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The Classification of Finite Simple Groups

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My aim in this lecture will be to try to convince you that the classification of the finite simple groups is nearing its end. This is, of course, a presumptuous statement, since one does not normally announce theorems as "almost proved". But the classification of simple groups is unlike any other single theorem in the history of mathematics, since the final proof will cover at least 5,000 journal pages. Moreover, at the present time, perhaps 80% of these pages exist either in print or in preprint form. One obtains a better perspective of the subject if instead of thinking of the classification as a single theorem, one views it as an entire field of mathematics—the *structure of finite groups.* Then when I say that there are some 4,000 pages in print, proving many general and specific results about simple groups, it should sound entirely reasonable, since one can make the same claim concerning many areas of mathematics. Thus my task is really to convince you that we have established so many results about simple groups and have developed sufficient techniques for completing the classification.

There are other reasons for skepticism besides my premature announcement of the impending completion of the classification. Indeed, to the nonspecialist, simple group theory appears to be in a rather chaotic state. Strange sporadic simple groups dot the landscape—26 at last count; and they appear to be widely unrelated to each other. The five Mathieu groups, 100 years old, examples of highly transitive permutation groups, the four groups of Janko, each arising from the study of centralizers of involutions, the three Conway groups, determined from the automorphisms of a certain integral lattice in 24-dimensional Euclidean space, etc. And now there comes the Fischer-Griess monster, of order over 10^{53}; to be precise:

$$2^{46}\quad 3^{20}\quad 5^{9}\quad 7^{6}\quad 11^{2}\quad 13^{3}\quad 17\quad 19\quad 23\quad 29\quad 31\quad 41\quad 47\quad 59\quad 71.$$

And to add to the confusion, we don't even know whether the group exists! If it does, it involves, inside of itself, approximately 20 of the 26 sporadic groups. But whatever the case may be, it is clear that at present we have no coherent explanation of these sporadic groups. How then will it be possible to classify the simple groups in the face of this reality?

There is another troubling problem. Who will accept a 5,000 page proof when it exists? For it seems humanly impossible to avoid local errors in very long papers, and there is no doubt that there are many such errors in the existing 4,000 pages. Most of us have been rushing ahead towards the finish line with little time to look backwards; but it is clear that the first major "postclassification" problem will be a reexamination of the entire proof.

The fact is that the chaos in the subject is apparent rather than real. In searching for new simple groups, any plausible direction is worth exploring. It is much like experimental science and there is an element of the haphazard about the whole process. One can compare the discovery of a new simple group with that of an elementary particle in physics.

This is quite the opposite of the idea of classification, which implies something *systematic*. If one studies all simple groups G with some property X: for example,

Property X: (a) G has odd order.
(b) G has order at most 1,000,000.
(c) G has abelian Sylow 2-subgroups.
(d) The normalizer of every nontrivial solvable subgroup of G is solvable.

Then the analysis must uncover every simple group having the specified property X. The major thrust of simple group theory during the past 25 years has been the development of methods which enable us to determine all simple groups with some such property X. Each of the above listed four problems has indeed been solved. The first is, of course, the celebrated Feit-Thompson theorem which asserts that all groups of odd order are solvable—equivalently, that every (nonabelian) simple group has even order. In fact, it is this landmark theorem which started the whole show!

Primarily the methods for dealing with such general classification problems are *internal*. They involve the study of the proper subgroup structure of the simple group G under investigation. This point of view is fundamental. These methods have as their goal the following objective:

Prove that the internal structure of G closely resembles that of some known simple group G^*.

In the extreme, this can be taken to mean that G and G^* have identical lattices of proper subgroups. However, in practice, one does not require such complete similarity. Often it is entirely sufficient for a *single* subgroup of G to resemble the corresponding subgroup of G^*—for example, the centralizer of an *involution* (i.e., an element of order 2 in G).

We must emphasize that this internal resemblance of G to G^* may have nothing whatsoever to do with the way that the group G^* was initially discovered. For example, suppose one of Conway's groups C has the specified property X. Then the analysis must yield C as a possible answer. But to assert that, say, the centralizer of an involution of G is isomorphic to that of C has no connection with 24-dimensional integral lattices. It says nothing more than that the internal structure of G resembles that of the group C. Naturally we would like to be able to conclude from the given resemblance that G must, in fact, be isomorphic to C.

This leads us to the first major chapter of simple group theory, which must be resolved before one can attempt to prove any classification theorems whatsoever. It is called *Recognition Theory* and concerns the following general question:

If a simple group G has an internal structure closely resembling that of a known simple group G^*, must G be isomorphic to G^*?

If so, we say that the group G^* is *characterized* by the given set of internal conditions.

At the present time, essentially every known simple group possesses such an internal characterization. What are the known simple groups? Obviously I have no time to do more than simply list them here. They are the trivial groups Z_p, the 26 sporadic groups, the alternating groups of degree $n \geq 5$, and the so-called groups of Lie type. These last are the finite analogues of the complex Lie groups; thus we have finite analogues of the complex linear, symplectic, and orthogonal groups, and of the five exceptional Lie groups G_2, F_4, E_6, E_7, E_8, as well as finite analogues of the unitary groups. In the finite case, it turns out that there are somewhat more families than in the complex case. But in any event, we have a complete list of the finite simple groups of Lie type.

We can think of the linear groups as the typical example of a group of Lie type.

General linear group: $\mathrm{GL}(n, q)$ is the group of all nonsingular $n \times n$ matrices with coefficients in the Galois field $\mathrm{GF}(q)$ with q elements.

Special linear group: $\mathrm{SL}(n, q)$ is the normal subgroup of $\mathrm{GL}(n, q)$ of matrices of determinant 1.

Projective special linear group: $L_n(q) = \mathrm{PSL}(n, q)$ is the factor group of $\mathrm{SL}(n, q)$ modulo scalar matrices of determinant 1.

Fact: $L_n(q)$ is simple if $n \geq 3$ or if $n=2$ and $q \geq 4$.

We cannot expect to have an internal characterization of the Fischer-Griess monster, since we do not even know if it exists. The same is true of Janko's most recently discovered fourth group J_4. I should say that the problem here of existence and uniqueness of these two groups will be dealt with by a high-speed computer. There remains perhaps a little more theoretical work to do to set these problems up for the computer. However, the main question will be simply whether the present generation of computers is fast enough to make the required calculations.

Apart from these two groups, every other known simple group with the exception of a single family of groups of Lie type discovered by Rimhak Ree has such an internal characterization. Ultimately the characterizations of the groups of Lie type rests on Tits' geometric descriptions of these groups in terms of apartments and buildings or on the so-called Steinberg presentation in terms of generators and relations.

The Ree groups are a troublesome family. They have no complex analogue and they exist only in characteristic 3.

Ree group $R(q)$: order $q^3(q-1)(q^3+1)$, q an odd power of 3. Also $R(q)$ is a doubly transitive permutation group on q^3+1 letters and a subgroup fixing three letters has order 2.

Problem: Prove that the groups $R(q)$ are the only doubly transitive permutation groups of this order satisfying the given conditions.

Let $G(q)$ be an arbitrary such group. With great effort, Thompson has proved the following results:

(1) Associated with any such group $G(q)$ is an automorphism θ of the field GF (q);

(2) If θ^2 is the automorphism: $x \to x^3$, $x \in \mathrm{GF}(q)$, then $G(q) \cong R(q)$;

(3) For each value of q and θ, there is at most one group $G(q)$.

Open Question: Must θ^2 be the cubing map for the group $G(q)$ to exist?

Theoretically, therefore, there may exist new simple groups corresponding to other values of the parameter θ. A recent Ph. D. student of Suzuki has shown with the aid of a computer that no other groups than $R(q)$ exist for $q=3^n$ when $n \leq 29$. In any event, the ambiguity here does not bother us too much—we simply allow for this degree of indeterminacy by speaking of a group of *Ree type* as any group satisfying all the specified conditions.

Likewise we have groups of *monster type*. They are simple groups of the order I have written above and which have the various properties already established for the Fischer-Griess monster. Even though we do not know whether such a group exists, we allow for its existence in our analysis. Similarly we have a group of *type* J_4.

Subject then to these precise indeterminacies, this chapter of simple group theory is complete. This means that we are ready to begin the classification of the finite simple groups. However, we must emphasize that these ambiguities will remain even after our present classification theorem is completed. They should be viewed as isolated problems, which hopefully will eventually be settled.

As the classification has evolved, it has broken down into four major categories, as follows:

A. Nonconnected groups.

B. Groups of component type.

C. Small groups of noncomponent type.

D. The general group of noncomponent type.

In the balance of the talk, I shall attempt to outline the results obtained to date in categories A and B. This is all that time will permit. Fortunately, Michael Aschbacher in his lecture will describe the current state of affairs in categories C and D. Taken together, these two talks should give you a good idea of how close we actually are to completing the classification of the finite simple groups.

Let me explain the term *connectedness.* Given any group X, consider the collection $\mathscr{K}$ of Klein four subgroups of X; i.e., of subgroups of X isomorphic to $Z_2 \times Z_2$. Construct a graph Γ, whose vertices are the elements of $\mathscr{K}$. Connect two vertices A, B of Γ, if A and B commute elementwise; i.e., if $[A, B]=1$. The group X is said to be *connected* if the resulting graph Γ is connected in the usual sense. It is this degree of freedom which is needed to carry out certain general lines of argument.

The meaning of category A is the following: *Determine all nonconnected simple groups.* This chapter of simple group theory has been completed. However, it has taken some 3,000 journal pages to achieve. The proof has been carried out in two major parts:

I. Determine all simple groups which possess a nonconnected Sylow 2-subgroup.
II. Determine all nonconnected simple groups with a connected Sylow 2-subgroup.

Subgroups and homomorphic images of nonconnected groups may be connected, so nonconnectedness is not a good inductive concept. The solution of I has been obtained by treating it as a special case of a more general classification problem which is inductive. This is based on the following proposition.

PROPOSITION. *Let* S *be a nonconnected* 2-*group. If* A *is any subgroup of* S *and* $\bar{A}$ *is any homomorphic image of* A, *then* $\bar{A}$ *does not contain a subgroup isomorphic to* $Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2$.

Such a group $\bar{A}$ is known as a *section* of S and so we rephrase the proposition by saying that a group with nonconnected Sylow 2-subgroup has *sectional 2-rank at most* 4. Thus I will be solved if we determine all simple groups of sectional 2-rank at most 4. The advantage of the latter condition is that it is preserved by subgroups and homomorphic images and so can be proved inductively. The resulting theorem will then stand in its own right, independent of whether the full classification is ever achieved. Most of the major results of simple group theory have a similar degree of independence.

I wish to state the sectional 2-rank ≤ 4 theorem in its entirety, for the answer is instructive. You will have to accept the fact that each of the terms I write down stands for some specific groups of family of groups.

THEOREM. *If* G *is a simple group of sectional* 2-*rank at most* 4, *then* G *is isomorphic to one of the groups on the following list:*

I. *Odd characteristic:* $L_2(q)$, $L_3(q)$, $U_3(q)$, $G_2(q)$, ${}^3D_4(q)$, Psp $(4, q)$, $L_4(q)$, $q \equiv 1$ (mod 8), $U_4(q)$, $q \equiv 7$ (mod 8), $L_5(q)$, $q \equiv 3$ (mod 4), $U_5(q)$, $q \equiv 1$ (mod 4), *or Ree type of characteristic* 3 *(Note the word "type" here).*

II. *Characteristic* 2: $L_2(8)$, $L_2(16)$, $L_3(4)$, $U_3(4)$, *or* Sz(8).
III. *Alternating:* A_n, $7 \leqslant n \leqslant 11$.
IV. *Sporadic:* M_{11}, M_{12}, M_{22}, M_{23}, J_1, J_2, J_3, Mc, *or* Ly

Thus, apart from certain families of groups of Lie type of odd characteristic of low dimension, there are precisely 19 other groups, half of them sporadic. You can see why the proof of this theorem must be a long one. If we think of each family as a single type of group, then there are some 30 distinct internal structures that can arise, 19 of them corresponding to individual groups. Thus our internal analysis of G must branch off into various directions, so that we can eventually show that G resembles internally one of these 30 types of groups. Each of these branches requires its own analysis.

Of course, groups of odd order correspond to "Case 0" of the theorem, which accounts for 250 pages of the argument!

To avoid repetition, we state the second part of the nonconnectedness theorem as follows:

THEOREM. *If G is a nonconnected simple group of sectional 2-rank at least 5, then G is isomorphic to one of the following groups:* $L_2(2^n)$, $U_3(2^n)$, *or* Sz(2^n).

Equivalently, G is of Lie type of characteristic 2 and "Lie rank 1." In particular, a Sylow 2-subgroup of G intersects its distinct conjugates only in the identity.

This last statement explains the structure of the graph Γ of G: each Sylow 2-subgroup of G corresponds to a distinct component of Γ.

The effect of having a complete solution to category A is that in all subsequent classification problems, one can assume at the outset that the group G under investigation is connected. I can only give the barest hint of the way this condition is used. Basically it helps us to analyze the *cores* of centralizers of involutions.

For any group X, the *core* of X is the unique largest normal subgroup of X of *odd* order. It is denoted by $O(X)$. By Feit-Thompson, cores are always solvable.

Fact. If G^* is a known simple group and t^* an involution of G^*, then $O(C_{G^*}(t^*))$ is a *cyclic* group.

Hence in studying arbitrary simple groups G and attempting to show that G internally resembles some known simple group, one of the first objectives is to prove that cores of centralizers of involutions of G are necessarily "small" (a cyclic group being a typical example of a small group). The methods we have developed for achieving this goal require G to be connected. This is all I can say here.

To describe the results in category B, I must now define a group of *component* type. To motivate the concept, let us examine briefly the general structure of the centralizer of an involution in a group of Lie type defined over GF(q), $q=p^n$, p a prime. As we shall see, we obtain quite distinct answers according as p is odd or $p=2$. This is, in fact, to be expected since in the Lie terminology an involution corresponds to a "semisimple" element when p is odd and to a "unipotent" element when $p=2$. We shall illustrate the situation using the groups $\mathrm{GL}(n, q)$.

odd characteristic

$$\textit{Involution } t: \begin{pmatrix} -1 & & & & & \\ & -1 & & & & \\ & & \ddots & & & \\ & & & -1 & & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} \qquad \textit{Centralizer } C_t: \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

(The first diagonal block consists of k entries -1, the second of $n-k$ entries 1.)

Here A is nonsingular $k\times k$ and B is nonsingular $(n-k)\times(n-k)$.

Structure $C_t \cong \mathrm{GL}(k, q)\times \mathrm{GL}(n-k, q)$.

characteristic 2

$$\textit{Involution } t: \begin{pmatrix} 1 & & & \\ 0 & 1 & 0 & \\ \vdots & & \ddots & \\ 0 & & & \\ 1 & 0 \ldots 0 & & 1 \end{pmatrix} \qquad \textit{Centralizer } C_t: \left\{ \begin{pmatrix} x_{11} & 0 & \ldots & 0 \\ x_{21} & & & \vdots \\ & & A & \\ \vdots & & & 0 \\ x_{n1} & x_{n2} & \ldots & x_{nn} \end{pmatrix} \right\}.$$

Here A is nonsingular $(n-2)\times(n-2)$, $x_{11}\neq 1$, $x_{nn}\neq 1$.

$$\text{Define } Q = \left\{ \begin{pmatrix} 1 & & & 0 \\ x_{21} & 1 & & \\ & 0 & \ddots & \\ x_{n1} & x_{n2} & \ldots & 1 \end{pmatrix} \right\} \quad \text{and} \quad K = \left\{ \begin{pmatrix} 1 & 0 \ldots & & 0 \\ 0 & & A & \vdots \\ \vdots & & & 0 \\ 0 & & \ldots 0 & 1 \end{pmatrix} \right\}$$

Q is a 2-group, $K \cong \mathrm{GL}(n-2, q)$.

Structure $C_t \sim Q\cdot K$; semidirect product; Q is normal in C_t; K acts faithfully on Q by conjugation.

We see then that when p is odd, the centralizer C_t is a product of groups of Lie type of lower dimension. Actually it is the $\mathrm{SL}(m, q)$ factors we are interested in rather than $\mathrm{GL}(m, q)$, for these are closer to being simple. In this example, each of these factors is normal, since the product is direct. However, in other groups, the centralizer may contain an element interchanging the factors, so these factors will only be what we call *subnormal*.

In general, a subgroup Y of a group X is called *subnormal* if there exists a chain of subgroups $Y=X_n, X_{n-1}, \ldots, X_1=X$ of X with each X_i normal in X_{i-1}.

On the other hand, when $p=2$, C_t has no such normal or subnormal subgroups of Lie type. The subgroup Q is an obstruction to the existence of such subgroups.

Only if one considers the factor group C_t/Q does one obtain a normal subgroup of Lie type.

This dichotomy is fundamental for understanding the general finite simple group, for it leads to a basic subdivision of simple groups into two distinct categories, one reflecting the odd characteristic phenomenon, the other the characteristic 2 phenomenon. To make the definition, we must take into account that the groups SL (m, q), need not be simple. Consider, for example, $X=\mathrm{SL}(2,q)$, q odd, $q\geq 5$; then $\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)$ is an element of X and commutes with every element of X, so is in the *center* of X. Hence certainly X is not simple. It is the factor group $X/\langle\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)\rangle\cong L_2(q)$ which is simple. Thus X is what we refer to as a *covering* group of a simple group.

The more precise term is given by the following definition.

Definition. A group X is said to be *quasisimple* if X is perfect (i.e., $X=[X, X]$) and X (center of X) is simple.

In the study of simple groups G, we have already observed that the core $O(C_t)$, t an involution of G, also acts as an obstruction to any statement we may wish to make about the structure of the centralizers of an involution. Hence the definition of a group of component type must be formulated in terms of $C_t/O(C_t)$ rather than of C_t itself.

Definition. A group G is said to be of *component* type if for *some* involution t of G, $C_t/O(C_t)$ possesses a quasisimple subnormal subgroup. In the contrary case, G is said to be of *noncomponent* type.

Now we see the meaning of category B and the contrapositive categories C and D.

I shall now state the goal of much of the research of the past ten years. Again to avoid repetition, I shall assume that G has sectional 2-rank at least 5.

Theorem (?). *Let G be a simple group of component type (of sectional 2-rank at least 5) and assume that for some involution t of G, $\bar{C}_t=C_t/O(C_t)$ possesses a quasisimple subnormal subgroup $\bar{L}$ which is a covering group of a known simple group. Then one of the following holds:*

I. *G is of Lie type of odd characteristic (of sectional 2-rank at least 5);*

II. *$G\cong A_n$, $n\geq 12$; or*

III. *$G\cong$ one of the following 13 sporadic groups:* HS, ON, He, Suz, Ru, *Conway .1 or .3, Fischer $M(22)$, $M(23)$, $M(24)'$, the baby monster F_2, Harada's group F_5 (a subgroup of the monster), or G is of monster type.*

The (?) here is to indicate that the proof is not quite complete. At this time, there still exist certain possibilities for $\bar{L}$ for which it has not been established. Here is the present list of open cases:

characteristic 2. $\bar{L}\cong {}^2F_4(2)'$, ${}^2F_4(2^n)$, n odd, $n>1$, $F_4(2)$, Sp $(6, 2)$, $U_6(2)$, $O_8^{\pm}(2)$, or a covering group of Sp $(6, 2)$, $U_6(2)$, $O_8^{\pm}(2)$.

characteristic 3. $\bar{L}\cong U_3(3)$, $U_4(3)$, $L_4(3)$, $G_2(3)$.

sporadic. $\bar{L}\cong$ Conway .2 and Thompson F_3.

Thus there is a single family of groups plus 16 individual possibilities for $\bar{L}$. It should be emphasized that the open list has been steadily shrinking as group theorists tackle the remaining cases. Moreover, the methods for treating these problems are well understood. It is, of course, possible that one or more of these cases may lead to a new simple group. If so, each such new group as well as all of its covering groups would then have to be "plugged in" for $\bar{L}$. The same applies if a new simple group of noncomponent type is discovered in the future. However, what is possible and what is probable are two different matters. The most likely conjecture is that *every* finite simple group is now known and the remaining cases of the component theorem will be finished within approximately a year's time!

In conclusion, I would like to state a magnificent theorem of Aschbacher which characterizes the groups of Lie type of odd characteristic among the groups of component type and which is completely proved.

Suppose, in the above theorem, that the group $\bar{L} \cong \mathrm{SL}(2, q)$, q odd. Then $\bar{L}$ has a center of order 2. The involution $\bar{t}$ is certainly in the center of $\bar{C}_t$ and so $\bar{t}$ is a possible candidate for the involution in the center of $\bar{L}$. If $\bar{t}$ does lie in $\bar{L}$, we say that $\bar{L}$ is an *intrinsic* SL $(2, q)$ and we call the involution t a *classical* involution.

THEOREM. *If* G *is a simple group which possesses a classical involution (and* G *has sectional* 2*-rank at least* 5*), then* G *is a group of Lie type of odd characteristic.*

This is a remarkable result because it asserts that the full structure of G is completely determined by a "tiny" piece of information in the centralizer of a single involution. It also shows the fundamental significance of the subgroups SL $(2, q)$ for the structure of a group of Lie type. I think you will agree that the theory of finite simple groups must be quite fully developed for us to be in a position to establish such a powerful conclusion from so little information!

Aschbacher's lecture on groups of noncomponent type will indicate that our results for the groups in categories C and D are rapidly approaching the same degree of finality as presently exists for the groups of component type in category B. In fact, if and when all the present work in progress is completed, there will remain only a very few, essentially isolated, problems of the type described above, to complete the entire classification of finite simple groups (these do not include the Ree group problem and the question of the existence and uniqueness of both the monster and Janko's fourth group, which as I have tried to make clear, may very well remain unresolved after the classification).

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