jectories nor the deformations would have been adequately defined. This is shown by an example due to Whitney.

Whitney's example. Whitney has shown the existence of a function F defined and continuous on a unit square in the (x, y)-plane, with F_x and F_y continuous, yet with F possessing a connected set X of critical points whose critical values *fill an interval.* One might say that the mission of the critical point theory was already fulfilled in the Whitney example since the existence of infinitely many critical points is granted. However, this view would be mistaken since the Morse theory also aims at the relations between the critical sets σ (each at one f-level and closed at that level) classified according to the nature of associated local relative homology groups. See Morse [2; 3]. We shall term a function f whose critical values fill some interval a function of Whitney type.

Limitations imposed by the use of singular cycles. The Morse theory aims to associate a critical set σ with each nontrivial homology class H on M (or Ω). Thus for a cycle $z \in H$, let $|z| \subset M$ be a compact carrier (always minimal) of z and set

$$c(H) = \inf_{z \in H} [\sup_{x \in |z|} f(x)] \tag{1.1}$$

$$N_\sigma = \sigma \cup x \mid (f(x) < f(\sigma)). \tag{1.2}$$

We say that H "*causes*" σ if $c(H) = f(\sigma)$, if $N_\sigma \supset z$ for some $z \in H$, and if no proper closed subset of σ has this property. If singular cycles (rather than Vietoris cycles) are used, an homology class H need cause no critical set when M and f are not analytic, even when N is a compact differentiable manifold of class $C^{(n)}$ and f of class $C^{(n)}$ on M, with n arbitrarily large. This is shown in Morse [5] and is not equivalent to the well-known fact that singular homology groups are not always isomorphic to the corresponding Vietoris homology groups.

A theorem on dim σ. Suppose that H_n and H_r are different homology classes of cycles of dimension n and r respectively. If $c(H_n) = c(H_r)$, simple examples will show that H_n and H_r may cause a common critical set σ consisting of just one point. If, however, $n > r$, there are simple conditions sufficient that the critical set σ caused by H_n be such that

$$\dim \sigma \geqq n - r. \tag{1.3}$$

In the special case in which the critical values of f are *isolated and f is of class* C'', such conditions have been given by Lusternik and Schnirelmann. If, however, f is a function of the Whitney type or if the orthogonal trajectories of the level manifolds of f fail to have the usual field properties, then the earlier proofs of (1.3) are not applicable. However, a general theorem with (1.3) as a conclusion can be stated in terms which are purely topological and proved without reference to the category theory. One may assume that the compact manifold M is merely locally euclidean (not in general differentiable) and that the function f is merely continuous. One uses a purely topological definition of a critical point of f. Such a theorem is stated for the first time in §5.

Multiple integrals. Prior to 1936 the theory had not been applied to obtain any general theorems on the existence of unstable (non-minimizing) extremals of multiple integrals J. The reasons for this are clear. There is as yet no workable generalization for J of the orthogonal trajectories of the level manifolds of f. In addition, the theorem on the existence of minimizing extremals in the small for positive definite regular simple integrals has no apparent counterpart for multiple integrals. Finally, it turned out that the concept of lower semi-continuity of a positive definite functional J, while adequate in the theory of the absolute minimum (if accompanied by conditions implying compactness of the set of admissible elements), requires the addition of the concept of *upper reducibility* of J (defined in §2) if unstable extremals are sought. The general grounds preparing for an attack on multiple integral problems were laid in Morse [2].

Minimal surfaces. Using these general concepts Morse and Tompkins, and Shiffman, independently and at essentially the same time, proved the existence of unstable minimal surfaces of disc type spanning a simple closed curve g. The conditions initially imposed on g were somewhat heavier than rectifiability. These conditions have been progressively reduced until Shiffman has established the existence of a minimal surface of minimax type for a rectifiable g. Morse has verified Shiffman's result by an independent proof not yet published. Some of the results of Morse and Tompkins cover more general topological aspects and have not yet been reduced to the hypothesis of rectifiability.

To escape the limitations of the earlier development the critical point theory has advanced at three levels in

(a) *the general causal theory* Morse [2],
(b) *the span theory* Morse [3],
(c) *the nondegenerate theory* Morse [4].

These three levels are distinguished by their objectives and hypotheses. They are all concerned with a positive lower semi-continuous function F on an abstract metric space S. We further distinguish these theories as follows.

2. The causal theory Morse [2]. This theory imposes minimum conditions on F and S and is concerned with critical sets as *caused* by homology or homotopy classes of various types. Extreme types of deformations are the isotopies (deformations in which the images at any one time are topological) and the F-deformations which require no derivatives of F for their definition. See §5. The function F is assumed positive and lower semi-continuous over the metric space S.

As distinguished from the span theory and nondegenerate theory, the causal theory does not aim at a complete set of *relations* between the classified critical sets. Its hypotheses are too general and the critical sets too numerous and complex in most problems to make a theory of relations feasible. On the other hand Whitney functions can be treated under this theory. The causal homology theory imposes two additional conditions on the function F and metric space S:

(i) *the F-accessibility of S*;

(ii) *the upper-reducibility of F*.

We shall define these conditions.

The set of points $x \in S$ on which $F(x) \leqq c < \infty$ will be denoted by S_c. The space S is termed *F-accessible* if any nonbounding Vietoris k-cycle z, given as homologous to zero mod S_{c+e} for each $e > 0$, is homologous to a k-cycle in S_c. If the subsets S_c are compact for each c, S can be shown to be F-accessible, making use of the lower semi-continuity of F.

A continuous deformation of a subset $A \subset S$ which replaces each point $x \in A$ by a point $x^t \in S$ at the time t is called an *F-deformation* of A if for each t $(0 \leqq t \leqq 1)$ and $x \in A$

$$F(x) - F(x^t) \geqq 0. \tag{2.1}$$

This deformation is termed *proper* over A if the difference (2.1) is bounded from zero whenever the distance $d(x, x^t)$ is bounded from zero. The integrals of variational theory are ordinarily lower semi-continuous but not upper semi-continuous. Upper reducibility in some form is, however, satisfied in general and serves in place of upper semi-continuity.

Let p be a point of S with $F(p) < \infty$. We term F *upper reducible* at p if for any set S_b, with $b > F(p)$, there exists an F-deformation D of a neighborhood N_b of p relative to S_b such that

$$\limsup_{(t,x)\to(1,p)} F(x^t) \leqq F(p) \qquad (x \in N_b, 0 \leqq t \leqq 1) \tag{2.2}$$

and such that D is a *proper* F-deformation of any subset of N_b on which $F(x)$ exceeds $F(p)$ by a positive constant. *Note that the deletion of t in* (2.2) *yields the definition of upper semi-continuity of F at p.* It is easy to show that lower semi-continuity and upper reducibility are independent conditions on F.

To state the principal theorem one must define a homotopic critical point. A point p of S at which F is finite will be called *homotopically ordinary* if some neighborhood of p, relative to some S_b with $b > F(p)$, admits a proper F-deformation $(0 \leqq t \leqq 1)$ which ultimately (for some t) displaces p. The point p will be termed *homotopically critical* if not homotopically ordinary.

THEOREM 2.1. *Suppose that F is positive, lower semi-continuous and upper reducible on an F-accessible metric space S. Let H be a nontrivial homology class of Vietoris cycles on S. If there is a k-cycle $z \in H$ on some set S_a with finite a, there is a least value of a such that there is a k-cycle of H on S_a. If c is this minimum value of a, there is at least one homotopic critical point p with $F(p) = c$.*

In establishing the existence of unstable minimal surfaces it was fundamental to show that the Douglas-Dirichlet integral

$$\iint_D \left[\sum_i \left(\frac{\partial x_i}{\partial u}\right)^2 + \left(\frac{\partial x_i}{\partial v}\right)^2\right] du\, dv \qquad (i = 1, 2, 3)$$

taken over the circular disc D was upper reducible, admitting harmonic surfaces $x_i(u, v)$, $i = 1, 2, 3$, spanning the given simple closed curve g. A modification of the above theorem was then used in which the homology class H was replaced by a suitable relative homology class. See Morse-Tompkins or Shiffman.

3. The span theory Morse [3]. In seeking the *totality* of relations between the critical values clarified by means of the associated groups of "caps," one alternative is to turn to nondegenerate functions (see §4) for which the critical points are isolated. Another alternative is to classify "caps" according to their "*spans*" e. It turns out very remarkably that on employing only caps of given fixed span greater than e, a consistent topological theory of critical values results which behaves formally as if F were a real analytic function for which the number of critical values $< c < 1$ is finite. An infinitely complex problem is thus reduced to an essentially finite problem.

It is convenient to suppose that $0 \leqq F \leqq 1$. If this were not the case, the functions

$$\frac{F}{1 + F}$$

could be used in place of F. Certain new terms needed here must be defined.

Let $F(p) = c < \infty$. The space S will be said to be *locally F-connected* of order r at p if corresponding to each positive constant e there exists a positive constant δ such that each singular r-sphere on the δ-neighborhood of p and on $S_{c+\delta}$ bounds an $(r + 1)$-cell of diameter at most e on S_{c+e}. We say that S is *F-reducible* at $c = 1$ if corresponding to any compact subset A of S there exists an F-deformation D^A of A into some subset S_c of S for which $c < 1$. The principal hypotheses in the span theory are then as follows.

(i) *The function F is positive and lower semi-continuous on S.*

(ii) *The sets S_c are compact for each $c < 1$, S is F-reducible at $c = 1$ and locally F-connected of all orders at each point x at which $F(x) < 1$.*

It remains to define *cap-spans*. Given a with $0 \leqq a < 1$, we say that a set A (for example, the compact carrier of a Vietoris cycle) lies definitely in S_a (written d-on S_a) if $A \subset S_{a-e}$ for some $e > 0$. The phrase d-mod S_a shall mean mod S_{a-e} for some $e > 0$.

k-cap-spans. Let u be a relative Vietoris k-cycle d-mod S_a, with a carrier $|u|$ in S_a. If $u \not\sim 0$ on S_a, d-mod S_a, u is called a *k-cap* with *cap-height* $a(u)$. The boundary βu of such a k-cap is d-on $S_{a(u)}$. The cycle u is termed *linkable* or *nonlinkable* according as $\beta u \sim 0$ or $\beta u \not\sim 0$, d-on $S_{a(u)}$. If u is linkable, set $\sigma(u) = \sup b$ for all b such that

$$u \not\sim 0 \qquad [\text{on } S_b,\ d\text{-mod } S_{a(u)}] \tag{3.1}$$

and set

$$\text{span } u = \sigma(u) - a(u) \geqq 0. \tag{3.2}$$

If u is nonlinkable, set $\tau(u) = \inf b$ for all b such that

$$\beta u \sim 0 \qquad [d\text{-on } S_{a(u)},\ d\text{-mod } S_b] \tag{3.3}$$

and set

$$\text{span } u = a(u) - \tau(u) > 0. \tag{3.4}$$

Corresponding to each nontrivial homology class H there is a k-cycle $z \in H$ which is also a k-cap and for which the cap-height $a(z)$ is a minimum for all k-cycle-caps $\in H$. This k-cycle z will be termed *canonical.* Morse [3].

Recall that the group of coefficients which we are using is a field G. We shall be concerned with classes A of k-caps or k-cycles such that the conditions $u \in A$, $\delta \in G$, and $\delta \neq 0$, imply $\delta u \in A$. A "*maximal group*" of elements in A is a group B every element of which, except the null element, is in A, while B is a proper subgroup of no other such group of elements in A. With this understood, let $e > 0$ be given and fixed. We introduce maximal groups

M_k^e of k-caps with span greater than e,

N_k^e of nonlinkable k-caps with span greater than e,

P_k of canonical k-cycles.

If A stands for any of these three defining properties, it is a theorem in Morse [3] that any two maximal groups with property A are isomorphic, with corresponding elements u and u' such that $u - u'$ does not have property A. It is also shown in Morse [3] that the group N_k^e is isomorphic with βN_k^e where $u \in N_k^e$ corresponds to βu. It is remarkable that dim N_k^e is finite, and that P_k is also a maximal group of nonbounding k-cycles. We have the following fundamental theorem. Morse [3, Corollary 12.2].

THEOREM 3.1. *The maximal groups N_k^e can be so chosen that the direct sum*

$$N_k^e + \beta N_{k+1}^e + P_k \qquad (k = 0, 1, \cdots) \tag{3.3}$$

is a maximal group of k-caps with span greater than e.

It is easy to show that the cap-heights $a(u)$ of k-caps with span greater than e have at most the cluster point $a = 1$. Moreover a maximal group of k-caps with cap-height a and span greater than e always has a *finite* dimension. A maximal group of k-caps with span greater than e is seen to be the direct sum of maximal groups of such k-caps with the respective cap-heights. On setting

$$\dim M_k^e = m_k^e, \qquad \dim N_k^e = n_k^e, \qquad \dim P_k = p_k$$

we have the following corollary.

COROLLARY. $m_k^e - p_k = n_k^e + n_{k+1}^e \quad (k = 0, 1, \cdots).$

The numbers p_k are of course the connectivities of S. One has the relations $m_k^e \geqq p_k$, and if one sets $E_k = m_k^e - p_k$ whenever $p_k < \infty$, and defines E_k as the right member in the corollary when $p_k = \infty$, the members E_k satisfy the infinite set of inequalities

$$E_n - E_{n+1} + \cdots + (-1)^n E_0 \geqq 0, \qquad (n = 0, 1, \cdots). \tag{3.4}$$

The preceding results hold for each $e > 0$. It is clear that m_k^e increases monotonically as e decreases. If F is an analytic function of the point on an analytic compact manifold, the numbers m_k^e are finite and independent of $e > 0$ for e sufficiently small. The span theory is a theory of *critical values*. It easily yields a theory of critical points and sets on adding the hypothesis of upper reducibility of F.

4. The nondegenerate theory Morse [4]. As distinguished from the causal theory, the objective here is to obtain the *totality of relations* between the critical points classified according to their indices. As distinguished from the span theory, the topological hypotheses that the critical points be nondegenerate makes it possible to treat *all the critical points together* rather than the generic subset of critical points associated with caps of span greater than e. The conditions on F are here necessarily more restricting, but there is a sense in which the nondegenerate F may be everywhere dense among all F admitted in the preceding sections. This has been established in important cases of considerable generality.

The *nondegenerate* theory owes much of its importance to the fact that it is through this theory that analysis extends topology as contrasted with the aid topology usually gives to analysis. The homology characters of Ω in (B) and (C) of §1 and of the symmetric square of the n-sphere were first obtained in this theory by a principle which we shall describe. Striking relations of this theory with homotopy theory have long been apparent and are now beginning to be explored. (See Morse [1, pp. 231–243] and Morse [6].)

The index. If there is just one nondegenerate critical point p at an F-level c, then as one passes from S_{c-e} to S_{c+e} for $e > 0$ sufficiently small only one homology group changes, and that by the addition of a k-cycle or subtraction of a $(k-1)$-cycle as a generator. We say that p then has the *index* k, and, if a k-cycle is added, that p is of *increasing* type. We state a fundamental theorem. Morse [1, p. 230] and Morse [4].

THEOREM 4. *If there exists on the abstract metric space S a nondegenerate function F all of whose critical points are of increasing type, then the k-homology group has a minimum base which includes just one k-cycle associated with each critical point of index k, and no other k-cycles.*

If f is a function of class C'' defined in an n-dimensional local coordinate system, an ordinary *differential critical point* p of f was termed nondegenerate if the Hessian H of f at p was nonvanishing; otherwise put, if no characteristic root of the determinant of H vanished. This generalizes for variational problems as follows. Given a critical extremal g in a variational boundary value problem, the Jacobi equations and the given boundary conditions give rise to a classical *characteristic value problem* associated with g. The critical extremal g is termed *nondegenerate* if and only if there is no vanishing characteristic value. The writer has shown, Morse [1, p. 230], that the integral of length $J(P, Q)$, along curves joining two fixed points P and Q on a compact differential n-manifold M_n of class C''' without boundary, admits no degenerate extremals for any fixed P and almost all Q on M_n. It is in this sense that nondegenerate curve-functions $J(P, Q)$ are dense among all admissible functions $J(P, Q)$.

A general problem. It turns out that a closed extremal g of J on M_n is degenerate in the above sense if and only if the Jacobi equations based on g have no non-null periodic solution. The question arises, is it possible to give meaning and validity to the statement: "Among admissible manifolds of class C''', near M_n in a suitably restricted sense, those manifolds on which every closed extremal is nondegenerate are everywhere dense"? The writer has established such a theorem when $n = 2$, but the case $n > 2$ is open. More generally it should be possible in the case of variational problems in the large of general type to show that in some sense the nondegenerate function is everywhere dense.

The most useful principle of this sort is the following. On the above manifold M_n point functions f of class C'' which are nondegenerate are everywhere dense among functions f of class C''. This follows from work of the writer (cf. Morse [6]) and will be more explicitly elaborated and used in a later paper. The theory of nondegenerate functions parallels the theory of analytic functions in many remarkable ways.

Nondegeneracy topologically defined. Morse [4]. We shall start with a homotopic critical point p of F when F is a positive lower semi-continuous function on the metric space S. Suppose that $F(p) = c < \infty$. We shall be concerned with an *F-bounded* neighborhood U of p, that is, a neighborhood of p relative to some S_b for which $b > F(p)$. If D^t is a deformation of U on S with time parameter t, $0 \leqq t \leqq 1$, the *terminal mapping* of U into S at the time 1 is D^1. We shall refer to a topological image K_r in S of a euclidean r-disc. We take K_0 as a point.

Definition D. *A homotopic critical point p of F will be termed nondegenerate if there exists a proper F-deformation D^t of some F-bounded neighborhood U of p such that*

(i). *D^t leaves p invariant and deforms U into a topological r-disc K_r which contains p as an interior point when $r > 0$, and on which $F(x) < F(p)$ when $x \neq p$.*

(ii). *The terminal mapping D^1, as applied to $K_r \cap U$, is F-deformable in K_r into the identity holding p fast.* Morse [4, p. 50].

It has been shown that an ordinary nondegenerate critical point of a point function f in a local n-dimensional coordinate system is nondegenerate in the above topological sense. See Morse [2, pp. 43–46]. The condition (ii) can be satisfied in the case of this f by choosing D^t so that the mapping in (ii) is the identity. It has also been shown that a critical extremal (an arc c) which is nondegenerate in the earlier sense of this section is also nondegenerate in our topological sense (Morse [4, p. 72]).

In Morse [4] the function F is termed nondegenerate if its homotopic critical points are topologically nondegenerate and finite in number below any finite F-level, and if certain F-deformations exist. All these conditions are topological.

The subscript r of the r-disc K_r appearing in the preceding definition is shown to be the index of the critical point. The homology theory used is the singular theory of Eilenberg. Let p_k be the kth connectivity of S and m_k the number of critical points of index k. Then $m_k \geqq p_k$ and there exist integers b_k, with $0 \leqq b_k \leqq \infty$, and $b_0 = 0$, such that

$$m_k - p_k = b_k + b_{k+1} \qquad (k = 0, 1, \cdots). \tag{4.1}$$

The numbers $E_k = m_k - p_k$, if finite, satisfy the relations (3.4).

Lacunary index sequences. One can obviously derive many properties of the connectivities p_k from the index sequence $I(F)$

$$m_0, m_1, m_2, \cdots. \tag{4.2}$$

If each integer $m_k \neq 0$ in (4.2) has vanishing adjacent integers, $I(F)$ will be termed *lacunary*. From (4.1) one obtains the new theorem:

THEOREM 4.1. *If (4.2) is a lacunary sequence, then $m_k = p_k$ $(k = 0, 1, \cdots)$ where p_k is the kth connectivity of S.*

An important use of this theory is the determination of the homology groups of the space $S = \Omega_M(P, Q)$ of sensed arcs joining two fixed points on a differentiable manifold M of class C'''. The homology groups are independent (up to isomorphisms) of the choice of P and Q so that one can take P and Q so that the length integral $F = J(P, Q)$ is nondegenerate. The index m_k is then the number of geodesics joining P to Q on which there are k conjugate points of P preceding Q. Cf. Morse [1, p. 229]. In the case of an n-sphere the index sequence is known to consist of zeros except that $m_k = 1$ when $k \equiv 0 \bmod (n - 1)$. Morse [1, p. 247]. Hence in the case of the n-sphere $(n > 2)$ the connectivities p_k of $\Omega_M(P, Q)$ are zero except that $p_k = 1$ when $k \equiv 0 \bmod (n - 1)$.

The question arises, what geometric manifolds admit a Riemann metric such that the nondegenerate length integral $J(P, Q)$ possess a lacunary sequence? There are infinitely many geometric manifolds with this property. In particular, the writer has shown in an unpublished paper that the cartesian product

$$S_{n_1} \times \cdots \times S_{n_r} = M \tag{4.3}$$

of any finite number of m-spheres with $n_i > 2$ admit such lacunary sequences. Thus a knowledge of the conjugate points of the geodesics joining P to Q on such M suffices to determine the homology groups of $\Omega_M(P, Q)$. This is consistent with, but not equivalent to, the theorem that the homology groups of $\Omega_M(P, Q)$ are obtained from the homology groups of $\Omega_{S_{n_i}}(P, Q)$, $i = 1, \cdots, r$, by the combinatorial processes usual for products. The latter theorem, proposed by the writer to Pitcher during the preparations of this report, was confirmed by Pitcher and later verified by the writer.

It is of interest to note that if $m_{n+1} = 0$, and if $E_i = m_i - p_i$ is finite for $i = 1, \cdots, n$, then

$$E_n - E_{n-1} + \cdots + (-1)^n E_0 = 0. \tag{4.4}$$

Thus the absence of geodesics g joining P to Q with $n + 1$ points on g conjugate to P implies (4.4).

5. A theorem on the dimension of a critical set. The results of this section will be published in detail in a later memoir. The principal theorem will be stated under much weaker conditions than the theorem on continuous functions on a manifold suggested at the end of §1. To this end, let F be defined over a metric space S, with

(i) *F positive, lower semi-continuous, and upper reducible,*

(ii) *the sets S_c compact for each $c < \infty$.*

Given a nonempty closed subset $A \subset S_c$, an infinite sequence

$$(D) = D_1, D_2, D_3, \cdots \tag{5.1}$$

of F-deformations will be regarded as *applicable* to A if D_1 is an F-deformation of A yielding a terminal image A_1 of A, if then D_2 is an F-deformation of A_1 yielding a terminal image A_2 of A_1 then, D_3 an F-deformation of A_2, etc. Let

$$\Delta_n = D_n D_{n-1} \cdots D_1 \tag{5.2}$$

be the resultant deformation of A, obtained on applying D_1 to A, D_2 to A_1, $\cdots$, D_n to A_{n-1}. Let T_n be the terminal transformation of A under Δ_n.

It is always possible to choose a sequence (5.1) of F-deformations applicable to A, together with a sequence (e_n) of numbers with $e_n > 0$ and $e_n \to 0$ as $n \uparrow \infty$, such that the following holds. If one sets

$$\lim_{n \uparrow \infty} [\sup_{x \in T_n(A)} F(x)] = v(A), \tag{5.3}$$

the set σ of homotopic critical points at the level $v(A)$ is not empty; if B_n is the subset of $T_n(A)$ on which $x > v(A) - e_n$, then B_n is not empty and

$$0 = \lim_{n \uparrow \infty} [\sup_{x \in B_n} d(x, \sigma)] \qquad (d = \text{distance}). \tag{5.4}$$

We suppose the sequence (5.1) and the sequence (e_n) so chosen, and term $v(A)$ an *F-barrier* of A.

We shall define an intrinsic property of a compact set A. Let r and n be integers with $0 < r < n$. We say that A is *(r, n)-admissible* if corresponding to an arbitrary closed subset $X \subset A$,

(1) *the bounding in A of each Vietoris $(n - r)$-cycle in X implies*

(2) *every Vietoris r-cycle in A is homologous in A to an r-cycle in $A - X$.*

The theory of manifolds contains explicit conditions for the existence of (r, n)-admissible sets A. It is clear, however, that A need not be restricted to manifolds. The theory of characters is involved. In this connection it should be recalled that our group of coefficients is a finite field. The fundamental theorem follows:

THEOREM 5.2. *Let A be a closed (r, n)-admissible subset of S_c with an F-barrier $v(A)$ and such that any $(n - r)$-cycle in A which bounds in S_c bounds in A. Let*

H be a nontrivial r-homology class in A. If $v(A) = c(H)$, then the set of homotopic critical points at the level $v(A)$ carries an $(n - r)$-Vietoris cycle which is nonbounding in S_c, so that

$$\dim \sigma \geqq n - r.$$

The proof of this theorem makes no use of category. In general, it seems to be possible to obtain many results on the minimum number of critical points which have been obtained by a use of the category theory without using that theory, and to add to these results a causal relation between various homology, homotopy, and isotopy classes and the respective critical points.

It should be noted that the critical point theory suggests many other numerical topological invariants in addition to the category: for instance, the minimum number N of *nondegenerate* critical points of a continuous nondegenerate function f defined over a geometric manifold, as f ranges over all such functions. If R is the minimum number of *isolated* homotopic critical points of a continuous function f defined over a geometric manifold, as f ranges over all such functions, it is clear that $N \geqq R$. Both N and R are topological invariants over geometric manifolds for which N is defined. When is $N > R$?

6. Other advances. I shall refer first of all to the unpublished work of E. Pitcher which makes use of the mechanism (Morse [1, pp. 244–247]) whereby the homology groups of the space $\Omega(P, Q)$ of §4 were determined for the space of curves joining P to Q on an n-sphere by explicitly giving a base for the nonbounding cycles. Pitcher's work makes it clear that these models will be useful in analyzing the homology groups $\pi_r(S^n, x)$ of the n-sphere. Here x is the point in S^n into which the fixed point of the antecedent r-sphere S^r is mapped. In results announced at the Congress, Pitcher has used these new geometric methods to verify the result of Whitehead that $\pi_5(S^3)$ yields the integers mod 2. The variational methods are capable of great extension in the direction of determining models for use in homotopy theories. For example, one can replace curves joining two fixed points on an n-sphere by disc-type surfaces spanning a circle.

The papers by Morse and Ewing introduce a new approach to the restricted problem of three bodies. The Jacobi least action integral J which is studied is neither regular nor positive definite. Nevertheless Morse and Ewing have established the upper reducibility of J under suitable conditions and prepared for the advances to follow. Ewing has used the Weierstrass generalized integral to give an essential simplification of one of the proofs. In the general direction of fundamental definition of integral and length see Menger, *What Paths Have Length?*

Special attention is called to the remarkable work of the Russian school. The recently published paper by L. Lusternik and Schnirelmann [4] gives more detail concerning early results and continues their program. The paper by Seifert cited refers to the general problem of periodic motion and runs into the same problem of one motion covering another.

Reference should be made to the forthcoming Colloquium Lectures, American Mathematical Society, New York, by Arnold Hedlund where his researches on flow, transitivity, symbolic dynamics, etc., have much to do with variational theory. A paper by Rauch to appear in the Annals of Mathematics uses variational theory in the large, and in particular uses generalized comparison theorems to obtain sufficient conditions on the variation in ratio of the Riemann curvature on a compact simply-connected manifold of positive curvature in order that the manifold be the topological image of an n-sphere.

A fundamental paper by C. B. Morrey first solves the problem of Plateau for a general Riemannian manifold in the case where the manifold is not coverable by a single coordinate system. This result should accelerate variational theory in the large for multiple integrals. In this direction is the penetrating work of Shiffman who has attacked the crucial problems of the multiple integral theory with great ingenuity and success. Courant has aided the general advance by his papers on minimal surfaces and conformal mapping. His book contains other references.

In a basic topological study Leray has initiated a theory of mappings which embraces part of the critical point theory and suggests unsuspected relations. One may expect striking developments along this line in the near future.

The extensive work of McShane and L. C. Young is in another direction but has introduced new power and completeness into the foundations of the theory of generalized curves and surfaces.

The recent work of Morse and Transue gives an abstract representation of a generalization of the second variation. The generalized Euler equations include classical Euler equations as well as integral and integro-differential equations of general type. The relation to the variational theory in the large is in connection with the unpublished characteristic value theory and index theory.

S. Bergman has made use of the theory of level manifolds and of critical points in his study of pseudo-conformal mapping. In particular the theory of equivalence of Reinhardt domains clearly calls for such analysis.

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Institute for Advanced Study,
Princeton, N. J., U. S. A.