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# Representations of Semisimple Lie Groups

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Fifty years ago, at the International Congress in Bologna, Hermann Weyl gave a report on representations of compact groups and, in particular, of compact Lie groups. Most of the important results had just been proved by him and by others, and at the time of his lecture, in 1928, the representation theory of compact Lie groups had become a very appealing subject. To a large extent, Weyl's theory has served as model and inspiration for the work on representations of noncompact, noncommutative groups, which was carried out in the last thirty years. To put the subject of my survey into perspective, I shall begin with a discussion of compact groups.

Initially, $G$ will denote a compact topological group, and $\hat{G}$ its set of isomorphism classes of irreducible unitary representations. To avoid complicated notation, I shall not distinguish between an isomorphism class and its members: each $\pi \in \hat{G}$ s to be thought of as a specific continuous[1] homomorphism

$$\pi: G \to U(H_\pi)$$

into the unitary group $U(H_\pi)$ of a specific Hilbert space $H_\pi$. The irreducibility of $\pi$, i.e. the nonexistence of a proper, closed $G$-invariant subspace, implies that $H_\pi$ is finite dimensional. According to the *Peter–Weyl theorem* [23], [30], there exists an isomorphism of Hilbert spaces

$$L^2(G) \cong \bigoplus_{\pi \in \hat{G}} H_\pi \otimes H_\pi^* \quad \text{(Hilbert space direct sum)}, \tag{1}$$

[1] Continuous with respect to the "weak topology" on $U(H_\pi)$ — the weakest topology which makes the functions $T \mapsto (Tu, v)$ continuous, for all $u, v$ in $H_\pi$.

which can be described explicitly, and which has the following crucial property: the action of $G$ on $L^2(G)$ induced by left translation corresponds to the action on the left factors $H_\pi$, whereas the right translation action corresponds to the dual action on the dual spaces $H_\pi^*$.

The statement of the Peter–Weyl theorem already points to the most fundamental reason for studying representation theory: to understand the representations of $G$ is to understand $L^2(G)$, as a left and right $G$-module. The theorem, an early success of "soft analysis", makes only a rather abstract assertion, however; it does not

(2)
(a) describe the set $\hat{G}$,
(b) give information about the structure of the irreducible representations.

These two problems must be dealt with if the Peter–Weyl theorem is to answer concrete questions about $L^2(G)$.

To get a grasp on the first problem, one associates to each finite dimensional representation $\pi$ the function

(3) $$\chi_\pi\colon g \mapsto \operatorname{trace} \pi(g),$$

the so-called character of $\pi$. As a formal consequence of the Peter–Weyl theorem, $\chi_\pi$ determines $\pi$ up to isomorphism. In particular, the passage from representations to their characters establishes a one-to-one correspondence

(4) $$\hat{G} \leftrightarrow \text{set of irreducible characters}.$$

It is usually easier to describe $\hat{G}$ indirectly, via this correspondence: characters, as functions on $G$, are less complicated objects than representations.

For only one large and significant class of compact, noncommutative groups does one understand the two problems (2) reasonably well—namely compact, connected Lie groups. I shall now assume specifically that $G$ belongs to this class. By a simple, but ingenious argument, which combines the Peter–Weyl theorem with basic properties of compact Lie groups, Herman Weyl was able to compute the irreducible characters of any such $G$. Implicitly the resulting Weyl character formula [27], [29] provides a parametrization of the set $\hat{G}$, and hence a solution of the problem (2a).

As for the second problem, the most useful technique is to study representations of $G$ by analyzing their restriction to a maximal torus $T \subset G$. Any two maximal tori are conjugate, and hence the particular choice of $T$ does not matter. Viewed as representation of $T$, each $\pi \in \hat{G}$ decomposes into a direct sum of one dimensional representations, which are called the weights of $\pi$. Among the weights, one is distinguished by being the "highest", in a certain definite sense; the highest weight occurs with multiplicity one and, most importantly, it characterizes $\pi$ up to isomorphism. This is the essence of Élie Cartan's *theorem of the highest weight* [26]. Virtually all general structural information about representations of compact Lie groups follows from it, at least indirectly. The theorem can be proved by infinitesimal methods, or alternatively, deduced from Weyl's character formula.

In a nutshell, the Peter–Weyl theorem, the Weyl character formula, and the theorem of the highest weight constitute the fundamentals of the representation theory of compact Lie groups. Taken together, they give a good grasp of $L^2(G)$, and hence also of $L^2(X)$, for any homogeneous space $X$ on which $G$ operates transitively. Indeed, every such homogeneous space $X$ can be represented as a quotient $X=G/U$, with $U$=isotropy subgroup at some point of $X$. Pulling back functions from $G/U$ to $G$, one finds

$$(5) \qquad L^2(X) = \text{space of right } G\text{-invariants in } L^2(G) \cong \bigoplus_{\pi\in\hat{G}} H_\pi \otimes (H_\pi^*)^U;$$

here $(H_\pi^*)^U$ denotes the subspace of all $U$-invariant vectors in $H_\pi^*$. The description (5) of $L^2(X)$ makes it a simple matter to determine the $G$-invariant subspaces of $L^2(X)$: they are of the form

$$(6) \qquad \bigoplus_{\pi\in\hat{G}} H_\pi \otimes W_\pi,$$

with suitably chosen subspaces $W_\pi \subset (H_\pi^*)^U$; conversely, every direct sum (6) is actually $G$-invariant.

Let me now consider a linear differential operator $D$ on $X$, $G$-invariant and, for simplicity, acting on scalar functions. One may extend $D$ to an unbounded operator on $L^2(X)$, by taking its closure. The kernel of $D$ then becomes an invariant subspace of $L^2(X)$:

$$(7) \qquad \operatorname{Ker} D \cong \bigoplus_{\pi\in\hat{G}} H_\pi \otimes W_\pi .$$

In our particular context, the $W_\pi$ can be identified as the kernels of a family of linear transformations,

$$(8) \qquad \begin{aligned} &W_\pi = \operatorname{Ker} D_\pi, \\ &D_\pi\colon (H_\pi^*)^U \to (H_\pi^*)^U, \end{aligned}$$

which are derived from $D$ in quite an explicit manner. Although $D$ was assumed to be a scalar operator, these remarks apply—mutatis mutandis—also to invariant systems of differential equations.

The preceding discussion, straightforward and formal as it is, should convey one salient point: the Peter–Weyl theorem and its companion statements make it possible, at least in principle, to solve invariant systems of differential equations on homogeneous spaces. Undoubtedly, this connection with the problem of solving invariant differential equations is one of the most important aspects of the representation theory of Lie groups.

To give a concrete example, I shall mention the *Borel–Weil–Bott theorem* [4], [21]. As before, $T\subset G$ denotes a maximal torus. One knows that the quotient $G/T$ can be made into a homogeneous complex manifold—a complex manifold such that $G$ acts, by left translation, as a group of holomorphic mappings. Moreover, each one dimensional representation

$$\sigma\colon T \to C^*$$

gives rise to a homogeneous holomorphic line bundle

$$\mathscr{L}_\sigma \to G/T, \tag{9}$$

i.e. a holomorphic line bundle to which the translation action of $G$ lifts; it is uniquely determined by the requirement that $T$ should operate on the fibre at the identity coset via $\sigma$. Since $G$ acts on the bundle (9), it also acts on the cohomology groups

$$H^k(G/T, \mathcal{O}(\mathscr{L}_\sigma)) \tag{10}$$

of the sheaf of germs of holomorphic sections $\mathcal{O}(\mathscr{L}_\sigma)$. The sheaf cohomology groups thus become finite dimensional representation spaces for $G$—finite dimensional because $G/T$ is compact. The Hodge theorem identifies the cohomology group (10) with the kernel of the ($G$-invariant) Laplace–Beltrami operator, acting on the $\mathscr{L}_\sigma$-valued $(0, k)$-forms. In particular, the present example fits into the framework of invariant differential equations.

The Borel–Weil–Bott theorem describes the cohomology groups (10): they vanish identically for certain special choices of $\sigma$; in all remaining cases, they are nonzero for exactly one integer $k=k(\sigma)$, and the representation of $G$ on this one non-zero cohomology group is irreducible, with a highest weight whose dependence on $\sigma$ can be made explicit. Every irreducible representation of $G$ arises in this fashion, even with $k(\sigma)=0$; for $k=0$, it should be noted, the group (10) is simply the space of holomorphic sections of the line bundle $\mathscr{L}_\sigma$. Most proofs reduce the Borel–Weil–Bott theorem to the theorem of the highest weight, through arguments in the spirit of (7)–(8). The theorem serves at least two purposes. It provides a realization of every $\pi\in\hat{G}$, on a concrete vector space, with a concrete $G$-action—in contrast to the Weyl character formula or the theorem of the highest weight, which enumerate the irreducible representations, without giving such a realization. Secondly, it computes certain cohomology groups which are of interest in complex analysis, and which were not understood before the advent of the Borel–Weil–Bott theorem.

The statement of the theorem also suggests another possible approach to the representation theory of compact Lie groups. One can use methods of differential geometry and complex analysis to prove the theorem directly, avoiding any reference to the Weyl character formula and the theorem of the highest weight. An application of the Atiyah–Bott fixed point formula then leads to the character formula, which thus becomes a consequence of the Borel–Weil–Bott theorem. This chain of arguments employs rather heavy machinery and may seem merely a curiosity. I mention it here because in the case of non-compact groups, analogous arguments turn out to be quite efficient.

So much for compact groups! The object of interest shall now be a locally compact group $G$, unimodular—i.e. the essentially unique left invariant measure is also right invariant— and of type I. The latter is a technical condition, satisfied by all the special classes of groups which are considered in this survey; it insures that $G$ has a "reasonable" representation theory. Again $\hat{G}$ stands for the set of isomorphism classes of irreducible unitary representations. In general, these will be infinite dimensional, since $G$ may not be compact. For the same reason, the Peter–Weyl theorem no longer applies; even the simplest noncompact examples show that $L^2(G)$ cannot be expressed as a direct sum of irreducibles. Its place is taken by the ***abstract Plancherel theorem*** [6], which essentially goes back to von Neumann: $L^2(G)$ decomposes into a Hilbert space direct integral,

$$L^2(G) \cong \int_{\pi \in \hat{G}} H_\pi \otimes H_\pi^* \, d\mu(\pi), \tag{11}$$

with respect to a measure $\mu$ on $\hat{G}$, the so-called Plancherel measure.

Just as in the case of the Peter–Weyl theorem, the isomorphism makes the left and right actions of $G$ correspond to the actions on the left and right factors of the integrand $H_\pi \otimes H_\pi^*$. The tensor product sign refers not to the algebraic tensor product, but rather to its completion. The notion of Hilbert space direct integral generalizes the notion of Hilbert space direct sum. For instance if $G$ is compact after all, the measure $\mu$ becomes discrete, and the direct integral (11) reduces to the direct sum[2] (1). The best known example of a direct integral, which is not actually a direct sum, is furnished by classical Fourier analysis on the real line: $L^2(\boldsymbol{R})$ may be viewed as a direct integral of a continuous family of one dimensional function spaces, namely those spanned by the unitary characters

$$x \mapsto e^{ixy}, \quad y \in \boldsymbol{R}.$$

These function spaces do not occur in $L^2(\boldsymbol{R})$ discretely, as subspaces, but only "infinitesimally".

Again, the description (11) of $L^2(G)$ raises some immediate questions:

(12)
- (a) what is the set $\hat{G}$?
- (b) what is the Plancherel measure $\mu$?
- (c) what can one say about the structure of the irreducible unitary representations?

Reasonably complete answers exist for only two major classes of noncompact, noncommutative groups—on the one hand, nilpotent Lie groups, and to some extent also solvable groups; on the other, semisimple Lie groups. The techniques which are appropriate in these two cases diverge widely, for quite fundamental reasons. I shall therefore limit the discussion to the semisimple case. The classical matrix groups $\mathrm{Sl}(n, \boldsymbol{R})$, $\mathrm{Sl}(n, \boldsymbol{C})$, $\mathrm{SO}(p, q)$, $\mathrm{SO}(n, \boldsymbol{C})$, $\mathrm{SU}(p, q)$, $\mathrm{Sp}(n, \boldsymbol{R})$, ..., which are of special interest in geometry, number theory, and physics, all fall into

[2] The measure $\mu$, which does not show up in (1), has been absorbed into the particular isomorphism. In fact, the isomorphism (11), and with it the measure $\mu$, are not uniquely determined. There is one natural choice, however.

this class (as do compact Lie groups with finite center)—ample justification for studying semisimple Lie groups in particular detail.

Let then $G$ be a connected semisimple Lie group, and $\pi$ an irreducible unitary representation of $G$, on a Hilbert space $H_\pi$. Typically $H_\pi$ is infinite dimensional. The definition of character, which proved so useful in the finite dimensional case, thus loses meaning, at least in its naive form (3): as unitary operators acting on an infinite dimensional space, the operators $\pi(g)$ do not have a trace in any obvious sense.

There exists a way around this difficulty, first discovered by Gelfand and Naimark in their study of the complex classical groups, later fully developed and systematically exploited by Harish-Chandra [10]. It proceeds from the following observation: for every compactly supported $C^\infty$ function $f$ on $G$, the operator-valued integral

$$\pi(f) = \int_G f(g)\pi(g)\,dg \tag{13}$$

is of trace class. In other words, if one represents $\pi(f)$ by an infinite matrix, relative to any orthonormal basis of $H_\pi$, the sum of the diagonal matrix entries converges absolutely. It then follows that the sum does not depend on the particular choice of basis, and one calls this sum the trace. The linear mapping

$$\Theta_\pi : f \mapsto \operatorname{trace} \pi(f), \tag{14}$$

which assigns to every $f \in C^\infty(G)_0$ the trace of the operator $\pi(f)$, turns out to be a distribution in the sense of L. Schwartz. It determines $\pi$ up to isomorphism and is, by definition, the character of $\pi$. If $\pi$ happens to be a finite dimensional representation, $\Theta_\pi$ is given by integration against the ordinary character. Thus Harish-Chandra's definition of character embraces the usual one.

Because of its definition in terms of a trace, $\Theta_\pi$ remains invariant under all inner automorphisms of $G$. A slightly more subtle argument, based on the irreducibility of $\pi$, shows that every bi-invariant linear differential operator maps $\Theta_\pi$ to a multiple of itself. In shorthand terminology, a distribution with these two properties is an *invariant eigendistribution.*

Distributions are decidedly more complicated objects than functions—more complicated to write down, more complicated to manipulate. At first glance, this appears to be a serious shortcoming of the notion of character in the infinite dimensional case. Fortunately, there is a remedy, Harish-Chandra's *regularity theorem* for invariant eigendistributions [13]–[15], [2]: every invariant eigendistribution, and in particular every character, can be expressed as integration against a locally $L^1$ function; this function is real-analytic on the complement of a real-analytic subvariety of $G$. Thus characters turn out to be functions, after all. The regularity theorem plays a crucial role in the representation theory of semisimple Lie groups; without it, the notion of character would be far less useful.

Let me now turn to the problem of describing $\hat{G}$. As a first step, it is helpful to consider a certain subset. An irreducible unitary representation $\pi$ is said to be *square-integrable* if the Plancherel measure assigns a positive mass to the single point $\pi \in \hat{G}$, i.e. if $\pi$ contributes discretely to the Plancherel decomposition (11) of $L^2(G)$. The isomorphism classes of all such irreducible, square-integrable representations constitute a subset $\hat{G}_{ds} \subset \hat{G}$, the *discrete series* of $G$.

To state Harish-Chandra's fundamental results on the discrete series **[1]**, **[16]**, I select a maximal compact subgroup $K \subset G$. Any two of them are conjugate, and this fact makes the particular choice of $K$ unimportant. According to Harish-Chandra's existence criterion, the discrete series of $G$ is nonempty if, and only if, $K$ has the same rank as $G$—equivalently, if any maximal torus $T \subset K$ is its own centralizer in $G$, or in more technical language, if $G$ contains a compact Cartan subgroup. Going back to the list of examples, one finds that $\mathrm{Sl}(n, \boldsymbol{R})$ has a discrete series only for $n=2$, $\mathrm{Sl}(n, \boldsymbol{C})$ never does, $\mathrm{SO}(p, q)$ has one precisely when $pq$ is even, and finally $\mathrm{SU}(p, q)$ and $\mathrm{Sp}(n, \boldsymbol{R})$ always have a discrete series.

In case $G$ satisfies the criterion, and subject to a minor restriction which will be mentioned presently, Harish-Chandra's parametrization of the discrete series establishes a bijection

$$\hat{K}' \leftrightarrow \hat{G}_{ds} \tag{15}$$

between a subset $\hat{K}'$ of $\hat{K}$ and the discrete series $\hat{G}_{ds}$. It assigns to the character $\chi$ of a representation in $\hat{K}'$ a discrete series character $\Theta$, whose restriction to $K$ is given by the formula

$$\Theta_K = \pm \frac{\chi}{D}; \tag{16}$$

here $D$ denotes a universal denominator, independent of $\chi$, and itself a linear combination of irreducible characters of $K$. The Weyl character formula for $K$ identifies $\hat{K}$ with a lattice, divided by the action of a finite linear group. In terms of this description, $\hat{K}'$ corresponds to the complement, in the lattice, of a finite number of hyperplanes. As was remarked already, the parametrization (15) does not apply to an arbitrary semisimple $G$; however, it does apply to some finite covering of any given $G$. This restriction is quite innocuous, since the discrete series for $G$ may be viewed as a subset of the discrete series for the covering group.

The discrete series provides a basic repertory of representations, from which others can be constructed. To be more concrete, I shall need the notion of Cartan subgroup. It is most easily defined for a linear semisimple group $G$: Cartan subgroups are then Abelian, they consist of group elements that can be diagonalized over $\boldsymbol{C}$, and are maximal subgroups with respest to these two properties. One can classify the conjugacy classes of Cartan subgroups **[20]**, **[28]**; in particular, they are finite in number.

To each conjugacy class, Harish-Chandra attaches a series of irreducible unitary representations **[17]**. If there exists a—necessarily unique—conjugacy class of compact Cartan subgroups, the corresponding series is the discrete series. The other series are obtained by an induced representation process, starting from discrete series representations of subgroups of $G$. In this construction, distinct conjugacy classes of Cartan subgroups lead to non-overlapping series of representations. Although the terminology is by no means standard, I shall call a representation *generic* if it belongs to one of the series, and otherwise *special*. Both types actually occur, unless $G$ is compact, in which case $\hat{G}=\hat{G}_{ds}$.

The crowning achievement of Harish–Chandra's program is a solution of problem (12b). Perhaps the explicit, somewhat complicated description of the Plancherel measure $\mu$ [17] matters less than the nature of the answer. To begin with, $\mu$ has the set of generic representations as support. Each of the various series is parametrized, roughly speaking, by the product of a lattice with a vector space, divided by the action of a finite linear group. It therefore carries a distinguished measure, namely the one derived from the invariant measures on the two factors. The restriction of $\mu$ to the series in question is completely continuous with respect to this distinguished measure. The ratio of the two measures reflects the rate at which the matrix coefficients of the representations in the given series decay at infinity, a fact which is a crucial ingredient of the actual computation of the measure.

Since the Plancherel measure completely disregards the special representations, these become irrelevant as far as the decomposition of $L^2(G)$ is concerned. They are quite important from other points of view, and I shall come back to them later. In any case, $L^2(G)$ is made up of generic representations, which are described, in Harish–Chandra's construction, in terms of their characters. Thus Harish–Chandra's theory accomplishes for semisimple Lie groups what Weyl's theory did for compact Lie groups. I should point out, however, that the technical difficulties are immensely greater. In my very condensed summary, I have broken down Harish–Chandra's program into three major components: the study of characters, the construction of the discrete series, and the determination of the Plancherel measure. Each of these is a large and elaborate edifice.

A brief historical remark: the idea that various series of representations should be attached to the conjugacy classes of Cartan subgroups made its first appearance in the work of Gelfand and his collaborators on the complex classical groups and the real special linear group. In general, it was conjectured—and of course later worked out—by Harish–Chandra. His address at the 1954 Congress in Amsterdam already gives a glimpse, in very rough outline, of his entire program.

To understand the structure of representations of compact Lie groups, one investigates their restrictions to a maximal torus. In the context of semisimple groups, there is a similar device, namely to break up representations under the action of a maximal compact subgroup $K\subset G$. When restricted to $K$, every $\pi\in\hat{G}$ becomes a direct sum of irreducibles, each occurring with finite multiplicity; in symbolic notation,

$$\pi|_K = \bigoplus_{\tau\in\hat{G}} n_\tau(\pi)\cdot\tau. \tag{17}$$

The analogy with the compact case, as well as examples of low dimensional noncompact groups [3], [5] suggest that the pattern of the multiplicities $n_\tau(\pi)$ is an important invariant of $\pi$.

Generic representations either belong to the discrete series, or are constructed, by a well understood procedure, from discrete series representations of subgroups of $G$. Structural questions about generic representations therefore come down to questions about the discrete series. A *conjecture of Blattner,* now a theorem [**18**], describes the $K$-multiplicities $n_\tau(\pi)$, for every discrete series representation $\pi$. According to the conjecture, of the various $\tau \in \hat{K}$ which appear in the restriction $\pi|_K$, one is lowest, in an appropriate sense, and occurs with multiplicity one. The *theorem of the lowest K-type,* counterpart to the theorem of the highest weight, asserts that this feature characterizes the discrete series representation $\pi$ uniquely, among all irreducible representations [**24**]. It can be quite difficult, if not impossible, to check directly whether a representation is square-integrable. The theorem of the lowest $K$-type provides a useful criterion, of an essentially algebraic nature.

The problem of realizing discrete series representations concretely is closely related to the theorem of the lowest $K$-type. If $G$ has a discrete series, it contains a compact Cartan subgroup $T$, which is in particular a maximal torus. Just as in the compact case, the quotient $G/T$ can be turned into a homogeneous complex manifold, noncompact of course, unless $G$ itself is compact. Every character

$$\sigma: T \to C^*$$

again determines a homogeneous holomorphic line bundle

$$\mathscr{L}_\sigma \to G/T.$$

By translation, $G$ acts unitarily on $\mathscr{H}(\mathscr{L}_\sigma)$, the Hilbert space of square-integrable, holomorphic sections. Long before the discrete series was fully understood, Harish–Chandra showed that for certain characters $\sigma$, $\mathscr{H}(\mathscr{L}_\sigma)$ is nonzero; the resulting representation is then necessarily irreducible and belongs to the discrete series [**12**]. Unfortunately this construction gives only a relatively small part of the discrete series, and for some groups $G$ it even gives nothing at all. To produce a realization of every discrete series representation, one must turn elsewhere.

The first explicit suggestion was made by Langlands: one should consider also the higher[3] $L^2$-cohomology groups $\mathscr{H}^k(\mathscr{L}_\sigma)$, i.e. the spaces of harmonic, square-integrable, $\mathscr{L}_\sigma$-valued $(0, k)$-forms on $G/T$. If $G/T$ happens to be compact, the Hodge theorem identifies $\mathscr{H}^k(\mathscr{L}_\sigma)$ with the $k$th sheaf cohomology group of $\mathscr{L}_\sigma$. Such a simple connection between $L^2$-cohomology and sheaf cohomology does not exist in general, for noncompact $G$, but in any case $\mathscr{H}^k(\mathscr{L}_\sigma)$ is a Hilbert space, on which $G$ operates as a group of unitary transformation. Guided by the analogy with the Borel–Weil–Bott theorem, and also by curvature computations of Griffiths, Langlands conjectured that all of the $L^2$-cohomology groups should vanish for

[3] For $k=0$, $\mathscr{H}^0(\mathscr{L}_\sigma)$ coincides with $\mathscr{H}(\mathscr{L}_\sigma)$.

special choices of $\sigma$; in the remaining cases, $\mathscr{H}^k(\mathscr{L}_\sigma)$ was to be nonzero for exactly one integer $k=k(\sigma)$, and $G$ was to act according to a representation of the discrete series. In this manner, one would be able to realize every discrete series representation, but usually not with $k(\sigma)=0$. The conjecture has in fact been proved, by an argument which reduces it to the theorem of the lowest $K$-type [25].

In order to verify the conjecture, one must somehow exhibit squareintegrable harmonic forms. The proof in [25] relies on Harish–Chandra's construction of the discrete series to overcome this analytic problem. Instead, one can also base a proof on Atiyah's $L^2$-index theorem, in combination with the Atiyah–Singer index theorem. An argument, showing that Langlands' conjecture accounts for all of the discrete series, goes hand in hand with the existence proof. The outcome[4] is an alternative, more geometrically oriented approach to the main results on the discrete series [1].

I shall conclude with some remarks about special representations. This will also give me an opportunity to touch on some important points which have been passed over so far. To describe what is known, one must look beyond the class of unitary representations. Indeed, techniques which are designed to deal with unitary representations very often lead naturally into questions about nonunitary representations.

A representation $\pi$ of $G$ on a Banach space is said to be admissible if every $\tau\in\hat{K}$ occurs at most finitely often in the restriction of $\pi$ to $K$. Irreducible unitary representations automatically have this property, it remains unknown whether all irreducible Banach representations are admissible. To simplify the terminology, when I speak of an irreducible representation, I shall always mean an irreducible admissible representation on a Banach space.

In the finite dimensional case, there exists a well understood, very useful relationship between representations of the group and those of the Lie algebra. Such a relationship exists also for infinite dimensional representations, but this involves some subtleties. Let $\pi$ be an irreducible representation, on a Banach space $B$. The analytic vectors—those vectors $v\in B$ for which

$$g\mapsto\pi(g)v$$

is a real analytic mapping from $G$ to $B$—form a dense subspace $B_\omega\subset B$. The Lie algebra $\mathfrak{g}$ of $G$ acts on $B_\omega$ by differentiation, but the resulting representation of $\mathfrak{g}$ is "too large"; in particular, it fails to be irreducible. The most natural way to pick out an irreducible subrepresentation is to pass to the space of $K$-finite vectors $B_0$, i.e. to the linear span of all finite dimensional $K$-stable subspaces of $B$. It is contained in $B_\omega$, dense in $B$, $\mathfrak{g}$-invariant, and algebraically irreducible as $\mathfrak{g}$-module. In this manner, one attaches to each irreducible representation of the group an irreducible representation of the Lie algebra [9].

A side remark is perhaps appropriate at this point. Hermann Weyl repeatedly

[4] For minor technical reasons, the construction in [1] works with harmonic spinors on $G/K$, rather than $L^2$-cohomology.

emphasized the fact that his methods were purely global; he studied representations of the group directly, without reference to the Lie algebra. My very brief account of Harish–Chandra's work may have fostered the impression that his approach is equally global. Quite to the contrary, infinitesimal arguments play an important role in many of his proofs.

The definition of character (14) seems to depend on the Hilbert space structure of the representation space. However, with relatively minor modifications, it makes sense also for representations on Banach spaces: every irreducible representation $\pi$ has a character $\Theta_\pi$, which is again an invariant eigendistribution. The characters of two irreducible representations $\pi_1, \pi_2$ coincide precisely when the corresponding Lie algebra representations are algebraically isomorphic **[10]**. In this case, $\pi_1$ and $\pi_2$ are said to be infinitesimally equivalent; loosely speaking, they look alike, except possibly for the topology on the representation spaces.

In the case of unitary representations, infinitesimal equivalence implies that the representations in question are actually isomorphic **[9]**. An irreducible representation on a Banach space can thus be infinitesimally equivalent to at most one unitary representation. If it is, one calls the representation unitarizable. To classify the irreducible unitary representations—or what amounts to the same, the special representations, since one already knows the others—, it suffices to

(18)
(a) classify the irreducible representations on Banach spaces, up to infinitesimal equivalence, and
(b) determine which are unitarizable.

The first of these two problems has been solved by Langlands **[22]**, and in somewhat different form also by Vogan **[31]**. Langlands classifies representations in terms of the growth behavior of their matrix coefficients. This has the effect of reducing the problem to the classification of a smaller class of representations, which was worked out by Knapp–Zuckerman **[19]**. Vogan's classification involves a very detailed study of how representations break up under the action of $K$. Each irreducible representation has a lowest $K$-type, which occurs with multiplicity one. The meaning of "lowest", however, is weaker than in the context of the discrete series, and the lowest $K$-type alone does not suffice to characterize an arbitrary irreducible representation uniquely.

In principle, (18b) is a purely algebraic problem, but apparently a difficult one, and no solution is yet in sight. If a representation is to be unitarizable, it must satisfy certain fairly obvious necessary conditions. This gives at least a partial answer to (18b), and hence some information about special representations—virtually the only information that is known at present.

Why is one interested in the special representations? The problem of decomposing function spaces on quotients of $G$ reveals perhaps the best explanation. Every quotient $G/U$, by a closed unimodular subgroup $U\subset G$, carries an invariant measure, and so $G$ acts unitarily on $L^2(G/U)$. Whenever $U$ is compact, one can embed $L^2(G/U)$ into $L^2(G)$,

$$L^2(G/U) \hookrightarrow L^2(G), \tag{19}$$

by pulling back functions from $G/U$ to $G$. In this way, the Plancherel decomposition of $L^2(G)$ leads to a decomposition of $L^2(G/U)$. If $U$ is non-compact, on the other hand, no inclusion (19) exists, and it becomes an entirely separate problem to express $L^2(G/U)$ as a direct integral of irreducible representations. In particular, special representations can and do contribute to the decomposition of $L^2(G/U)$.

The most important case is that of a discrete subgroup $\Gamma \subset G$, such that $G/\Gamma$ has finite volume; an arithmetic subgroup, for example. One can then use the Selberg trace formula to analyze $L^2(G/\Gamma)$. Although much work is being done in this direction, to understand $L^2(G/\Gamma)$ remains a distant goal.

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