LINEARIZATION AND DELINEARIZATION

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1. Introduction

My terms of reference, as prescribed by our President, are to survey problems of applied mathematics which still challenge the pure mathematician. This is an agreeable exercise for it enables me to range over a wide field, to select such topics as fancy and caprice may dictate, and above all to shun the rigours of precise proof and detailed definition.

The group of problems which I propose to describe belong to that Cinderella of pure mathematics—the study of differential equations. The closely guarded secret of this subject is that it has not yet attained the status and dignity of a science, but still enjoys the freedom and freshness of such a pre-scientific study as natural history compared with botany. The student of differential equations—significantly he has no name or title to rank with the geometer or analyst—is still living at the stage where his main tasks are to collect specimens, to describe them with loving care, and to cultivate them for study under laboratory conditions. The work of classification and systematization has hardly begun.

This is true even of differential equations which belong to the genus technically described as 'ordinary, linear equations'. The morphology of this genus has progressed only as far as equations which possess three or at most four regular singularities. In the case of non-linear equations, Lie's theory of transformation groups has done little but suggest a scheme of classification. An inviting flora of rare equations and exotic problems lies before a botanical excursion into the non-linear field.

I propose today to speak of some linear and non-linear differential equations as they arise in mathematical physics, with an eye to the unsolved analytical problems which they present.

The history of mathematical physics during the last century may be divided into two periods—the linear period and the non-linear period. In those happy far-off times of the linear period, all differential equations were linear and the principle of superposition reigned supreme. In the present distressful times most differential equations are non-linear and no effective general method of solution has yet been proposed. We have, however, two practical expedients—the method of linearization by which non-linear equations are forcibly reduced to an associated, approximate

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linear form, and the method of delinearization by which the non-linearities are partially restored.

Linearization and delinearization are the main topics of my address, especially in relation to the equations of fluid dynamics, but perhaps it is desirable to illustrate the nature of the problems involved by some trivial examples.

2. Regular and singular perturbations

Consider the ordinary differential equation of the first order

$$du/dx = F(x, u, \alpha),$$

in which α is a small parameter. The classical existence theorem can be easily proved by the use of dominant functions (Goursat[7]). It shows that, if F is an analytic function of x, u and α in the neighbourhood of a point $x = x_0$, $u = u_0$ and $\alpha = \alpha_0$, then the differential equation possesses a solution $u = u(x, \alpha)$, which is analytic in some neighbourhood of the point $x = x_0$, and such that $u_0 = u(x_0, \alpha)$, if α is in some neighbourhood of α_0. In the type of problem which we wish to study we are especially interested in the solution for small values of α, and therefore in the 'reduced equation'

$$du/dx = F(x, u, 0),$$

which, in practice, is often much simpler than the original 'perturbed equation', $du/dx = F(x, u, \alpha)$. The solution of the reduced equation is called the 'basic solution', $f_0(x) = u(x, 0)$. It is clear that the perturbed equation will possess a solution of the form

$$u = f_0(x) + \alpha f_1(x) + \ldots + \alpha^n f_n(x) + \ldots,$$

convergent in some interval $|\alpha| < \rho$, and reducing to u_0 at $x = x_0$, if the function F is analytic in a neighbourhood of $x = x_0$, $u = u_0$, $\alpha = 0$. Also the leading term $f_0(x)$ will then satisfy the reduced equation. In this case the perturbation is said to be 'regular' at (x_0, u_0). But if the function F is not analytic in a neighbourhood of $x = x_0$, $u = u_0$, $\alpha = 0$, the perturbation is said to be 'singular'. The classical existence theorem then applies no longer. This is the interesting case which frequently arises in applied mathematics.

There is one obvious method of dealing with singular perturbations—it is to find a transformation which will result in an equation (or equations), for which the perturbation is regular.

Consider, for example, the trivial equation

$$(x + \alpha)\, du/dx + u = 0,$$

with the reduced equation

$$x\,du/dx+u = 0,$$

and the initial conditions $\quad x = 1, \quad u = 1.$

The full perturbation equation is regular everywhere in the x, u-plane except at $x = 0$. The solution of the perturbed equation is

$$u(x,\alpha) = (1+\alpha)/(x+\alpha),$$

while the basic solution is $\quad f_0(x) = x^{-1}.$

The relation between the basic solution and the perturbed solution is that

$$u(x,\alpha)-f_0(x) = \frac{\alpha(1-x)}{x(x+\alpha)} = O(\alpha),$$

but the approximation indicated by the order term is not uniformly valid for all values of x. In fact it is uniformly valid only in domains which exclude $x = 0$ and $x = -\alpha$.

If, however, we express the equation and solution in inverted form as

$$u\,dx/du+x+\alpha = 0$$

and

$$x(u,\alpha) = -\alpha+(1+\alpha)\,u^{-1},$$

then the perturbation is regular, the basic solution is

$$x(u,0) = u^{-1},$$

and

$$x(u,\alpha)-x(u,0) = -\alpha+\alpha u^{-1} = O(\alpha),$$

uniformly in a neighbourhood of $\alpha = 0$.

3. Neighbouring solutions

If $u = u(x,\alpha)$ is an integral curve of

$$du/dx = F(x,u,\alpha),$$

which passes through a point (x_0, u_0) in a region D in which the differential equation is regular, then

$$u(x,\alpha)-u(x,0) = O(\alpha) \tag{3.1}$$

uniformly in D. But, as the preceding example shows, this is no longer true if D contains points at which the differential equation is singular.

The significance of the relation (3.1) is that the integral curves $u = u(x,\alpha)$ and $u = u(x,0)$ are 'neighbouring curves', with ordinates differing by $O(\alpha)$ in D. But the preceding example suggests at once a more general concept of neighbourliness.

Elementary geometrical considerations applied to a system of curves

$$\phi(x, u, \alpha) = 0$$

suggest that the curves

$$\Gamma \quad \text{or} \quad \phi(x, u, 0) = 0,$$

and

$$C \quad \text{or} \quad \phi(x, u, \alpha) = 0,$$

should be regarded as 'neighbouring' in a region D, if, with any point (ξ, η) on Γ, we can associate a point (x, u) on C such that

$$x - \xi = O(\alpha),$$

and

$$u - \eta = O(\alpha),$$

uniformly in D. In the preceding example

$$u = \eta, \quad x - \xi = -\alpha(1-\xi).$$

This then suggests that the whole system of curves

$$\phi(x, u, \alpha) = 0$$

should be regarded as a system of neighbouring curves if they can be represented in the parametric form

$$x = X(z, \alpha), \quad u = U(z, \alpha),$$

where X and U are analytic functions of z and α reducing to ξ and η respectively when $\alpha = 0$.

Since ξ and η are connected by the relation $\phi(\xi, \eta, 0) = 0$ this representation is equivalent to

$$x = \xi + \sum_1^\infty \alpha^n x_n(\xi), \quad u = \eta(\xi) + \sum_1^\infty \alpha^n u_n(\xi),$$

in a region where $d\eta/d\xi$ is bounded.

Although there is no *a priori* reason to assert that the solutions of a given singular perturbation problem must form a system of neighbouring curves, the preceding ideas do provide a powerful and flexible technique for searching for solutions and approximations which are uniform within a region containing singular points.

This technique is due to Lighthill and has received numerous applications in fluid dynamics. It is reminiscent of the method of small perturbations employed by Poincaré[16], but the motivation of Poincaré's work was not any singularity in the perturbation but practical convenience in calculating the period of non-linear oscillations.

4. Uniformization

If the original perturbation equation

$$du/dx = F(x, u, \alpha) \quad (u = u_0 \text{ at } x = x_0),$$

possesses a system of neighbouring solutions

$$x = x(\xi, \alpha), \quad u = \eta(\xi, \alpha),$$

then the equations for $x(\xi, \alpha)$ and $\eta(\xi, \alpha)$ must be regular in a neighbourhood of $\xi = x_0$, $u = \eta(x_0, 0)$. The search for systems of neighbouring solutions therefore depends upon the introduction of a new variable ξ and the replacement of the original equation

$$du/dx = F(x, u, \alpha)$$

by two new equations,

$$dx/d\xi = X(\xi, x, u, \alpha), \quad du/d\xi = U(\xi, x, u, \alpha),$$

regular in α.

This process may be called the 'uniformization' of the original equation, and it is equivalent to the method introduced by Lighthill[13].

Thus a typical equation discussed by Lighthill

$$(x + \alpha u)\, du/dx + q(x)\, u = r(x)$$

possesses the uniformizing equations

$$\frac{du}{d\xi} = r(x) - q(x)\, u,$$

$$\frac{dx}{d\xi} = x + \alpha u.$$

These equations are manifestly analytic in α, and in fact their solutions are precisely those given by Lighthill if we write $\xi = \log z$.

Consider for example, the equation

$$(x + \alpha u)\, du/dx + (2 + x)\, u = 0,$$

with the condition $\quad u = e^{-1} \quad \text{at} \quad x = 1.$

The reduced equation $\quad x\, du/dx + (2 + x)\, u = 0$

has the solution $\quad u = x^{-2} e^{-x}.$

To obtain the solution of the perturbed equation which is valid near $x = 0$ we must therefore uniformize by introducing an auxiliary variable.

To facilitate comparison with Lighthill's solution[13] (p. 1190) we write

$$z\,dx/dz = x+\alpha u, \quad z\,du/dz = -(2+x)\,u,$$

with

$$x = 1, \quad u = e^{-1} \quad \text{at} \quad z = 1.$$

These equations are analytic in α and possess solutions of the form

$$x = x_0+\alpha x_1+\ldots, \quad u = u_0+\alpha u_1+\ldots,$$

where

$$x_0 = z, \quad u_0 = z^{-2}\,e^{-z}, \quad x_1 = z\phi(z),$$

$$u_1 = -z^{-2}e^{-z}\int_1^z \phi(t)\,dt, \quad \phi(z) = \int_1^z s^{-4}e^{-s}\,ds.$$

Hence, near $z = 0$,

$$x = z-\tfrac{1}{3}\alpha z^{-2}+O(\alpha^2/z^4),$$

$$u = z^{-2}-\tfrac{1}{6}\alpha z^{-4}+O(\alpha^2/z^6),$$

and, at $x = 0$,

$$u = (3/\alpha)^{\frac{2}{3}}+O(\alpha^{-\frac{1}{3}}).$$

The method of uniformization suggested here systematizes Lighthill's method of expansion in powers of an auxiliary variable. Its main advantage is that it establishes the *existence* of a solution which is analytic in the small parameter, without becoming embroiled in the details of its *computation.*

5. Singular boundary conditions

The singular perturbation equations which arise in fluid dynamics are often of a rather different character from those discussed above. In the first place they are usually of the second order, and in the second place the singularity is not in the equation but in the boundary conditions.

The first difference is of little importance. An equation of the second order can be replaced by a pair of equations of the first order, e.g.

$$F(x, u, v, du/dx, \alpha) = 0, \quad dv/dx = u.$$

The process of uniformization then consists in introducing an auxiliary variable z in such a way that the original system of equations is replaced by a system

$$dx/dz = X(z, x, u, v, \alpha),$$

$$du/dz = U(z, x, u, v, \alpha),$$

$$dv/dz = uX,$$

which is analytic in α.

The second difference is much more significant and a systematic examination of this question is lacking.

A survey of those problems of compressible fluid flow which can be reduced to ordinary differential equations has been given by Lighthill[12]. Some of these require the location of a shock wave and involve singular boundary conditions.

A striking example given by Lighthill[13] refers to the waves produced in still air by the slow uniform expansion of a circular cylinder with radial velocity αa_0, a_0 being the speed of sound in the undisturbed air, and α a small parameter. The velocity potential has the form

$$\phi = a_0^2 t f(x),$$

where t is the time since the cylinder was of zero radius, and $x = r/(a_0 t)$. The disturbed region is bounded externally by a shock wave at $r = Ma_0 t$ or $x = M$, and internally by the surface of the cylinder $r = \alpha a_0 t$ or $x = \alpha$. The main problem is to calculate M for small values of α.

Bernoulli's equation gives the local speed of sound in the form

$$a = a_0\{1-(\gamma-1)(f-xf'+\tfrac{1}{2}f'^2)\}^{\frac{1}{2}},$$

while the potential equation is

$$a^2 \operatorname{div}\operatorname{grad}\phi = \ddot{\phi}+2\phi_r\dot{\phi}_r+\dot{\phi}_r^2\phi_{rr}.$$

Hence $$\{1-(\gamma-1)(f-xf'+\tfrac{1}{2}f'^2)\}(f''+x^{-1}f') = (x-f')^2 f''.$$

The boundary conditions are

(1) at $x = \alpha$, $\quad f'(\alpha) = \alpha$,

(2) at $x = M$, $\quad f(M) = 0$,

and $$f'(M) = 2(M-M^{-1})/(\gamma+1).$$

To put the differential equation in standard form we write $f' = u$, $f = v$, whence

$$P\,du/dx+Qu/x = 0, \quad dv/dx = u,$$

where $$P = 1-x^2+(\gamma+1)xu-(\gamma-1)v-\tfrac{1}{2}(\gamma+1)u^2,$$

and $$Q = 1+(\gamma-1)(xu-v-\tfrac{1}{2}u^2).$$

If we linearize this equation we find that

$$(1-x^2)\,du/dx+u/x = 0,$$

whence $$|u| = C\,|x^{-2}-1|^{\frac{1}{2}},$$

and $$C = \alpha^2(1-\alpha^2)^{-\frac{1}{2}}.$$

It is then obvious that this approximation fails as we approach the upper limit $x = M$. We therefore proceed to uniformize the equation by writing

$$z\,dx/dz = x/Q, \quad z\,du/dz = -u/P,$$

and we construct solutions of the form

$$x = z + \alpha^2 x_1 + \alpha^4 x_2 + \ldots,$$

$$u = \alpha^2 u_0 + \alpha^4 u_1 + \ldots,$$

$$v = \alpha^2 v_0 + \alpha^4 v_1 + \ldots.$$

This preserves the solution of the linearized equation in the leading terms, with

$$u_0 = (z^{-2} - 1)^{\frac{1}{2}}.$$

The solution then follows the lines of Lighthill's argument[13] (p. 1191) and finally yields

$$M = 1 + \tfrac{3}{8}(\gamma + 1)^2 \alpha^4 + \ldots.$$

6. Perturbations which are singular almost everywhere

A specially interesting type of perturbation equation is one which is singular everywhere in the x, u-plane except on a certain curve C, e.g. the equation

$$\alpha\,du/dx = F(x, u).$$

A classical example occurs in the theory of relaxation oscillations of the type studied by van der Pol. Here the perturbation equation can be expressed in the form

$$\alpha u\,du/dx = u - \tfrac{1}{3}u^3 - x = F(x, u).$$

The periodic solution is represented approximately by a closed curve in the x, u-plane, consisting of certain arcs of the curve $F(x, u) = 0$ and of certain straight lines parallel to the x-axis (Stoker[18], p. 128).

The basic equation

$$F = 0 \quad \text{or} \quad x = u - \tfrac{1}{3}u^3$$

provides an approximate solution except near the points where

$$dx/du \equiv 1 - u^2$$

vanishes, i.e. at

$$x = \pm\tfrac{2}{3}, \quad u = \pm 1.$$

Near these points uniformization is easily carried out by employing the Carrier 'two-way stretch'[4],

$$x = \pm\tfrac{2}{3} + \alpha^m \xi, \quad u = \pm 1 + \alpha^n \eta,$$

with suitable exponents m and n, chosen so as to make the resulting equation regular. The simplest choice is

$$m = 1, \quad n = 0,$$

which yields the regular equation

$$(\pm 1 + \eta)\, d\eta/d\xi = \mp \eta^2 - \tfrac{1}{3}\eta^3 - \alpha\xi.$$

7. The thin aerofoil problem

Although a number of interesting and important problems in fluid dynamics involving singular perturbations of *partial* differential equations have been examined and uniformized by Lighthill[13], Carrier[4] and Whitham ([21] and numerous subsequent papers), the theory is in a much less advanced state than the corresponding theory for *ordinary* equations. It therefore seems preferable to give just a few specific examples.

In the first place we consider the problem of a thin two-dimensional symmetric aerofoil (or strut) with profile

$$y = \pm \alpha f(x) \quad (0 \leqslant x \leqslant 1),$$

placed in a uniform stream of incompressible, inviscid fluid with velocity components $(U\cos\alpha,\ U\sin\alpha)$ at infinity[14]. The potential $\alpha\phi$ of the disturbance velocity satisfies the equation

$$\phi_{xx} + \phi_{yy} = 0,$$

and the boundary conditions

$$\phi = O(R^{-1})$$

for large

$$R = (x^2 + y^2)^{\frac{1}{2}},$$

and

$$(U\sin\alpha + \alpha\phi_y) = \pm (U\cos\alpha + \alpha\phi_x)\,\alpha f'(x),$$

on the surface of the aerofoil. Now near the leading edge

$$[f(x)]^2 = c^2 x + O(x^2) \quad (c \neq 0),$$

and

$$f'(x) = O(x^{-\frac{1}{2}}).$$

The surface boundary condition is therefore singular at the leading edge.

The reduced boundary condition is

$$(U + \phi_y) = \pm U f'(x),$$

to be satisfied on the x-axis, $y = 0$, $0 \leqslant x \leqslant 1$, and it is this equation together with the potential equation for ϕ which forms the basis of 'thin aerofoil theory'. The main problem is to improve this approximation without making a completely fresh start.

The boundary conditions can be uniformized by introducing parabolic co-ordinates ξ, η, where

$$x+iy-\tfrac{1}{4}\alpha^2c^2 = c^2(\xi+i\eta)^2.$$

Then
$$x = c^2(\xi^2-\eta^2+\tfrac{1}{4}\alpha^2), \quad y = 2c^2\xi\eta,$$

and the parabola $\eta = \frac{1}{2}\alpha$ osculates the leading edge section

$$y^2 = \alpha^2c^2x+O(x^2).$$

Hence in parabolic co-ordinates the profile $y = \alpha f(x)$ has an equation of the form

$$\eta = \tfrac{1}{2}\alpha+\alpha P(\xi)$$

$$= \tfrac{1}{2}\alpha+\alpha \sum_{n=1}^{\infty} c_n \xi^n,$$

and the exact boundary condition becomes

$$\{-2Uc^2\eta\cos\alpha+2Uc^2\xi\sin\alpha+\alpha\phi_\eta\}$$
$$= \alpha P'(\xi)\{2Uc^2\xi\cos\alpha+2Uc^2\eta\sin\alpha+\alpha\phi_\xi\}.$$

This condition is regular, and hence the problem admits a solution of the form

$$\phi = \phi_0(\xi,\eta)+\alpha\phi_1(\xi,\eta)+\dots.$$

8. The boundary layer on a flat plate

Another problem which exemplifies the techniques of both Lighthill and Carrier is that of the steady flow of an incompressible, viscous fluid past a semi-infinite flat plate

$$y = 0, \quad x \geqslant 0,$$

placed parallel to the main stream. The natural units of length, velocity and pressure are ν/U, U and ρU^2, where ν is the kinematic viscosity, U the main stream velocity and ρ the density. In terms of these units the Navier–Stokes equations for the pressure p and the components of fluid velocity are

$$uu_x+vu_y = -p_x+\Delta u,$$

$$uv_x+vv_y = -p_y+\Delta v,$$

where
$$\Delta u = u_{xx}+u_{yy}.$$

The boundary conditions are

$$u \to 1, \quad v \to 0 \quad \text{as} \quad x^2+y^2 \to \infty,$$

except on the flat plate $y = 0$, $x \geqslant 0$ where

$$u = 0, \quad v = 0.$$

There is no parameter in these equations or boundary conditions, but it is known from experiment that derivatives of u and v with respect to y are small compared with derivatives with respect to x, except at the leading edge, $x = 0$, $y = 0$, where presumably the dominant derivative is in the radial direction. These conditions are conveniently expressed in terms of parabolic co-ordinates ξ, η such that

$$x+iy = (\xi+i\eta)^2,$$

or

$$\xi = [\tfrac{1}{2}(r+x)]^{\frac{1}{2}}, \quad \eta = [\tfrac{1}{2}(r-x)]^{\frac{1}{2}},$$

where

$$r = [x^2+y^2]^{\frac{1}{2}}.$$

The stream function ψ is defined by the equations

$$u = \psi_y, \quad v = -\psi_x,$$

$\psi \sim y = 2\xi\eta$ for large $\xi^2+\eta^2$ ($\eta \neq 0$!), and itself satisfies the equation

$$\rho^2\Delta^2\psi - 4(\xi\Delta\psi_\xi+\eta\Delta\psi_\eta-\Delta\psi) = -\rho^2(\psi_\xi\Delta\psi_\eta-\psi_\eta\Delta\psi_\xi)+2(\eta\psi_\xi-\xi\psi_\eta)\Delta\psi,$$

where $\rho^2 = \xi^2+\eta^2$.

To identify the dominant terms we write

$$\eta = \epsilon\bar{\eta}, \quad \psi = \epsilon^{-1}\bar{\psi},$$

thus introducing a small parameter ϵ and thus obtaining a regular equation with parameter ϵ. On retaining the terms of lowest order (i.e. those in ϵ^{-5}) we obtain the reduced equation

$$\xi^2\bar{\psi}^{\text{iv}} = -\xi^2(\bar{\psi}_\xi\bar{\psi}'''-\bar{\psi}'\bar{\psi}_\xi'')-2\xi\bar{\psi}'\bar{\psi}'',$$

where accents indicate differentiation with respect to η. If we now write

$$\bar{\psi} = \xi f(\xi,\bar{\eta}),$$

we find that by a remarkable and unexpected simplification the function f satisfies the *ordinary* differential equation

$$f^{\text{iv}} = -ff'''-f'f'',$$

which integrates at once, in virtue of the boundary conditions at infinity, to

$$f'''+ff'' = 0.$$

This is the well-known Blasius equation, with the boundary conditions

$$f \sim 2\eta \quad \text{for large } \eta,$$

$$f = 0, \quad f' = 0 \quad \text{at} \quad \eta = 0.$$

The independent variable however is now

$$\eta = r^{\frac{1}{2}} \sin \tfrac{1}{2}\theta,$$

whereas in the classical Blasius problem it is

$$y/x^{\frac{1}{2}} = r^{\frac{1}{2}} \sin\theta/\sqrt{(\cos\theta)}.$$

The preceding analysis is due to Carrier and Lin[5] and there can be no doubt of the superiority of their solution of this problem over the classical solution given by Blasius[2]. A somewhat similar investigation, carried to the next order of approximation has been given by Kuo[11].

9. Accuracy of approximations

The preceding brief accounts of some methods and problems of interest to applied mathematicians will doubtless suggest many questions for the analyst, but the question of outstanding importance is surely that of the accuracy of the approximations obtained. The existence theorems which have been invoked do little more than guarantee the existence of solutions in the form of power series in the perturbation parameter α. The following questions arise at once:

(1) What is the radius of convergence of the power series?

(2) What is the rapidity of convergence?

(3) Is it possible to prescribe an upper bound to the absolute magnitude of the error which is involved in truncating the power series after N terms? And, in particular, can we do this for the 'basic solution' where $N = 1$?

A classical example of this problem is provided by the Blasius equation which is obtained as the 'reduced equation' from the Navier–Stokes equations for the flow of an incompressible, inviscid fluid past a semi-infinite flat plate. In this case, as in so many other physical problems, even the reduced equation is not linear.

The Blasius equation, as obtained in § 8, is

$$f''' + ff'' = 0,$$

primes indicating differentiation with respect to η, and a solution is required for the range $0 \leqslant \eta \leqslant \infty$, with the boundary conditions

$$f = 0 \quad \text{and} \quad f' = 0 \quad \text{or} \quad \eta = 0$$

and
$$f' \to 2 \quad \text{as} \quad \eta \to \infty.$$

There is a power series solution (obtained by Blasius[2]) in the form

$$f = c\left\{\frac{(c\eta)^2}{2!} - \frac{(c\eta)^5}{5!} + \frac{11(c\eta)^8}{8!} - \ldots\right\},$$

where
$$c^3 = f''(0).$$

Weyl[20] showed that the radius of convergence R of this power series in η satisfies the inequalities

$$18 < c^3R^3 < 60,$$

and Punnis[17] obtained the closer limits

$$3{\cdot}11 < cR < 3{\cdot}18,$$

by showing that the power series has a simple pole at $\eta = -R$. There is therefore a real problem for the analyst to determine the value of c so as to satisfy the condition $f' \to 2$ as $\eta \to \infty$, although the practical computer has little difficulty in obtaining the approximate value

$$f''(0) = 1{\cdot}328\ldots.$$

Quite another approach to this problem is provided by Weyl's transformation[20] of the differential equation into an integral equation of the form

$$\log F''(\eta) = \Phi(F'') = -\frac{1}{2}\int_0^\eta (\eta - s)^2 F''(s)\, ds,$$

where
$$f(\eta) = cF(c\eta),$$

and, as before,
$$c^3 = f''(0).$$

If an iteration process is specified by the conditions

$$F_0'' = 0, \quad F_{n+1}'' = \Phi(F_n''),$$

then
$$F_0'' < F_1'', \quad F_1'' > F_2'', \quad F_2'' < F_3'', \quad \text{etc.}$$

The sequence $\{F_n''\}$ converges and any two consecutive members form upper and lower bounds to the limit function. Moreover,

$$F_2''(\eta) = \exp\left(-\tfrac{1}{6}\eta^3\right),$$

is found to be an adequate approximation to the limit.

The use of integral equations of Weyl's type has been successfully exploited by Meksyn[15] in numerous papers on boundary layer theory, although the convergence of the iteration process still requires examination. As examples of other important investigations on approximate solutions of partial differential equations we may cite papers by Westphal[19] and Görtler[6].

10. Conclusion

In the preceding paper the name of 'delinearization' has been given to the process whereby we endeavour to return from a linearized approximation to the original non-linear equation. There is, however, another kind of delinearization, which, I venture to predict, will become increasingly important—namely a process whereby an exact linear equation is replaced by an exact non-linear equation. This apparently retrograde step is sometimes advantageous because good approximate solutions may be obtainable more easily for the non-linear equation than for the original linear equation.

One example is provided by the so-called Wentzel–Kramers–Brillouin method of solving the Schrödinger wave equation

$$\epsilon^2\psi'' + f(x)\,\psi = 0$$

for small values of the parameter ϵ. This method, due to Jeffreys[8, 9], consists in writing

$$\psi = \exp\left\{i\epsilon^{-1}\int \chi\,dx\right\},$$

and thus obtaining the Riccati equation

$$-\chi^2 + i\epsilon\chi' + f = 0,$$

with the series solution

$$\chi = \chi_0 - i\epsilon\chi_1 + \dots, \quad \chi_0 = f^{\frac{1}{2}}, \quad \chi_1 = -\tfrac{1}{2}\chi_0'/\chi_0.$$

Another example is derived from the new theory of diffraction problems, suggested by Birkhoff[1] and recently developed by Keller, Lewis and Seckler[10]. Here the wave equation for monochromatic light,

$$\operatorname{div}\operatorname{grad} u + k^2 u = 0,$$

is solved in the form

$$u \sim e^{ik\psi} \sum_{n=0}^{\infty} (ik)^{-n} v_n,$$

where $\psi, v_0, v_1, \dots$ satisfy the non-linear equations

$$(\operatorname{grad}\psi)^2 = 1,$$

$$2\operatorname{grad} v_n \,.\, \operatorname{grad}\psi + v_n \operatorname{div}\operatorname{grad}\psi = -\operatorname{div}\operatorname{grad} v_{n-1} \quad (v_{-1} \equiv 0\,!).$$

These examples do suggest that the eras of linear equations and of linearized non-linear equations may be succeeded by the era of delinearized linear equations.

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