SOME ASPECTS OF LINEAR AND NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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The speaker's committee has asked me to speak on some of the recent work in nonlinear partial differential equations. I would like to start by quoting from Prof. G. Temple's address [58] at the Edinburgh Congress—"the closely guarded secret of this subject (differential equations) is that it has not yet attained the status and dignity of a science but still enjoys the freedom and freshness of pre-scientific study.... The work of classification and systemization of specimens has hardly begun." Indeed most of the specimens considered still deal directly with equations arising from physical theory or from geometric problems, and the work seems rather special and indeed peculiar to some mathematicians from other fields. In the past decade, however, the general theory of linear partial differential equations with constant coefficients has been greatly developed; this progress is described elsewhere in this Congress.

In this talk which is necessarily limited by my own restricted knowledge and inclinations I shall not deal with equations arising from physical problems. Thus, in particular, and also because of time restrictions, I will not describe the intensive attack that has been made again in recent years, after the basic work by J. Leray in the 30's, on the existence problems for the Navier-Stokes equations of fluid mechanics, except to make reference to the recent book by O. A. Ladyzhenskaya [28] (soon to appear in English translation) for a clear exposition of much of the recent developments, as well as expository articles by G. Prodi [52], J. Serrin [55] and R. Finn [10] where further references may be found. (Rather than have a lengthy bibliography, I have tried to limit it to papers containing fairly comprehensive lists of references.)

I wish to describe some recent developments in existence and regularity theory for nonlinear boundary value problems, mainly for elliptic differential equations, with side remarks for parabolic and hyperbolic equations. Since this lecture is directed to nonexperts, I would like to give some indication of the techniques used and not merely list the latest and strongest results which are often quite complicated to formulate.

§ 1. Most results for nonlinear problems are still obtained via linear ones, i.e. *despite* the fact that the problems are nonlinear not *because* of it. So we shall begin with the simplest question for a nonlinear problem, a perturbation problem, in which the problem differs slightly from a linear one; this simply involves the implicit function theorem in a suitable framework.

To fix notation we usually treat functions $u(x)$, $x=(x_1,...,x_n)$ defined in a bounded domain in E^n with smooth boundary; u may represent a system of functions. Differentiation is denoted by $D=(D_1,...,D_n), D_j=\partial/\partial x_j$,

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$D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$ for $\alpha = (\alpha_1, \dots \alpha_n)$ with the α_j integers $\geqslant 0$, and D^α has order $|\alpha| = \Sigma \alpha_j$. We also write the first and second derivatives of u as $u_{x_i}, u_{x_i x_j}$. For $k \geqslant 0$ an integer we use the norm

$$|u|_k = \sum_{|\alpha| \leqslant k} \sup |D^\alpha u|$$

for the space C^k.

Consider a nonlinear partial differential equation of order m depending on some parameters ε

$$F(\varepsilon, x, u, D^\alpha u) = 0, \tag{1.1}$$

we are interested in solutions satisfying, say, homogeneous conditions on the boundary (or part of the boundary) assuming that u_0 is a solution for $\varepsilon = 0$. If $L_0 = L(u_0)$ is the linearized operator about the solution u_0, the first variation of F:

$$L_0 v = L(u_0) v = \sum_\alpha \frac{\partial F}{\partial D^\alpha u} (0, x, D^\alpha u_0) D^\alpha v,$$

then one writes the nonlinear equation (by writing F as a linear term in $(u - u_0)$ plus a remainder term of higher order)

$$L_0(u - u_0) = R[u, \varepsilon],$$

where R involves higher order terms in $u - u_0$ and its derivatives. If the linearized problem

$$L_0 v = f$$

with the homogeneous boundary conditions has a solution (say unique) $v = L_0^{-1} f$, such that if f belongs to a certain class of functions then the m^{th} order derivatives of the solution v belong to this same class, the Picard iteration:

$$u_{n+1} = L_0^{-1} R[u_n, \varepsilon]$$

will yield a solution. This is just the usual implicit function theorem. In applying it the main thing to verify is the property of the solution $L_0^{-1} f$—that it is sufficiently differentiable. Otherwise one loses differentiability in each iteration step.

For elliptic partial differential operators L_0 and for a wide class of boundary conditions it has been shown that L_0^{-1} has the right properties, in a suitable function space. This is a consequence of the, so-called, Schauder type estimates for such equations; see S. Agmon, A. Douglis, L. Nirenberg [1] (these estimates have been derived also for general elliptic systems in part 2 soon to appear), see also F. Browder [3].

We remark that the class of boundary conditions for which such estimates hold are those satisfying certain algebraic relations involving their leading parts and the leading part of the operator L_0. (A simple way of describing these conditions at the boundary point x_0 is to consider the leading part L_0' of L_0, and leading parts of the boundary differential operators with coefficients having their value at x_0, i.e. constant, and to imagine the boundary as flat at x_0, and require that the only bounded finite sum of exponentials v satisfying $L_0' v = 0$ in the half space, and satisfying the leading part of the boundary conditions on the boundary of the half space, and tending to zero as we go to infinity normal to the boundary, is $v \equiv 0$.) Recently there have occurred some situations in which the lower order terms also play a crucial role, in particular, in the work of C. B. Morrey [39] and its extension by J. J. Kohn [27] on complex analytic manifolds; the general situation here has yet to be cleared up. For planar boundaries and constant coefficients, in the case of one operator, L. Hörmander [22] has resolved the problem of regularity at the boundary.

Let us return to the general perturbation problem or implicit function theorem. The Picard iteration method described there converges like a geometric series. Namely $\|u_{n+1}-u_n\| \leqslant \theta \|u_n - u_{n-1}\|, \theta < 1$. Here $\| \; \|$ denotes some norm. It may occur in practice that the operator L_0^{-1} "loses" derivatives. For instance, if we work with the spaces C^k (which is not very suitable for elliptic problems) we want L_0^{-1} to map C^ϱ boundedly into $C^{\varrho+m}$ but it may only map C^ϱ boundedly into $C^{\varrho+m-\sigma}$ (thus a loss of σ derivatives). Then Picard iteration doesn't work. In 1956 J. Nash [46] was able to treat a special situation just of this kind occurring in his work on isometric embedding of Riemannian manifolds. He did this by means of a rather remarkable but complicated and, to me, mysterious scheme involving a combination of approximation and "smoothing" of functions. This method was set into a general framework by J. Schwartz [54] who proved a general implicit function theorem. Recently J. Moser [44] has succeeded in giving a clear and conceptually straightforward proof of this and I would like to describe the idea of this important method—sticking still to a partial differential equation although the method works for general nonlinear functional equations. It involves two devices:

(1) An iteration scheme—Newton's instead of Picard's. Newton's method is the following

$$u_{n+1} - u_n = -L(u_n)^{-1} F(\varepsilon, x, D^\alpha u_n) = -L(u_n)^{-1} F[u_n].$$

Thus we assume not only that $L(u_0)$ is invertible but also $L(u)$, for u sufficiently "close" to u_0. By the mean value theorem we have

$$|F[u_n] \underbrace{- F[u_{n-1}] - L(u_{n-1})(u_n - u_{n-1})}_{0}|_\varrho \leqslant \text{const}\, |u_n - u_{n-1}|^2_{m+p}. \tag{1.2}$$

If $L(u_n)^{-1}$ were a bounded linear map of $C^{\varrho+m}$ into C^ϱ we would consequently have

$$|u_{n+1} - u_n|_{m+\varrho} \leqslant \text{const}\, |u_n - u_{n-1}|^2_{m+\varrho}$$

and hence very rapid convergence of the iteration scheme.

Moser uses this scheme together with a smoothing device (also used by Nash), an operator T_N depending on a parameter N:

(2) For every N large T_N is a linear mapping of functions into C^∞ functions with the properties (here $\delta \geqslant 0$ is a fixed number):

For all integers $p, q \geqslant 0$

(i) $|T_N u|_{p+q} \leqslant \text{const}\; N^{q+\delta} |u|_p$,

(ii) $|u - T_N u|_p \leqslant \text{const}\; N^{-q+\delta} |u|_{p+q}$.

The constants depend on p and q. Condition ii) expresses the fact that the approximation $T_N u$ is very close to u if u itself is very smooth. (In case we consider functions u defined in all E^n [or periodic in all variables] such smoothing operators may be obtained by multiplying the Fourier transform of u [or Fourier coefficients of u] by functions with compact support, or, what is equivalent, by convoluting u with *suitable* kernels $K(Nx)$ such that $\int K(x)dx=1$. One even obtains (i) and (ii) with $\delta=0$.)

Let us see how to obtain a solution u of (1) in some class $C^{m+\varrho}$ assuming that the inverse of the linear operator $L(u)$ is defined for $|u-u_0|_m$ sufficiently small and is a bounded map of C^ϱ into $C^{\varrho+m-\sigma}$, i.e. it loses σ derivatives. Moser's iteration scheme is to choose $N_{n+1}=N_n^{3/2}$, and set

$$u_{n+1}-u_n=-T_{N_{n+1}}L(u_n)^{-1}F[u_n]. \tag{1.3}$$

With $\mu=3(\sigma+\delta), l=35(\sigma+\delta+1)$ one shows inductively that

$$|u_{n+1}-u_n|_{m+\varrho}\leqslant \text{const } N_{n+1}^{-\mu},$$

$$|u_n-u_0|_{m+\varrho+l}\leqslant \text{const } N_n^l.$$

These estimates are not very difficult to prove using (i) and (ii) and assuming that u_0, the initial solution, belongs to $C^{m+\varrho+l}$. The iterates u_n converge to a solution u in $C^{m+\varrho}$; u is thus considerably less smooth than the initial solution u_0. In working with general nonlinear operator equations Moser imposes a third condition (condition (3) in [44]) which is usually satisfied in practice, and which we have omitted here.

This result and its proof are set within the framework of linear spaces. For nonlinear problems this seems slightly unnatural, but there is as yet no systematic theory operating without these spaces.

Moser has applied similar techniques in studying stability of C^k solutions of ordinary differential equations. In this work he is not able to operate in a linear function space. Furthermore the nonlinear character of the equations is used in an essential way, indeed he obtains results *because* of the nonlinearity not despite it. This work is related to the work of Kolmogorov and Arnold on stability for analytic nonlinear differential equations; the use of Newton's method was suggested by that work (see Moser's report at the Congress).

§ 2. We turn to more special equations—second order quasilinear equations of the form (using summation convention)

$$a_{ij}(x,u,Du)\,u_{x_ix_j}=f(x,u,Du) \tag{2.1}$$

a_{ij} positive definite matrix, or equations in divergence from

$$\frac{\partial}{\partial x_i}a_i(x,u,Du)=f(x,u,Du), \tag{2.2}$$

$\partial a_i/\partial u_{x_j}$ positive definite matrix, such as arise from regular variational problems

$$\delta\int F(x,u,Du)\,dx=0, \tag{2.3}$$

for which the Euler equation is

$$\left(\frac{\partial F}{\partial u_{x_i}}\right)_{x_i}=\frac{\partial F}{\partial u}. \tag{2.4}$$

One may also write (2.2) in the "weak" from

$$\int (a_1(x, u, Du)\zeta_{x_i} + f(x, u, Du)\zeta)\,dx = 0 \tag{2.2'}$$

for all $\zeta \in C_0^\infty$, i.e. for all functions $\zeta \in C^\infty$ with compact support in the domain. Since Hilbert formulated the problem of showing regularity of solutions of (2.3) or (2.4) these equations have received much study concerned also with existence theorems for boundary value problems.

Much of the work in connection with these equations has been directed to finding a priori bounds for solutions of the equations, i.e. bounds for a solution and its derivatives, assuming that a smooth solution exists. The search for such bounds goes back to the fundamental work of *S*. Bernstein, and their use has been fully clarified in the basic work of J. Leray and J. Schauder [33].

We can give a brief indication, to those who are not familiar with the field, how these bounds may be used in proving existence of a solution of, say, (2.1) with given boundary values. Assuming that we know how to solve linear elliptic problems (elliptic here means that a_{ij} is positive definite), insert in the coefficients of (2.1) a function v (and its first derivatives), and solve the corresponding linear elliptic equation for a function u taking on the given boundary values. This defines a transformation $u = T[v]$ which, because u belongs to a higher differentiability class than v, can be shown to be a compact operator. If we had a priori bounds for solutions of such linear elliptic equations which involve practically no knowledge of the coefficient, say that u and some derivatives are bounded, then we could assert that T maps the set of functions satisfying these conditions into itself. Because T is compact it would follow, by the Schauder fixed point theorem, that T has a fixed point—which is then a solution of (2.1). Thus we see why it is useful to obtain *a priori* bounds for solutions of linear equations under minimal assumptions on the coefficients.

A more general procedure is to obtain a priori bounds for solutions of the nonlinear problem by using some special features or structure of the equation. Then one connects the equation by a one parameter family of equations $E_t, 0 \leqslant t \leqslant 1$, to an equation E_1, which one can solve. If a priori bounds for solutions of all these equations can be obtained, e.g. that some norm $\|u\|$ of the solutions u remain bounded, $\|u\| \leqslant K$, then in the sphere $\|u\| \leqslant K$ in the Banach space with $\|\ \|$ as norm one may try to use the Leray-Schauder [33] theory of degree of mapping (see also Leray [32] and M. Nagumo [45]) to show that all the equations E_t possess solutions starting from E_1, since the solutions cannot cross the boundary of the sphere $\|u\| \leqslant K$.

Before taking up any special equations let me mention a few tools of the calculus which are used; these sometimes enable one to treat certain lower order terms in an equation as minor perturbation terms. In the following we use $|D^j u|_{L_q}$ to denote the sum of the L_q norms of all derivatives of u of order j, and denote constants by c; also $j < m$ below.

Sobolev inequalities:

$$|D^j u|_{L_q} \leqslant c(|D^m u|_{L_p} + |u|_{L_p})$$

if
$$\frac{1}{q} = \frac{1}{p} - \frac{m-j}{n} > 0.$$

Interpolation inequalities:

$$|D^j u|_{L_q} \leqslant c\,|D^m u|_{L_p}^{a}\,|u|_{L_r}^{1-a} + c\,|u|_{L_r}, \quad \text{for} \quad \frac{j}{m} \leqslant a \leqslant 1$$

if
$$\frac{1}{q} = \frac{j-am}{n} + \frac{a}{p} + \frac{1-a}{r} > 0.$$

The latter inequalities enable one to say that the set of bounded functions with derivatives of order m in L_p form an algebra—a fact that is useful in nonlinear problems. (Proofs of these may be found in L. Nirenberg [48], lecture 2, and in E. Gagliardo [13].) The interpolation inequalities are special cases of general abstract interpolation inequalities that have been found in the last few years which are generalizations of the Riesz-Thorin convexity theorem. These should prove useful in nonlinear problems. We content ourselves here with the following references: E. Gagliardo [14], A. P. Calderon [4], J. L. Lions [34] where further references may be found. Some applications to nonlinear problems are indicated in chapters 4, 8 and 10 of J. L. Lions [35].

Another concept that is used in the study of nonlinear differential equations is that of "weak" or "generalized" solution. Using W_p^m, (or $H_{m,p}$) to denote the completion in the norm $\sum_{|\alpha|\leqslant m} |D^\alpha u|_{L_p}$ of C^∞ functions (the preceding inequalities extend to functions lying in these spaces) one says, say, that $u \in W_p^1$ is a weak solution of

$$(a_{ij} u_{x_i})_{x_j} = f$$

if
$$\int (a_{ij} u_{x_i} \zeta_{x_j} + \zeta f)\, dx = 0 \quad \text{for all} \quad \zeta \in C_0^\infty.$$

In seeking to prove existence of regular solutions of nonlinear elliptic equations it is often convenient (and usually simple) to prove first the existence of a weak solution. Then one attempts to show that this solution is regular.

§ 3. In considering second order equations I shall divide the results to be described into two classes; those similar to results for linear equations and others which are more nonlinear—starting with the former. In seeking bounds for solutions of equations (2.1) and (2.2) one has first, in case the maximum principle applies, a bound for the solution itself. In the 1930's Schauder developed an extensive theory for linear equations with Hölder continuous(1) coefficients (this has been extended in [1]). Applying this theory one obtains bounds for all derivatives of a solution of (2.1) if a bound

(1) A function v is Hölder continuous, or satisfies a Hölder condition if

$$\sup \frac{|v(x)-v(y)|}{|x-y|^\alpha} < \infty, \quad \alpha < 1.$$

for its first derivatives and their Hölder continuity is known. So we shall restrict ourselves to these. For two dimensional, ($n=2$) problems, after Schauder and Leray (see C. Miranda [36] for other references) the basic work in getting bounds for solutions, from which much further work stems is due to C. B. Morrey [37–38]. This work is connected with the theory of quasi-conformal mappings (see also L. Bers, L. Nirenberg [2]). In higher dimensions little was known (see, however, H. O. Cordes [5] where other references may be found) until in 1957 E. de Giorgi [6] (and in 1958 John Nash [47], for parabolic equations, which we mention later) succeeded in estimating the Hölder continuity in compact subsets of solutions u of linear elliptic equations in divergence, i.e. in the weak, form

$$\int a_{ij} u_{x_i} \zeta_{x_j} dx = 0 \quad \text{for all } \zeta \in C_0^\infty, \tag{3.1}$$

assuming the equations to be uniformly elliptic, i.e. to satisfy

$$m \sum_j \xi_j^2 \leqslant a_{ij} \xi_i \xi_j \leqslant M \sum \xi_j^2,$$

for suitable positive constants m, M. They showed that a solution u in W_2^1 is Hölder continuous. De Giorgi used this result to prove the analyticity of solutions of uniformly elliptic variational problems,

$$\delta \int F(Du)\, dx = 0.$$

Any first derivative u_{x_k} of such a solution satisfies an equation of the form (3.1), namely

$$\int F_{u_i u_j} u_{x_k x_i} \zeta_{x_j} dx = 0 \quad \text{for all } \zeta \in C_0^\infty$$

and u_{x_k} is easily seen to belong to W_2^1. Therefore u_{x_k} is Hölder continuous, and the analyticity of u then follows from previously known results. The proof of Hölder continuity of solutions of (3.1) involves two steps:

(a) Obtaining a bound for u, of the form

$$|u(x)|^2 \leqslant \frac{\text{const}}{R^n} \int_{|y-x|<R} |u(y)|^2 dy,$$

(b) Then deriving a Hölder condition for u.

After de Giorgi simpler proofs have been given by J. Moser [42–43] and G. Stampacchia [56]. Stampacchia's proof of (a), which is closer to de Giorgi's, involves choosing for ζ in (3.1) a C_0^∞ function ψ times a truncation of $u, u^k = \max(u-k, 0)$. This choice gives immediately a bound on the L_2 norm of grad u^k. Using Sobolev's inequalities one obtains a bound for the measure σ_k of the set where $u > k$. Repeating this argument again one obtains an estimate for σ_h in terms of σ_k, $h > k$ and one finds that for $h \geqslant$ some $k_0, \sigma_h = 0$, i.e. $u \leqslant h$ almost everywhere. Moser's proof of (a) works with the L_p norm of u in place of σ_h; he chooses for ζ a function $\psi \in C_0^\infty$ times a power of u (note that though this is a linear problem nonlinear operations are performed), and shows that the L_p norm of u on a compact subset remains bounded as $p\to\infty$.

Stampacchia (see his Congress lecture and [57] has used his method of proof to prove the following

Maximum principle: Let $u\in W_2^1$ be a weak solution in a domain $\mathfrak{D}$ of

$$(a_{ij}u_{x_i}+f_j)_{x_j}\geqslant 0,\ a_{ij}\xi_i\xi_j\geqslant m\xi_i^2,\quad u=\phi\quad\text{on boundary}\tag{3.2}$$

i.e.
$$\int(a_{ij}u_{x_i}+f_j)\zeta_{x_j}dx\leqslant 0\ \text{for}\ \zeta\geqslant 0,\quad \zeta=0\quad\text{on boundary.}\tag{3.2$'$}$$

then
$$u\leqslant\max\phi+\frac{K(\mathfrak{D},p)}{m}\sum_j|f_j|_{L_p}\quad\text{for}\quad p>n.$$

We may give a brief sketch of the proof. Take for ζ the function $\max(u-k,0)$ with $k>\max\phi$. Then from (3.2′) we find, letting D_k denote the set where $u>k$, and σ_k its measure

$$m\int_{D_k}|\operatorname{grad}u|^2dx\leqslant\int_{D_k}|\sum f_ju_{x_j}|\,dx\leqslant\left(\int_{D_k}|\operatorname{grad}u|^2dx\right)^{1/2}\left(\int\sum|f_j|^2dx\right)^{1/2}$$

or, by Hölder's inequality,

$$m\left[\int_{D_k}|\operatorname{grad}u|^2dx\right]^{1/2}\leqslant(\sum|f_j|_{L_p})\,\sigma_k^{(p-2)/2p}.$$

By Sobolev's inequality it follows that for some constant C

$$\left(\int_{D_k}|u-k|^qdx\right)^{1/q}\leqslant\frac{C}{m}\sum|f_j|_{L_p}\sigma_k^{(p-2)/2p},\quad\text{for}\quad q=\frac{2n}{n-2}.$$

Therefore restricting the integration on the left to D_h, for $h>k$ we find

$$\sigma_h^{(n-2)/2n}\leqslant\frac{\frac{C}{m}\sum|f_j|_{L_p}}{h-k}\,\sigma_k^{(p-2)/2p}\cdot h>k.$$

For $p>n$ the power of σ_k on the right is greater than that of σ_h on the left and one shows consequently, fairly readily, that

$$\sigma_h=0\quad\text{for}\quad h\geqslant\max\phi+\frac{\text{const}}{m}\sum|f_j|_{L_p}$$

which is the desired result.

The most elegant proof of the Hölder continuity (b) has been given by Moser in [43] (his second proof). It is based on a

Harnack inequality: any nonnegative solution u of

$$(a_{ij}u_{x_i})_{x_j}=0,\quad m\sum\xi_j^2\leqslant a_{ij}\xi_i\xi_j\leqslant M\sum\xi_j^2$$

in a unit sphere, satisfies in any concentric smaller sphere S

$$\max_S u \leqslant C \min_S u,$$

where C depends only on m, M and S.

This yields easily the fact that if u is a bounded solution of (3.1) in a sphere then its oscillation (max-min) in a concentric sphere of half the radius is bounded by $\theta(<1)$ times its oscillation in the full sphere. Hölder continuity then follows easily in turn from this result.

(Moser's proof makes use of a lemma by F. John and L. Nirenberg [25] which has also been used by F. John in an interesting paper [24] studying (nonlinear) mappings of a domain in E^n into E^n which differ little from an isometry.

LEMMA. *Suppose u is an integrable function in a cube C_0; denote by u_c the average of u in a parallel subcube C. Assume that for every such subcube C the inquality*

$$(|u-u_C|)_C \leqslant 1$$

holds then $u \in L_p$ for avery p, in fact $\int_{C_0} e^{a|u-u_{C_0}|}dx \leqslant$ const, for some suitable constant $a>0$.)

The results of de Giorgi have also been extended to hold in the full domain (not just compact subsets) for solutions satisfying various boundary conditions. These extensions yield existence theorems for uniformly elliptic equations (2.2) and (2.4). The results have also been extended to certain nonuniformly elliptic equations. The main extensions of this kind, and applications to boundary value problems, have been given by C. B. Morrey, O. A. Ladyzhenskaya and N. N. Uraltseva, O. A. Oleinik and C. N. Kruzhkov. The equations are not required to be uniformly elliptic but, say for (2.2), the coefficients a_i are permitted to behave like polynomials in the u_{x_i}. The conditions on the coefficients are too complicated to be stated here, but there are examples showing that these conditions are rather natural. I mention here just one condition involved in one case for (2.2), here $V^2=1+|u|^2+|\text{grad } u|^2$, $\xi=(\xi_1,\dots,\xi_n)$ a real vector;

$$mV^k|\xi|^2 \leqslant \frac{\partial a_i}{\partial u_{x_j}}\xi_i\xi_j \leqslant MV^k|\xi|^2 \quad (k>-1),$$

and a_i is permitted to grow like V^{k+1}. In case (2.2) comes from a variational problem (2.3) the further conditions express the requirement that the functional $\int F(x,u,Du)\,dx$ be of class C^2 in the space W^1_{k+2}.

This work, together with complete references, may be found in the clear expository article by Ladyzhenskaya and Uraltseva [29], in the papers [40–41] by Morrey and in Oleinik, Kruzhkov [50].

I will just mention one result from the paper [29]. Consider the simple nonlinear equation:

$$\Delta u = \text{quadratic in the } u_{x_i}, u=\phi \text{ on boundary}$$

(here Δ represents the Laplace operator, this is a very special case of a result in [29]). It is known, and in fact not difficult to derive with the aid of the interpolation inequalities mentioned above, that if ϕ is small, or if the coefficients in the quadratic are small, or even if the oscillation of u in a fixed region is small, then one can obtain a priori estimates for all derivatives of u and prove the existence of a solution. If the quadratic is replaced by something of higher degree in u_{x_i} then a regular solution need not exist, while for lower degree it is quite easy to prove existence. It has been an open question for some time to see if there is existence in the case of a general quadratic. This is settled in the affirmative in [29] by Ladyzhenskaya and Uraltseva. The main step is an estimate of the Hölder continuity of u, and its derivation is similar to their extension of de Giorgi's work, and is therefore not very simple.

G. Stampacchia [57] and D. Gilbarg [16] have also obtained existence theorems with the aid of de Giorgi's result and its extensions. In particular for equations in a strictly convex domain they have independently obtained some very clean results. To mention one, consider a variational problem

$$\delta \int F(u_{x_i})\,dx = 0, \quad u = \phi \text{ (smooth) on boundary,}$$

which is supposed to be regular, i.e. for which the Euler equation is elliptic, $\partial^2 F/\partial u_{x_i}\partial u_{x_j}$ is positive definite—then F is a convex function. They prove that there exists a unique (regular) solution of this problem. The proof is based on the a priori estimate

$$|u| + |\text{grad}\, u| \leqslant K(\mathfrak{D},\phi)$$

where the constant K depends only on the (strictly convex) domain $\mathfrak{D}$ and the boundary function ϕ, K does not depend on the function F. The estimate is easily derived with the aid of the maximum principle (which holds also for grad u) and a comparison function which is linear. Using the estimate Stampacchia's existence proof proceeds as follows: He modifies the function F for $|\text{grad u}| \geqslant K$ so that the new function $\tilde{F}$ is uniformly regular, i.e. the eigenvalues of the Hessian matrix are bounded from above and below by fixed positive constants; this indeed is the main step of the proof. For the variational problem $\delta\int \tilde{F}(u_{x_i})dx = 0, u = \phi$ on boundary, it is very easy to find a weak solution in W_2^1. Using the extension of de Giorgi's result to the boundary for uniformly elliptic problems it follows that this solution is regular. But it must also satisfy the a priori estimate above, and hence is seen to be a solution of the original problem.

A word about parabolic equations. Using quite different methods J. Nash [47] proved the analogue of de Giorgi's result for parabolic equations in full space of the form

$$\frac{\partial u}{\partial t} = (a_{ij} u_{x_i})_{x_j}.$$

This has been generalized by Oleinik and Kruzhkov [50], and rederived and extended by Ladyzhenskaya and Uraltseva [30] by adapting de Giorgi's methods. The analogue of the Schauder theory for linear equations had been established by A. Friedman [11] who treated also nonlinear problems [12]. Various existence (as well as nonexistence theorems), a priori estimates and stability theorems, have also been given by A. F. Filippov [8] and S. Kaplan [26]. Unfortunately we cannot describe these results here.

What about higher order elliptic equations? Very little is known, but recently M. I. Vishik [59–60] has made an interesting beginning by proving existence of weak solutions for certain classes of quasilinear systems (written in divergence form) whose coefficients behave essentially like polynomials in the unknown functions and their derivatives. The analogue of the de Giorgi results, giving regularity, has yet to be found.

§ 4. I would like to turn now to second order elliptic equations which, are "truly nonlinear", in the sense that one has new phenomena occurring which have no analogue for linear equations. The best known specimen (of which most of the others are variations and generalizations) is the minimal surface equation, expressing the fact that the surface is a solution of the variational problem

$$\delta \int dA = 0,$$

where dA is element of surface area. If the surface has simple projection on the (x,y) plane, i.e admits the representation $u=u(x,y)$ in (x,y,u) space, then u satisfies

$$(1+q^2)u_{xx} - 2pqu_{xy} + (1+p^2)u_{yy} = 0, \quad p = u_x, \quad q = u_y.$$

Geometrically this asserts that the spherical image mapping of the surface by the unit normal to the unit sphere is conformal. For this equation have been known for some years a variety of results showing marked difference from linear equations. To mention just a few (references can be found in the papers listed later):

1. S. Bernstein's classical theorem that the only solution defined over the whole plane is a linear function.
2. Removability of isolated singularities (L. Bers).
3. It is possible to estimate the Gauss curvature of the surface $u=u(x,y)$ defined in $x^2+y^2<R^2$ at $(0,0)$ in terms of the gradient of u at $(0,0)$. (E. Heinz, E. Hopf.) On letting $k \to \infty$ this estimate yields another proof of Bernstein's result 1.

In recent years these results have been extended in various directions. R. Osserman [51] showed that Bernstein's theorem holds in general for a *complete* (i.e. geodesics can be extended to have infinite length) minimal surface which need not have a simple projection on a whole plane, provided that the spherical image of the surface deletes an open set on the sphere. This was done by showing that such a (simply connected) surface, when regarded as a Riemann surface, is of parabolic type. He also obtained estimates for the Gauss curvature analogous to those in 3.

R. Finn has treated boundary value problems, obtained estimates as well as the removable singularity theorem, and other results, for a class of equations which he calls of minimal surface type (see [9] and other papers by Finn)

$$a(p,q)u_{xx} + 2b(p,q)u_{xy} + c(p,q)u_{yy} = 0$$

of the form

$$(A(p,q))_x + (B(p,q))_y = 0.$$

In this work one uses results for quasi-conformal mappings. Some of his results are new also for minimal surfaces.

I would like to call attention to an interesting paper by H. B. Jenkins [23] in which he treats a class of *nonparametric* variational problems of the form

$$\delta \int F(\text{unit normal})\, dA = 0,$$

where F is homogeneous function of three variables of degree zero such that $o < m \leqslant F \leqslant M$ and the surface $|x| F(x) = 1$, is closed convex with Gauss curvature > 0; $F = 1$ for minimal surfaces. For such variational problems, to which he shows that the theory of Finn applies, he derives a variety of interesting results including Osserman's extensions of Bernstein's result. More recently Jenkins together with J. Serrin have derived a priori estimates for derivatives of solutions of such variational problems, improving Finn's results and, in particular, have derived an interesting *Harnack inequality* for solutions with simple projection on a disc, i.e. solutions of the form

$$u = u(x, y) > 0 \quad \text{in } \textit{unit circle};$$

namely $$\phi_1(r, u(0)) \leqslant u(x, y) \leqslant \phi_2(r, u(0)),\ r = \sqrt{x^2 + y^2},$$

where however $\phi_2(r, u(0)) \to \infty$ *as* $r \to r_0(u(0)) < 1$.

This behavior of ϕ_2 is also shown to be appropriate even for minimal surfaces. Furthermore they show that

$$r_0(t) \to 0 \text{ as } t \to \infty \quad \text{and} \quad r_0(t) \to 1 \quad \text{as} \quad t \to 0.$$

Finn has also recently obtained extensions of his earlier results.

Finally I would like to mention the deep and difficult work of E. Heinz in a series of papers [18–20] on elliptic Monge–Ampère equations in the plane

$$u_{xx} u_{yy} - u_{xy}^2 + \text{quasilinear second order expressions} = 0.$$

In these papers Heinz extends the important work of Hans Lewy on such equations which are analytic to non analytic equations, deriving estimates for derivatives of solutions. This work really deserves more time here but the results are complicated to state. Let me just say that Heinz makes essential use of characteristic coordinates for the equation, i.e. new coordinates (depending on the solution) which reduce the leading part of the equation regarded as linear to the Laplace operator.

To mention just one of his simpler results, proved with the aid of characteristic coordinates and some theorems on quasi-conformal mappings, he shows [21] that if

$$0 < \alpha \leqslant u_{xx} u_{yy} - u_{xy}^2 \leqslant \beta < \infty \quad \text{and} \quad |u| \leqslant \delta$$

in a circle then in any compact subset the first derivatives of u (which are easily estimated a priori because of convexity of $u = u(x, y)$) satisfy also a fixed Hölder condition.

I might also mention that in [17] Heinz proved the existence of surfaces of constant mean curvature spanning a given closed curve in space.

§ 5. I have said nothing about nonlinear hyperbolic equations. Recently using extensions of Sobolev's inequalities L. Gårding and Leray [15] and P. A. Dionne [7] have treated the existence and uniqueness of regular solution for the initial value problem and have reduced it to the possibility of deriving a priori estimates of a certain number of derivatives of the solutions —in case one wants global solutions.

Besides that much work has been done in recent years on the very important initial value problem involving discontinuous initial data. Here, in contrast to elliptic problems one seeks some form of generalized solution admitting discontinuities—shocks to be exact. The basic problem of finding the appropriate class of generalized solution in which one has existence and uniqueness has still not been solved except in special, though very interesting cases. In particular in the case of more than one space variable essentially nothing is known. There are expository papers on the subject by P. D. Lax [31], O. A. Oleinik [49], and B. L. Rozhdestvenskii [53].

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