Whitney in his revealing proof of Hopf's theorem). Let Y be a 1-connected space and P an arbitrary polytope. Let us denote by P^m the m-dimensional skeleton of P, that is, the union of all simplexes of P of dimensions less than or equal to m. Consider now two continuous mappings f and g of P into Y. An attempt to deform f continuously into g can be carried out stepwise, each step involving considerations in one dimension only. Suppose we have succeeded in deforming f into a mapping f' which agrees with g on the $(m - 1)$-dimensional skeleton P^{m-1}. For each oriented simplex σ^m of P the images $f'(\sigma^m)$ and $g(\sigma^m)$ (which coincide on the boundary of σ^m) yield, in an obvious fashion, a continuous image of an m-sphere. Let us denote by $\varphi(\sigma^m)$ the element of the homotopy group $\pi_m = \pi_m(Y)$ determined by this spherical image. The function φ can be regarded as an m-dimensional cochain of P with coefficients in the group π_m. This cochain turns out to be a cocycle. Its cohomology class is called the homotopy obstruction for the couple (f', g). The notation is justified by the following theorem: *If the obstruction is zero (that is, if the cocycle φ is cobounding), the deformation process can be pushed one step further so as to deform f' into a mapping f'' which agrees with g on the m-dimensional skeleton P^m. Moreover the deformation can be carried out in such a way that the image of P^{m-2} (but not necessarily of P^{m-1}) remains unchanged.*

If the cohomology group $H(P, \pi_m)$ of P with coefficients in π_m is trivial (this will be the case, for instance, if $\pi_m = \pi_m(Y)$ vanishes), all obstructions in dimension m are zero, and hence any two mappings which coincide on P^{m-1} are homotopic on P^m.

The author derived further results by connecting homotopy obstructions with so called *homology obstructions*. Let $H_m = H_m(Y)$ be the mth homology group of Y with integers as coefficients, in the sense of the singular homology theory, and let $H^m(P, H_m)$ be the mth cohomology group of P with coefficients in H_m. The natural homomorphism of π_m into H_m yields a homomorphism of $H(P, \pi_m)$ into $H(P, H_m)$. Under this homomorphism the homotopy obstruction of the couple (f', g) is sent into an element of $H(P, H_m)$ which is called the homology obstruction of the couple (f', g). The homology obstruction is zero if the mappings f' and g, or—what amounts to the same thing—the mappings f and g belong to the same homology class.

Under certain conditions, homotopy obstructions coincide with homology obstructions. This is, for instance, the case if $\pi_m(Y)$ is isomorphic to $H_m(Y)$ under the natural homomorphism. Under such circumstances the homotopy problem in dimension m is completely reducible to the corresponding homology problem.

5. The groups $\pi_n(Y)$ are a special case of more general invariants called *relative homotopy groups*, which are in many respects analogous to relative homology groups.

Let Y be a topological space and Z a subset of Y. Both Y and Z are assumed to be arcwise connected. For every integer $n \geqq 2$ we shall define the relative homotopy group $\pi_n(Y, Z)$. Let E_n be a fixed n-cell with the boundary S_{n-1}. Let us select a point x_0 of S_{n-1} and a point z_0 of Z. An element of $\pi_n(Y, Z)$ is determined by a continuous mapping of E_n into Y satisfying the boundary conditions

$$f(S_{n-1}) \subset Z, \qquad f(x_0) = z_0.$$

Two mappings determine the same element of $\pi_n(Y, Z)$ if they can be continuously deformed into each other in such a way that the boundary conditions are satisfied during the entire process of deformation. The composition law is defined by partitioning an n-cell into two cells with an $(n - 1)$-cell in common and shrinking this $(n - 1)$-cell into a single point x_0. It is evident that no reasonable composition can be defined when $n = 1$. An alternative definition describes relative homotopy groups as fundamental groups of suitably defined functional spaces.

For $n > 2$, $\pi_n(Y, Z)$ is abelian. The group $\pi_2(Y, Z)$ is in general nonabelian, and this accounts for some of the peculiar difficulties encountered in the homotopy theory of two-dimensional spaces.

In exactly the same way as in the case of absolute homotopy groups, one defines a natural homomorphism of the relative homotopy group $\pi_n(Y, Z)$ into the relative homology group $H_n(Y, Z)$ (with integer coefficients). We shall call the couple (Y, Z) *n-connected* ($n \geqq 2$) if (a) the group $\pi_1(Z)$ is isomorphic to $\pi_1(Y)$ under the natural homomorphism and (b) $\pi_m(Y, Z)$ vanishes for

$2 \leqq m \leqq n$. Without using homotopy groups, the definition can be formulated as follows: the couple (Y, Z) is n-connected if given any n-dimensional polytope P with a subpolytope Q and any continuous mapping f of P into Y satisfying $f(Q) \subset Z$, f can be deformed, without changing the image $f(Q)$, into a mapping g satisfying $g(P) \subset Z$. In analogy with the equivalence theorem for absolute homotopy groups we have:

If the couple (Y, Z) is $(n - 1)$-connected, the homotopy group $\pi_n(Y, Z)$ is isomorphic to the homology group $H_n(Y, Z)$ under the natural homomorphism.

Relative homotopy groups play a basic role in the study of fibre spaces and fibre bundles.

An important generalization of relative homotopy groups has been developed recently by A. L. Blakers and W. S. Massey. They define homotopy groups of a so-called "triad," that is, of a space Y supplied with two closed sets U and V whose union is Y. Roughly speaking, the elements of the n-dimensional homotopy group of a triad are defined by mappings of an n-dimensional cell into Y such that one of the two hemispheres of the boundary of the cell is mapped into U and the other one into V. The theory of homotopy triads helps greatly to understand Freudenthal's so-called "suspension homomorphism" which is the basic tool in the discussion of homotopy groups of spheres.

6. The problem of classifying mappings of one space into another space is closely related to the problem of classifying spaces themselves according to their homotopy properties. Two spaces X and Y are said to have the same *homotopy type* if there exists a continuous mapping f of X into Y and a continuous mapping g of Y into X such that the combined mappings $f \circ g$ and $g \circ f$ are homotopic to identities. Two spaces which have the same homotopy types have isomorphic cohomology rings and isomorphic homotopy groups in all dimensions. As has been shown recently by J. H. C. Whitehead, a necessary and sufficient condition for X and Y to have the same homotopy type is the existence of a continuous mapping f of X into Y which induces isomorphic mappings of the fundamental group and the homology groups of X into the corresponding groups of Y.

J. H. C. Whitehead has succeeded in completely describing the homotopy types of simply connected four-dimensional polytopes in terms of their homology invariants. This description involves in addition to cohomology rings the so-called "Pontrjagin squares."

7. So far we have been concerned mainly with the problem of reducing homotopy properties of mappings and spaces to their homology properties. In certain cases, however, one is led to the converse problem of obtaining information about homology properties of a space from its known homotopy properties. A typical example is an *aspherical space*. By this is meant a space whose homotopy groups vanish in all dimensions $n \geqq 2$. It is known that the homotopy type and hence all homology invariants of an aspherical space are determined by its fundamental group. An analogous result holds for spaces which have only one nonvanishing homotopy group. The algebraical process by which in cases of this type the homology invariants of the space are determined by its homotopy groups has been studied extensively by Eilenberg-MacLane. Their research resulted in a fruitful theory of homology invariants associated with abstract groups. This theory has interesting applications in algebra and in the theory of Lie groups.

8. At present the main effort in homotopy theory seems to concentrate on the problem of determining homotopy groups of spheres. The tools used in this research are predominantly of algebraical nature, like generalized Hopf invariants studied by G. Whitehead, or "cup products" introduced by N. Steenrod. Important advances have been made, most significant of which is the result established recently by G. Whitehead and Pontrjagin, to the effect that $\pi_{n+2}(S_n)$ is a group of order 2 for $n \geqq 3$. Nevertheless our knowledge of homotopy groups of spheres remains meager.

Perhaps the present trend of research does not put enough emphasis on tools that could be provided by the geometrical structure from the point of view of differential geometry, like properties of geodesic lines, study of critical points, etc. Recent work of E. Pitcher seems to indicate that some progress can be expected from this direction.

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