EXTREMUM PROBLEMS AND VARIATIONAL METHODS IN CONFORMAL MAPPING

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1. Introduction

One fundamental problem in the classical theory of conformal mapping was the study of the various types of canonical domains upon which any domain, arbitrarily given in the complex plane, can be mapped conformally. The first question to be settled was, therefore, the existence of various types of canonical mapping functions. From the beginning, methods of the calculus of variations were applied in order to establish the necessary existence theorems. The role of the Dirichlet principle in the attempted proof of Riemann's mapping theorem for simply connected domains is well known and also the influence of its initial failure upon the critical period of the calculus of variations and upon the development of the powerful modern direct methods in this important branch of analysis. The existence proofs for canonical conformal mappings by means of extremum problems like the Dirichlet principle are so difficult because they characterize the sought mapping function, which is analytic and univalent, as the extremum function in a much wider class of admissible competing functions. The latter class is so large that the main labour in the proof is spent in establishing the existence of an extremum function of the variational problem considered.

The theory of conformal mapping advanced considerably when one started a systematic study of the univalent analytic functions in a given domain; that is, the class of those functions which realize the various conformal mappings of that domain. The main result of this theory is that all univalent functions in a given domain form a normal family. This fact leads easily to the consequence that for each reasonable extremum problem within the family of univalent functions there exists at least one element of the family which attains the extremum considered[20]. On the basis of this theory, very elegant proofs could be derived for the Riemann mapping theorem and for the existence of numerous other canonical mappings. The characteristic difficulty of the new approach, that is to study extremum problems within the family of univalent functions, lies in the fact that the univalent functions form no linear space; hence, it is not at all easy to characterize an extremum function by comparison with its competitors by infinitesimal variation.

In each particular existence proof a special comparison method had to be devised and the essential step of the whole proof was the characterization of the extremum function by this particular variation.

It is possible to develop a systematic infinitesimal calculus within the family of univalent functions. In 1923 Löwner gave a now classical partial differential equation which has as solutions one-parameter families of univalent functions which admit a very simple geometric interpretation[18]. I showed in 1938 that the univalent extremum functions do satisfy in very many cases a first-order differential equation and gave a standard variational procedure for establishing these ordinary differential equations[26]. In the following years, Schaeffer and Spencer applied this variational procedure systematically to the coefficient problem for functions univalent in the unit circle and developed an extensive theory for it[22, 23, 24]. Golusin applied the same variational technique to numerous questions of geometric function theory[7, 8]. The significance of extremum problems for the general theory of conformal mapping is evident. The great number of possible conformal mappings of a given domain precludes the study of all of them; however, important individual mappings can be singled out as solutions of extremum problems and can be described geometrically and analytically just because of their extremum property. The remaining amorphous mass of conformal mappings is subjected to all the inequalities which flow from the solutions of the various extremum problems and is, thus, at least partially characterized.

In the present paper we shall try to give a brief survey of the basic methods of variation within the family of univalent functions. By discussing a few important extremum problems, we will show the flexibility of the technique. It will appear that the variational method provides very often an elegant and useful transformation of the extremum problem but leads sometimes to functional equations whose solution is a deep problem again. It is clear that the field of research described is by no means completely explored and exhausted and that, because of its interest from the point of view of applied as well as of pure mathematics, it deserves the continued attention of mathematicians.

2. Variation of the Green's function

The simplest approach to the calculus of variations for univalent functions seems to lead through the theory of the Green's function of a domain and its variational formula. Let D be a domain in the complex z-plane whose boundary C consists of n closed analytic curves and let

$g(z, \zeta)$ be its Green's function with the source point ζ. We consider the conformal transformation

$$z^*(z) = z + \frac{e^{i\alpha}\rho^2}{z - z_0} \quad (z_0 \in D, \, \rho > 0). \tag{1}$$

This mapping is univalent in the domain $|z - z_0| > \rho$; hence, for small enough ρ it will be univalent on C and transform it into a new set C^* of n closed analytic curves which bounds a new domain D^*. We denote the Green's function of D^* by $g^*(z, \zeta)$ and wish to express it in terms of $g(z, \zeta)$. We observe that $\gamma(z, \zeta) = g^*(z^*(z), \, \zeta^*(\zeta))$ is a harmonic function in the domain D_ρ which is obtained from D by removal of the circle $|z - z_0| < \rho$. $\gamma(z, \zeta)$ has a pole for $z = \zeta$ and vanishes on the boundary curves C of D_ρ. We choose two fixed points ζ and η in D_ρ and apply Green's identity in the form

$$\frac{1}{2\pi} \int_{C+c} \left[\frac{\partial}{\partial n} g(z, \eta) \, \gamma(z, \zeta) - g(z, \eta) \frac{\partial}{\partial n} \gamma(z, \zeta) \right] ds = \gamma(\zeta, \eta) - g(\zeta, \eta). \tag{2}$$

Here, c denotes the small circumference $|z - z_0| = \rho$. Observe now that the integration takes place only over the circumference c since both g and γ vanish on C.

In order to simplify (2), we introduce the analytic functions of z whose real parts are $g(z, \eta)$ and $g^*(z, \zeta)$, respectively, and denote them by $p(z, \eta)$ and $p^*(z, \zeta)$. These functions have logarithmic poles at η or ζ and have also imaginary periods when z circulates around a boundary continuum. It is now easy to express (2) in the form

$$g^*(\zeta^*, \eta^*) - g(\zeta, \eta) = \operatorname{Re} \left\{ \frac{1}{2\pi i} \oint_c p^*(z^*, \zeta^*) \, dp(z, \eta) \right\}. \tag{3}$$

This integral equation for $g^*(\zeta^*, \eta^*)$ in terms of $g(\zeta, \eta)$ must now hold for the most general domain D which possesses a Green's function at all. Indeed, such a domain D may be approximated arbitrarily by domains D_ν with analytic boundaries C_ν for which the identity (3) is valid. If D and D_ν go under the variation (1) into the domains D^* and D_ν^*, then the D_ν^* will likewise approximate D^*. Since (3) holds for all approximating domains and since at ζ, η and on c the Green's functions of D_ν and D_ν^* converge uniformly to the Green's functions of D and D^*, respectively, the formula (3) must remain valid in the limit and is thus generally proved[31].

We may apply Taylor's theorem in the form

$$p^*(z^*, \zeta^*) = p^*(z, \zeta^*) + p^{*\prime}(z, \zeta^*) \frac{e^{i\alpha}\rho^2}{z - z_0} + O(\rho^4), \tag{4}$$

where the residual term $O(\rho^4)$ can be estimated equally for all domains D which contain a fixed subdomain Δ which, in turn, contains the point ζ and the circle c. Thus, inserting (4) into (3) and using the residue theorem we obtain after an easy transformation

$$g^*(\zeta^*, \eta^*) = g(\zeta, \eta) + \mathrm{Re}\{e^{i\alpha}\rho^2 p'(z_0, \zeta)\, p'(z_0, \eta)\} + O(\rho^4), \tag{5}$$

where again $O(\rho^4)$ can be estimated equally as above. Finally, using Taylor's theorem again, we can reduce (5) to[27, 28]

$$g^*(\zeta, \eta) = g(\zeta, \eta) + \mathrm{Re}\left\{e^{i\alpha}\rho^2\left[p'(z_0, \zeta)\, p'(z_0, \eta) - \frac{p'(\zeta, \eta)}{\zeta - z_0} - \frac{p'(\eta, \zeta)}{\eta - z_0}\right]\right\} + O(\rho^4). \tag{6}$$

In the preceding, we have restricted ourselves to the particular variation (1) for the sake of simple exposition. It is clear that a corresponding formula can be established for each variation $z^* = z + \rho^2 v(z)$, where $v(z)$ is analytic on the boundary C of the varied domain. On the other hand, such a general variation can be approximated arbitrarily by superposition of elementary variations of the type (1). Indeed, for most applications the formulas (1) and (6) are entirely sufficient.

A remarkable transformation of (6) is possible if the boundary C of D is a set of smooth curves. Indeed, we may express (6) in the form

$$g^*(\zeta, \eta) - g(\zeta, \eta) = \mathrm{Re}\left\{e^{i\alpha}\rho^2 \frac{1}{2\pi i}\int_C \frac{p'(z, \zeta)\, p'(z, \eta)}{z - z_0}\, dz\right\} + O(\rho^4). \tag{7}$$

We observe that the real part of $p(z, \zeta)$ is the Green's function $g(z, \zeta)$ and that it vanishes, therefore, on C. Let $z' = dz/ds$ denote the tangent vector to C at the point $z(s)$; it is easy to see that

$$p'(z, \zeta)\, z' = -i\frac{\partial g(z, \zeta)}{\partial n}$$

and hence (7) may be given the real form

$$g(\zeta, \eta) = -\frac{1}{2\pi}\int_C \frac{\partial g(z, \zeta)}{\partial n_z}\frac{\partial g(z, \eta)}{\partial n_z}\,\delta n\, ds \tag{8}$$

with
$$\delta n = \mathrm{Re}\left\{\frac{1}{iz'}\frac{e^{i\alpha}\rho^2}{z - z_0}\right\}. \tag{9}$$

Clearly, δn denotes the shift along the interior normal of the boundary point $z \in C$ under the variation (1).

By linear superposition of elementary variations (1), formula (8) can be proved for very general δn-variations of the boundary curves C.

This formula was first given by Hadamard in 1908[11] and has been very frequently used in applied mathematics because of the very intuitive and geometric significance of the normal displacement of the boundary points. We may mention, in particular, Lavrentieff's systematic use of boundary deformations in many problems of fluid dynamics and conformal mapping[16, 17].

If D is a simply connected domain there exists a close relationship between the Green's function of D and the univalent function $\phi(z)$ which maps the domain D onto the exterior of the unit circle. In fact, we have

$$g(z,\zeta) = \log\left|\frac{1-\phi(z)\,\overline{\phi(\zeta)}}{\phi(z)-\phi(\zeta)}\right|. \tag{10}$$

Julia used this interrelation in order to derive from the Hadamard formula (8) a variational formula for univalent functions[15]. This very intuitive and elegant formula, however, cannot be applied directly to the study of extremum problems in the theory of conformal mapping. In fact, one cannot assert *a priori* that the extremum domain D will possess a boundary C which is smooth enough to admit a variation of the Hadamard–Julia type.

3. Infinitesimal variations and extremum problems

We are now in a position to construct, by means of the fundamental formula (6), in any given domain D, univalent mappings which are arbitrarily close to the identity mapping. We have to assume only that the boundary C of D contains a non-degenerate continuum Γ. Let $D(\Gamma)$ denote the domain of the z-plane which contains the domain D and the point at infinity and which is bounded by Γ; let $g(z,\zeta)$ denote now the Green's function of $D(\Gamma)$. We choose an arbitrary but fixed point $z_0 \in D$ and subject $D(\Gamma)$ to a variation (1) which transforms it into the varied domain $D(\Gamma^*)$ with the Green's function $g^*(z,\zeta)$. The relation between $g^*(z,\zeta)$ and $g(z,\zeta)$ is given by the variational formula (6).

Let $w = \phi(z)$ be univalent in $D(\Gamma)$, normalized at $z=\infty$ by the requirement $\phi'(\infty) = 1$, and let it map $D(\Gamma)$ onto the domain $|w| > 1$. Analogously, we define $w = \phi^*(z)$ with respect to the domain $D(\Gamma^*)$. By virtue of the relation (10), we have obviously

$$g(z,\infty) = \log|\phi(z)|, \quad g^*(z,\infty) = \log|\phi^*(z)|; \tag{11}$$

these relations permit us to connect $\phi^*(z)$ with $\phi(z)$ by use of (6).

The function

$$v(z) = \phi^{*-1}[\phi(z)] \tag{12}$$

is analytic and univalent in $D(\Gamma)$ and hence, *a fortiori*, in D. A simple calculation based on (6) and (11) shows that

$$v(z) = z + e^{i\alpha}\rho^2\left[\frac{1}{z-z_0} - \frac{\phi'(z_0)^2\,\phi(z)}{\phi'(z)\,\phi(z_0)\,[\phi(z)-\phi(z_0)]}\right] + e^{-i\alpha}\rho^2\frac{\overline{\phi'(z_0)^2}\,\phi(z)^2}{\phi'(z)\,\overline{\phi(z_0)}\,[1-\overline{\phi(z_0)}\,\phi(z)]} + O(\rho^4). \tag{13}$$

Since ρ can be made arbitrarily small, we have in (13) the representation for a large class of univalent variations of the domain D considered. We will now show that this set of variations is general enough to characterize the extremum domains for a large class of extremum problems relative to the family of univalent functions.

We shall consider extremum problems of the following type. Let T be a domain in the complex t-plane which contains the point at infinity and which is analytically bounded. We denote by F the family of all analytic functions $f(t)$ in T which are univalent there, have a simple pole at $t = \infty$ and which are normalized by the condition $f'(\infty) = 1$. Let $\phi[f]$ be a real-valued functional defined for all analytic functions $f(t)$ in T. We suppose that $\phi[f]$ is differentiable in the sense that for an arbitrary analytic function $g(t)$ defined in T

$$\phi[f+\epsilon g] = \phi[f] + \mathrm{Re}\{\epsilon\psi[f,g]\} + O(\epsilon^2) \tag{14}$$

holds, where ψ is a complex-valued functional of f and g, linear in g. We suppose that the residual term $O(\epsilon^2)$ can be estimated equally for all analytic functions $g(t)$ which are equally bounded in a specified subdomain of T. Thus, we require for $\phi[f]$ the existence of a Gâteaux differential with the above additional specifications.

We assume also that $\phi[f]$ has an upper bound within the family F. Then, in view of the normality of this family, it is easy to show that there must exist functions $f(t) \in T$ for which $\phi[f]$ attains its maximum value within F. We can characterize each extremum function by subjecting it to infinitesimal variations and comparing $\phi[f]$ with the functional values of the varied univalent elements of the family. Indeed, by means of the functions (13) we can construct the competing functions in F

$$f^*(t) = v[f(t)]\,.\,v'(\infty)^{-1}, \tag{15}$$

where $z = f(t)$ maps the domain T onto the extremum domain D in the z-plane. An easy calculation yields

$$\phi[f^*] = \phi[f] + \mathrm{Re}\{e^{i\alpha}\rho^2 A + e^{-i\alpha}\rho^2 B\} + O(\rho^4), \tag{16}$$

with

$$\left.\begin{aligned} A &= \psi\left[z, \frac{1}{z-z_0} - \frac{\phi'(z_0)^2\,\phi(z)}{\phi'(z)\,\phi(z_0)\,[\phi(z)-\phi(z_0)]}\right] \\ B &= \psi\left[z, \frac{\overline{\phi'(z_0)^2}\,\phi(z)^2}{\phi'(z)\,\overline{\phi(z_0)}\,[1-\overline{\phi(z_0)}\,\phi(z)]} + \frac{\overline{\phi'(z_0)^2}}{\overline{\phi(z_0)^2}}\,z\right] \end{aligned}\right\} \quad (z = f(t)). \tag{17}$$

Since the extremum property of f requires $\phi[f^*] \leqslant \phi[f]$ and since ρ and $e^{i\alpha}$ are at our disposal, we can easily conclude $A + \bar{B} = 0$, that is

$$\begin{aligned} \psi\left[f(t), \frac{1}{f(t)-z_0}\right]\frac{\phi(z_0)^2}{\phi'(z_0)^2} &= \overline{\psi\left[z, \frac{\phi(z)}{\phi'(z)} - z\right]} \\ &+ \psi\left[z, \frac{\phi(z)\,\phi(z_0)}{\phi'(z)\,[\phi(z)-\phi(z_0)]}\right] + \overline{\psi\left[z, \frac{\phi(z)\,\overline{\phi(z_0)}^{-1}}{\phi'(z)\,[\phi(z)-\overline{\phi(z_0)}^{-1}]}\right]}. \end{aligned} \tag{18}$$

Before discussing the consequences of (18), we introduce some more elementary variations in F which will allow us to simplify the result (18). We map the domain $D(\Gamma)$ onto $|w| > 1$ by means of the function $w = \phi(z)$; we then turn this circle into itself by the linear mapping $w_1 = e^{i\epsilon} w$ and return to the z-plane through $\phi^{-1}(w_1)$. Thus, the function

$$v_1(z) = e^{-i\epsilon}\phi^{-1}[e^{i\epsilon}\phi(z)] \tag{19}$$

is univalent in $D(\Gamma)$ and hence in D. For small ϵ, we have the series development in ϵ

$$v_1(z) = z + i\epsilon\left[\frac{\phi(z)}{\phi'(z)} - z\right] + O(\epsilon^2). \tag{20}$$

Since $f^*(t) = v_1[f(t)]$ is an admissible competing function in F, we deduce easily from the extremum property of $f(t)$ and from the freedom in the choice of the real parameter ϵ

$$\psi\left[z, \frac{\phi(z)}{\phi'(z)} - z\right] = \text{real}. \tag{21}$$

Another possible infinitesimal variation is obtained by

$$v_2(z) = (1+\epsilon)^{-1}\phi^{-1}[(1+\epsilon)\,\phi(z)] \quad (\epsilon > 0). \tag{22}$$

In fact, we may map $D(\Gamma)$ onto $|w| > 1$, magnify the unit circle by a factor $(1+\epsilon)$ and return through $\phi^{-1}(w)$ to the z-plane. The function $f^*(t) = v_2[f(t)]$ lies also in F and from the extremum property of $f(t)$ we deduce by use of (21) the inequality

$$\psi\left[z, \frac{\phi(z)}{\phi'(z)} - z\right] \leqslant 0. \tag{23}$$

We return now to formula (18) and observe that in view of (21)

$$\lim_{z_0 \to \Gamma} \psi\left[f(t), \frac{1}{f(t)-z_0}\right] \frac{\phi(z_0)^2}{\phi'(z_0)^2} = \text{real}. \tag{24}$$

In order to simplify the discussion we shall assume that

$$\psi\left[z, \frac{1}{z-z_0}\right] = W(z_0)$$

is a meromorphic function of z_0; this is, indeed, the case in most applications. We put $z = \psi(w)$, where ψ is the inverse function of $w = \phi(z)$, and obtain from (24) the boundary relation

$$\lim_{|w| \to 1} W[\psi(w)]\, w^2 \psi'(w)^2 = \text{real}, \tag{25}$$

for the function $\psi(w)$ which is analytic in $|w| > 1$ and maps this circular domain onto $D(\Gamma)$. By the Schwarz reflection principle, the function $W[\psi(w)]\, w^2\psi'(w)^2$ can then be continued analytically into the domain $|w| \leqslant 1$. Thus, $\psi(w)$ satisfies a first-order differential equation with analytic coefficients in the entire w-plane. This fact shows that Γ is composed of analytic arcs and the same holds for the boundary C of the extremum domain D: *C is composed of analytic arcs.*

In order to complete the argument we need a last elementary variation. We again map $D(\Gamma)$ onto the domain $|w| > 1$ by means of $\phi(z)$. The function

$$\omega = p(w) = w + \frac{w_0^2}{w} \quad (|w_0| = 1) \tag{26}$$

maps the circular region $|w| > 1$ onto the ω-plane slit along the segment between the points $-2w_0$ and $+2w_0$. It is then easily seen that, for $\epsilon > 0$,

$$w_1 = p^{-1}[(1+\epsilon)\, p(w) + 2\epsilon w_0] = w + \epsilon \frac{w(w+w_0)}{w-w_0} + O(\epsilon^2) \tag{27}$$

provides a mapping of $|w| > 1$ onto the same circular region from which a small radial segment issuing from the periphery point w_0 has been removed. The function

$$\begin{aligned} v_3(z) &= (1+\epsilon)^{-1} \phi^{-1}\left[\phi(z) + \epsilon \frac{\phi(z)\,[\phi(z)+\phi(z_0)]}{\phi(z)-\phi(z_0)} + O(\epsilon^2)\right] \\ &= z + \epsilon\left[\frac{\phi(z)}{\phi'(z)} \frac{\phi(z)+\phi(z_0)}{\phi(z)-\phi(z_0)} - z\right] + O(\epsilon^2) \quad (z_0 \in \Gamma) \end{aligned} \tag{28}$$

is then normalized at infinity and univalent in D. Hence, $f^*(t) = v_3[f(t)]$ is again a competing function in our extremum problem, whence

$$\operatorname{Re}\left\{\psi\left[z, \frac{\phi(z)}{\phi'(z)} \frac{\phi(z)+\phi(z_0)}{\phi(z)-\phi(z_0)} - z\right]\right\} \leqslant 0. \tag{29}$$

But observe that the left side of (29) coincides with the right-hand term of (18) since $z_0 \in \Gamma$. Hence, we have proved

$$\psi\left[f(t), \frac{1}{f(t)-z_0}\right]\frac{\phi(z_0)^2}{\phi'(z_0)^2} \leqslant 0 \quad (z_0 \in \Gamma). \tag{30}$$

Since $|\phi(z)| = 1$ for $z \in \Gamma$ we have $\log\phi(z) = \text{imaginary}$ on Γ and, consequently, we can write on each analytic arc of Γ

$$z'\frac{\phi'(z)}{\phi(z)} = \text{imaginary}, \quad z' = \frac{dz}{ds}. \tag{31}$$

Thus, we may express (30) also in the form

$$\psi\left[f(t), \frac{1}{f(t)-z}\right]\left(\frac{dz}{ds}\right)^2 \geqslant 0 \quad \text{on} \quad C. \tag{32}$$

In this final form the characterization of the extremum domain has become independent of the choice of the subcontinuum Γ. The boundary arcs of C are determined by a first-order differential equation involving the meromorphic function $W(z)$ defined above.

Under our assumptions made regarding the functional $\psi[z, 1/(z-z_0)]$ it is also easy to prove that the extremum domain cannot possess exterior points. For, suppose z_0 were an exterior point of an extremum domain D. In this case, the mapping (1) itself would be an admissible univalent variation for ρ small enough and the extremum property of $f(t)$ would imply

$$\operatorname{Re}\left\{e^{i\alpha}\rho^2\psi\left[z, \frac{1}{z-z_0}\right]\right\} + O(\rho^4) \leqslant 0, \tag{33}$$

whence easily

$$\psi\left[z, \frac{1}{z-z_0}\right] = 0. \tag{34}$$

But if, as supposed, ψ is a specific meromorphic function $W(z_0)$, not identically zero, this result is impossible since (34) would imply by analytic continuation that $W(z_0) \equiv 0$[19]. Thus, we have proved the

Theorem. The extremum domain of the extremum problem $\phi[f] = \max$ *within the family* F *is a slit domain bounded by analytic arcs. Each satisfies the differential equation*:

$$\psi\left[f(t), \frac{1}{f(t)-z(\tau)}\right]\left(\frac{dz}{d\tau}\right)^2 = 1, \tag{35}$$

where τ *is a properly chosen real curve parameter.*

This theorem was proved originally[26] by means of rather deep theorems of measure theory. It can be derived in elementary manner from the variational formula for the Green's function as shown here. It permits now a systematic and unified treatment of numerous extremum problems of conformal mapping. The extremum domain can be determined either by integrating the differential equation (35) for the boundary slits or by solving the differential equation implied by (25) for the functions $\psi(w)$ which map the circular domain $|w| > 1$ onto the domains $D(\Gamma)$. The latter procedure is particularly convenient in the case that the original domain T is simply connected.

4. The coefficient problem

The best studied extremum problem in conformal mapping is without any doubt the coefficient problem for the functions univalent in the unit circle. We consider all power series

$$f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots, \tag{36}$$

which converge for $|z| < 1$ and which represent univalent functions. Bieberbach stated the conjecture

$$|a_n| \leqslant n. \tag{37}$$

Since the 'Koebe function'

$$\frac{z}{(1-z)^2} = z + 2z^2 + \ldots + nz^n + \ldots \tag{38}$$

is indeed such a univalent power series, this function would seem to be the solution of an infinity of extremum problems. Because of its simple formulation the conjecture (37) has attracted the attention of many analysts. Bieberbach himself proved (37) in 1916 for $n = 2$[2]; Löwner proved the case $n = 3$ in 1923[18] and Garabedian and Schiffer proved the case $n = 4$ in 1955[5]. These proofs are to be considered as tests for our technique in handling extremum problems of conformal mapping and the main significance of the coefficient problem is indeed that it raises a challenge to our various methods in this field. We want to give a brief survey of variational methods applied in this problem.

We define a sequence of polynomials $P_n(x)$ of degree $(n-1)$ by means of the generating function

$$\frac{f(z)}{1 - xf(z)} = \sum_{n=1}^{\infty} [a_n + P_n(x)]\, z^n \quad (P_1(x) = 0). \tag{39}$$

We note down the first few polynomials

$$P_2(x) = x, \quad P_3(x) = 2a_2 x + x^2, \quad P_4(x) = (2a_3 + a_2^2)\, x + 3a_2 x^2 + x^3. \tag{39'}$$

A simple application of the reasoning in the preceding section leads to the following result. Let $f(z)$ be a univalent function which maximizes $|a_n|$; we can make the permissible assumption $a_n > 0$. Then $f(z)$ satisfies the differential equation[27]

$$\frac{z^2 f'(z)^2}{f(z)^2} P_n\left[\frac{1}{f(z)}\right] = \frac{1}{z^{n-1}} + \frac{2a_2}{z^{n-2}} + \frac{3a_3}{z^{n-3}} + \ldots + \frac{(n-1)\,a_{n-1}}{z}$$
$$+(n-1)\,a_n + (n-1)\,\bar{a}_{n-1} z + \ldots + 3\bar{a}_3 z^{n-3} + 2\bar{a}_2 z^{n-2} + z^{n-1}. \quad (40)$$

The right side as well as the polynomial $P_n(x)$ depends on the coefficients of the unknown function $f(z)$; hence, (40) represents a rather complicated functional equation for the extremum function sought which has been solved until now only in the cases $n \leqslant 4$.

We may attack the functional equation (40) as follows. It is easily shown in all cases $n \leqslant 4$ that the extremum function $w = f(z)$ maps the domain $|z| < 1$ onto the entire w-plane slit along a single analytic arc Γ which runs out to infinity. We consider then the analytic functions

$$w = f(z,t) = e^t[z + a_2(t)\,z^2 + \ldots + a_n(t)\,z^n + \ldots], \quad (41)$$

which map $|z| < 1$ onto the w-plane slit along infinite subarcs Γ_t of Γ. We can read off from (40) that Γ satisfies the differential equation

$$\frac{w'(\tau)^2}{w(\tau)^2} P_n\left[\frac{1}{w(\tau)}\right] + 1 = 0 \quad (\tau = \text{real parameter}), \quad (42)$$

and evidently the subarcs Γ_t satisfy precisely the same equation. Using next the Schwarz reflection principle, we can show that the functions $f(z,t)$ satisfy differential equations which are very similar to (40); namely

$$\frac{z^2 f'(z,t)^2}{f(z,t)^2} P_n\left[\frac{1}{f(z,t)}\right] = \sum_{\nu=-(n-1)}^{n-1} A_\nu(t)\,z^\nu = q(z,t),\ A_{-\nu}(t) = \overline{A_\nu(t)}. \quad (43)$$

We may transform (43) into

$$\int_{f(z_0,t)}^{f(z,t)} \sqrt{\left[P_n\left(\frac{1}{w}\right)\right]}\frac{dw}{w} = \int_{z_0}^{z} \sqrt{[q(z,t)]}\frac{dz}{z}. \quad (44)$$

Löwner has shown[18] that the functions $f(z,t)$ which represent the unit circle on a family of slit domains with growing boundary slits Γ_t of the above type satisfy the partial differential equation

$$\frac{\partial f(z,t)}{\partial t} = z\frac{1+\kappa(t)\,z}{1-\kappa(t)\,z}\frac{\partial f(z,t)}{\partial z} \quad (\kappa(t) \text{ continuous, } |\kappa| = 1). \quad (45)$$

Thus, differentiating (44) with respect to t and using (43) and (44) we find

$$\sqrt{[q(z,t)]}\frac{1+\kappa z}{1-\kappa z}-\sqrt{[q(z_0,t)]}\frac{1+\kappa z_0}{1-\kappa z_0}=\frac{1}{2}\int_{z_0}^{z}\frac{\partial q(z,t)}{\partial t}\frac{1}{\sqrt{[q(z,t)]}}\frac{dz}{z}. \tag{46}$$

Differentiating (46) again with respect to z, we find after simple rearrangement

$$\frac{\partial q(z,t)}{\partial t}=z\frac{1+\kappa z}{1-\kappa z}\frac{\partial q(z,t)}{\partial z}+\frac{4\kappa z}{(1-\kappa z)^2}q(z,t). \tag{47}$$

On the other hand, $q(z,t)$ is a simple rational function of z as is seen from its definition (43). When we insert its expression into (47) and compare the coefficients of equal powers of z on both sides, we obtain

$$\frac{dA_\nu(t)}{dt}=\nu A_\nu(t)+2\sum_{\mu=-(n-1)}^{\nu-1}(2\nu-\mu)\,A_\mu\kappa^{\nu-\mu}. \tag{48}$$

In order that $A_\nu(t)\equiv 0$ for all $\nu\geqslant n$ it is necessary and sufficient that

$$\sum_{\mu=-(n-1)}^{n-1}A_\mu\kappa^{-\mu}\equiv 0,\qquad \sum_{\mu=-(n-1)}^{n-1}\mu A_\mu\kappa^{-\mu}\equiv 0\quad\text{identically in } t. \tag{49}$$

These conditions guarantee also that $A_{-\nu}\equiv\bar{A}_\nu$ is fulfilled for all values of t.

We observe that the equations (48) for $\nu=-1,-2,\ldots,-(n-1)$ give $(n-1)$ differential equations for the corresponding functions $A_\nu(t)$; their coefficients depend in a very simple manner on $\kappa(t)$. The function $\kappa(t)$, in turn, can be determined from the $A_\nu(t)$ by means of the second equation (49), which can be written in the form

$$\operatorname{Im}\left\{\sum_{\mu=-(n-1)}^{-1}\mu A_\mu\kappa^{-\mu}\right\}=0. \tag{50}$$

Thus, $A_{-1},A_{-2},\ldots,A_{-n-1}$ and κ satisfy a well-determined system of ordinary differential equations.

Let us start with the case $n=3$. The differential system to be considered is

$$\left.\begin{aligned}\frac{dA_{-2}(t)}{dt}=-2A_{-2}(t),\quad \frac{dA_{-1}(t)}{dt}=-A_{-1}(t),\\ \operatorname{Im}\{2A_{-2}\kappa^2+A_{-1}\kappa\}=0.\end{aligned}\right\} \tag{51}$$

We can integrate immediately and find

$$A_{-2}(t)=\alpha_2e^{-2t},\quad A_{-1}(t)=\alpha_1e^{-t}. \tag{52}$$

Since for $t=0$ the function $q(z,0)$ coincides with the right side of (40) for $n=3$, we determine the constants of integration as follows: $\alpha_2=1$, $\alpha_1=2a_2$. Thus, $\kappa(t)$ satisfies the equation

$$e^{-2t}\kappa^2+a_2e^{-t}\kappa=\text{real}. \tag{53}$$

From the general Löwner theory it is well-known that

$$a_2 = -2\int_0^\infty \kappa\, e^{-t}\, dt. \tag{54}$$

We have to utilize now the inequality $|a_3 - a_2^2| \leqslant 1$ which follows from the elementary area theorem. Since we assume $a_3 \geqslant 3$ we can assert $\operatorname{Re}\{a_2^2\} \geqslant 2$ and see that the left side of (53) cannot vanish for $0 \leqslant t < \infty$.

We wish to show next that $a_2^2 =$ real in consequence of (53) and (54). Indeed, if a_2^2 were not real, equation (53) would exclude the possibility $\kappa = \pm \operatorname{sgn} a_2$ and the expression $\operatorname{Im}\{\overline{\kappa(t)}\, a_2\}$ could never change its sign. Consequently

$$\int_0^\infty \operatorname{Im}\{\overline{2\kappa(t)}\, a_2\}\, e^{-t} dt = -\operatorname{Im}\{|a_2|^2\} \tag{55}$$

could not be zero, which yields a contradiction. Thus, $a_2^2 =$ real and in consequence of the area theorem even $a_2^2 > 2$ holds; hence, we conclude $a_2 =$ real. From (53) follows then easily that κ must be real throughout and it can be shown that $a_3 = 3$.

The above proof for $|a_3| \leqslant 3$ is more complicated than Löwner's original proof which made use only of the formula (45). It can, however, be generalized to the problem of a_4 though it becomes in this case still more complicated. The differential system becomes now

$$\left.\begin{aligned} &\frac{dA_{-3}}{dt} = -3A_{-3}, \quad \frac{dA_{-2}}{dt} = -2A_{-2} - 2A_{-3}\kappa, \quad \frac{dA_{-1}}{dt} = -A_{-1} + 2A_{-3}\kappa^2, \\ &\qquad\qquad \operatorname{Im}\{3A_{-3}\kappa^3 + 2A_{-2}\kappa^2 + A_{-1}\kappa\} = 0. \end{aligned}\right\} \tag{56}$$

We find $A_{-3} = \alpha_3 e^{-3t}$ and, since $A_{-3}(0) = 1$, we have $A_{-3} = e^{-3t}$. We set up

$$A_{-2}(t) = \alpha_2(e^{-t})\, e^{-2t}, \quad A_{-1}(t) = \alpha_1(e^{-t})\, e^{-t}; \tag{57}$$

inserting into (56) and putting $\sigma = e^{-t}$, we arrive at the differential system

$$\left.\begin{aligned} &\frac{d\alpha_2(\sigma)}{d\sigma} = 2\kappa, \quad \frac{d\alpha_1(\sigma)}{d\sigma} = -2\kappa^2\sigma \quad (0 \leqslant \sigma \leqslant 1), \\ &\operatorname{Im}\{3\kappa^3\sigma^3 + 2\alpha_2(\sigma)\,\kappa^2\sigma^2 + \alpha_1(\sigma)\,\kappa\sigma\} = 0. \end{aligned}\right\} \tag{58}$$

A simple calculation leads to the boundary conditions

$$\left.\begin{aligned} \alpha_2(0) &= 3a_2, \quad \alpha_1(0) = 2a_3 + a_2^2, \\ \alpha_2(1) &= 2a_2, \quad \alpha_1(1) = 3a_3. \end{aligned}\right\} \tag{59}$$

Those for $\sigma = 1$, $t = 0$ are obvious; those for $\sigma = 0$, $t = \infty$ follow by comparison of coefficients of powers of e^{-t} in (43) and by passage to the limit $t = \infty$.

The differential system (58), together with the boundary conditions (59), represents a typical Sturm–Liouville boundary value problem. We have to start integration of (58) with such initial values $\alpha_1(0)$ and $\alpha_2(0)$ that we end up at the other end of the interval considered with

$$\alpha_1(1) = \tfrac{3}{2}[\alpha_1(0) - \tfrac{1}{9}\alpha_2(0)^2], \quad \alpha_2(1) = \tfrac{2}{3}\alpha_2(0). \tag{60}$$

The difficulty of the problem lies in the non-linear character of the equations and of the boundary conditions. Each possible set $\alpha_1(0)$, $\alpha_2(0)$ determines a set of possible values a_2, a_3. Clearly, $a_2 = 2$, $a_3 = 3$ and $\kappa(\sigma) \equiv -1$ is an admissible solution which leads to the Koebe function (38), the conjectured extremum function.

The question arises now whether the corresponding special values $\alpha_1(0)$, $\alpha_2(0)$ connected with the conjectured extremum function might not be imbedded into a one-parameter family of initial values such that all of them lead to the boundary relations (60). For this purpose, we have to study the variational equations of the system (58) and of the boundary conditions (60). If we denote the derivatives of α_1, α_2 and κ with respect to the parameter by β_1, β_2 and $i\lambda$, we find easily

$$\left.\begin{aligned} &\frac{d\beta_1}{d\sigma} = 4i\lambda\sigma, \quad \frac{d\beta_2}{d\sigma} = 2i\lambda, \quad \lambda = \frac{1}{2p(\sigma)}\operatorname{Im}\{\beta_1 - 2\beta_2\sigma\}, \\ &\beta_1(1) = \tfrac{3}{2}[\beta_1(0) - \tfrac{4}{3}\beta_2(0)], \quad \beta_2(1) = \tfrac{2}{3}\beta_2(0); \quad p(\sigma) = 8\sigma^2 - 12\sigma + 5. \end{aligned}\right\} \tag{61}$$

We are thus led to a linear differential system with linear boundary conditions which can be treated by the standard Sturm–Liouville methods.

It is immediately seen from (61) that λ is real and that $\beta_1(\sigma)$ and $\beta_2(\sigma)$ must be pure imaginary. When we introduce the new unknowns

$$u(\sigma) = \operatorname{Im}\{\beta_1 - 2\sigma\beta_2\}, \quad v(\sigma) = \operatorname{Im}\{\beta_2\}$$

the system (61) simplifies to

$$\frac{du}{d\sigma} = -2v, \quad \frac{dv}{d\sigma} = \frac{1}{p(\sigma)}u, \tag{62}$$

with the boundary conditions

$$u(1) = \tfrac{3}{2}u(0) - \tfrac{10}{3}v(0), \quad v(1) = \tfrac{2}{3}v(0). \tag{63}$$

From the differential system we derive by integration by parts the equality

$$2\int_0^1 v^2\,d\sigma = \int_0^1 pv'^2\,d\sigma + \tfrac{20}{9}v(0)^2; \quad v(1) = \tfrac{2}{3}v(0). \tag{64}$$

We may now apply the calculus of variations in order to estimate the ratio

$$\left[\int_0^1 pv'^2 d\sigma + \tfrac{20}{9}v(0)^2\right].\left[\int_0^1 v^2 d\sigma\right]^{-1} = R[v] \tag{65}$$

under the given boundary condition on $v(\sigma)$. Even when we replace in (65) the polynomial $p(\sigma)$ by a piecewise constant function which is nowhere larger than $p(\sigma)$ the minimum value of the new ratio, which can now be computed explicitly, comes out to be larger than 2. Hence, *a fortiori*, we can assert that $R[v] > 2$ for all admissible $v(\sigma)$ and that (64) is impossible. We have thus shown that the solution $a_2 = 2$, $a_3 = 3$, $\kappa(\sigma) \equiv -1$ cannot be imbedded into a one-parameter family of solutions which can be differentiated continuously with respect to this parameter.

By a more careful analysis we may now treat differences of solution systems $\alpha_1(\sigma)$, $\alpha_2(\sigma)$, $\kappa(\sigma)$ instead of differentials. We can then delimit an entire neighborhood of the point $a_2 = 2$, $a_3 = 3$, $\kappa = -1$ in which no other solution point could be located. On the other hand, one can combine the area theorem with various relations between the coefficients of the extremum function which arise from the differential equation (40), in order to estimate the values $|a_2 - 2|$ and $|a_3 - 3|$ in the extremum case. It can be seen by elementary if very tedious calculations that the point a_2, a_3, $\kappa(1)$ must lie precisely in the neighborhood in which 2, 3, -1 is the only solution point. This proves that the Koebe function (38) is, indeed, the extremum function and establishes the inequality $|a_4| \leqslant 4$ for all univalent functions (36).

The actual labor in the proof sketched here lies in the very extensive elementary estimations and could probably be reduced considerably by extending the uniqueness neighborhood through greater attention to the theory of the differential system (58), (59).

It may be remarked, finally, that the Koebe function (38) satisfies the functional equation (40) which characterizes the extremum function for every $n \geqslant 2$. This fact tends, of course, to strengthen the evidence for the Bieberbach conjecture. The following fact should be mentioned, however, in order to caution against too great reliance on this evidence. One may consider the family of functions

$$f(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots + \frac{b_n}{z^n} + \dots, \tag{66}$$

which are univalent in the outside $|z| > 1$ of the unit circle and one may ask for $\max |b_n|$. The same variational technique as above yields for the extremum functions $f_n(z)$ of this 'exterior' problem a differential-functional equation which is analogous to (40). It is easy to show that the functions

$$F_n(z) = \left[z^{n+1}+2+\frac{1}{z^{n+1}}\right]^{1/(n+1)} = z+\frac{2}{n+1}\frac{1}{z^n}+\ldots \tag{67}$$

belong to the family considered and satisfy the extremum condition for the corresponding $f_n(z)$. For $n = 1$ and $n = 2$ these functions are, indeed, the extremum functions of the exterior coefficient problem. The estimate $|b_1| \leqslant 1$ was discovered together with the area theorem[2] and $|b_2| \leqslant \frac{2}{3}$ was established in 1938 by Golusin[6] and myself[25]. It was conjectured that $|b_n| \leqslant 2/(n+1)$ was the best possible estimate for the nth coefficient for all values of n. However, in 1955 Garabedian and I[4] showed that the precise value of the maximum for $|b_3|$ is not $\frac{1}{2}$ as expected but $\frac{1}{2}+e^{-6}$. Thus, in spite of the fact that the function $F_3(z)$, defined in (67), satisfies the rather restrictive extremum condition, it is not the extremum function $f_3(z)$. Since e^{-6} is a small number, this example shows also how little empirical numerical evidence can be trusted in problems of this kind. Recently, Waadeland[36] has shown that quite generally

$$\max |b_{2k-1}| \geqslant \frac{1}{k}(1+2e^{-2[(k+1)/(k-1)]}), \tag{67'}$$

while for $n = 2k$ no counter example to the conjecture $|b_n| \leqslant 2/(n+1)$ seems to be known.

There are, of course, numerous cross-relations between the coefficient problem for univalent functions and the general theory of conformal mapping. Two examples may serve as illustrations. There is a well-known problem in the theory of conformal mapping: given n points in the complex plane, to find a continuum which contains these points and has minimum capacity[10]. From the topology of the extremum continuum, one can derive by an elementary variation the coefficient inequality $|b_2| \leqslant \frac{2}{3}$[25]. Here, the general theory of conformal mapping helped to solve a coefficient problem. Conversely, de Possel[21] formulated a simple extremum problem for the coefficients of univalent functions in a multiply-connected domain and showed that the extremum functions mapped the domain onto a parallel slit domain. Since the existence of an extremum function is assured, an elegant existence proof for an important canonical mapping was thus established.

5. Fredholm eigenvalues

The problem of conformally mapping a given plane domain D can often be reduced to a boundary value problem for the functions harmonic in D. If the boundary C of D is sufficiently smooth, the latter problem can be attacked through the Poincaré–Fredholm integral equation

$$m(z) = \mu(z) + \frac{1}{\pi}\int_C \frac{\partial}{\partial n_\zeta}\left(\log\frac{1}{|z-\zeta|}\right)\mu(\zeta)\,ds_\zeta \quad (z \in C). \tag{68}$$

In order to solve this fundamental integral equation of two-dimensional potential theory one has to consider the corresponding homogeneous integral equation

$$\phi_\nu(z) = \frac{\lambda_\nu}{\pi}\int_C \frac{\partial}{\partial n_\zeta}\left(\log\frac{1}{|z-\zeta|}\right)\phi_\nu(\zeta)\,ds_\zeta \quad (z \in C), \tag{69}$$

its eigenfunctions $\phi_\nu(z)$ and its eigenvalues λ_ν. The eigenvalue $\lambda = 1$ occurs always and has as eigenfunctions a set of easily described functions on C; we shall call this eigenvalue the trivial eigenvalue of the domain. The non-trivial eigenvalues λ_ν satisfy the inequality $|\lambda_\nu| > 1$. It is easily seen that with each non-trivial eigenvalue λ_ν also the value $-\lambda_\nu$ will occur as eigenvalue of (69) with the same multiplicity. We shall restrict ourselves, therefore, to the positive non-trivial eigenvalues λ_ν and assume them ordered in increasing magnitude. These eigenvalues λ_ν are called the Fredholm eigenvalues of the domain D and they are of importance for the potential theory and the function theory of the domain considered.

It is, for example, of great interest to obtain a lower bound for the first eigenvalue λ_1 of a given domain. Such information would enable us to estimate the speed of convergence of the Neumann–Liouville series which solves the basic equation (68). The larger λ_1 can be asserted to be, the easier the numerical work for the solution of the boundary value problems in the potential theory for D. Thus, the λ_ν seem to be a set of functionals of D which deserves a careful study.

The λ_ν are also closely related to the theory of the Hilbert transformation

$$F(z) = \frac{1}{\pi}\iint_D \frac{\overline{f(\zeta)}}{(\zeta-z)^2}\,d\tau_\zeta, \tag{70}$$

which carries each analytic function in D into a new analytic function in the same domain. There exists a set of eigenfunctions $w_\nu(z)$ which are analytic in D and which satisfy

$$w_\nu(z) = \frac{\lambda_\nu}{\pi}\iint_D \frac{\overline{w_\nu(\zeta)}}{(\zeta-z)^2}\,d\tau_\zeta \quad (\lambda_\nu > 1). \tag{71}$$

The eigenvalues λ_ν are precisely the Fredholm eigenvalues defined above. We shall assume the $w_\nu(z)$ to be normalized by the usual convention

$$\iint_D |w_\nu(z)|^2 \, d\tau = 1. \tag{72}$$

The eigenfunctions $w_\nu(z)$ form an orthonormal set of analytic functions in D and play an interesting role in the theory of the kernel function of D[33].

In order to establish a unified theory for the treatment of extremum problems for the functionals λ_ν of D it is necessary to determine the variation of each λ_ν for a variation of the defining domain D. If we assume the variation to be of the special type (1) with $z_0 \in D$ and if λ_ν is non-degenerate, we have

$$\lambda_\nu^* = \lambda_\nu + (1-\lambda_\nu^2)\,\pi \operatorname{Re}\{e^{i\alpha}\rho^2 w_\nu(z_0)^2\} + O(\rho^4). \tag{73}$$

An analogous, though slightly more complicated, formula can be given for the variation of degenerate eigenvalues.

When one wishes to apply the variational formula (73) to the solution of extremum problems, one runs immediately into a serious difficulty. The entire theory of the Fredholm eigenvalues has been established under certain smoothness conditions for the boundary and one has to be sure that the extremum domain does possess a boundary of this type. One has to introduce a class of domains which possess admissible boundaries and which is compact; within such a class the calculus of variations based on (73) and the theory of extremum problems become possible.

For this purpose, we introduce the concept of *uniformly analytic curves*. A curve is called analytic if it can be obtained as the image of the unit circumference $|z| = 1$ by means of a function $t(z)$ which is analytic and univalent on $|z| = 1$. A set of curves is said to be uniformly analytic with the modulus of uniformity (r, R) (where $r < 1 < R$) if all of them are obtained by means of mapping functions $f(z)$ which are analytic and univalent in the fixed annulus $r \leqslant |z| \leqslant R$. This concept of uniform analyticity seems to be quite useful in the variational theory of domain functionals.

We can now formulate the theorem:

If a simply connected domain is bounded by a curve which is analytic with the modulus (r, R), then its lowest Fredholm eigenvalue λ_1 satisfies the inequality:

$$\lambda_1 \geqslant \frac{r^2+R^2}{1+r^2R^2}. \tag{74}$$

This estimate is the best possible for every modulus (r, R).

Frequently, the boundary curve C of a domain is given in a parametric representation from which the modulus (r, R) can be readily deduced. Thus, the estimate (74) is often convenient to predict the convergence of the Neumann–Liouville series which solve the various boundary value problems in the domain.

We may also connect with a given domain D the Fredholm determinant

$$D(\lambda) = \prod_{\nu=1}^{\infty} \left(1 - \frac{\lambda^2}{\lambda_\nu^2}\right) \tag{75}$$

of the integral equation (68) and consider, for fixed λ, $D(\lambda)$ as a functional of the domain D. The following extremum problem suggests itself: Let D_0 be a given multiply-connected domain; consider all smoothly bounded domains D which are conformally equivalent to it and ask for those domains in this equivalence class which yield the maximum value of $D(\lambda)$.

This problem has been solved in the case $\lambda = 1$. The main difficulty in the investigation was again the non-compactness of the class of domains considered. It could be overcome by considering maximum sequences of domains and their limit domain; all domains of the sequence were subjected to the same variation (1) and from the fact that they formed a maximum sequence it could be shown that their limit domain is analytically bounded. Then, the existence of a maximum domain is easily established and it can be shown that it is bounded by circumferences. We obtain thus a new proof of Schottky's famous circular mapping theorem and also a characterization of this canonical mapping by an extremum property. Methodologically, the proof is of interest since the method of variation is not applied to the extremum domain, whose existence is not yet known, but to the extremum sequence. This procedure seems to be of very great applicability.

The solution of the maximum problem for general $D(\lambda)$ is not yet known and well deserves additional study.

The Fredholm eigenvalues represent an instructive example for the flexibility of the variational method in dealing with extremum problems for rather difficult types of domain functionals. The great formal elegance of the variational formula (73) enabled us to overcome the quite serious difficulties which arise from the fact that these functionals are defined only for a restricted and non-compact class of domains.

6. Further applications

We have restricted ourselves to a few fundamental problems in order to exhibit clearly the basic ideas of the variational method. It may, however, be applied to much more general function-theoretic problems. It can be used in problems of mapping of domains on Riemann surfaces[29, 33] and leads there to existence theorems for various canonical realizations of Riemann domains. It can be applied to the theory of multivalent functions in a given domain[27], their coefficient problems and distortion theorems. Some interest has been devoted to the problem of developing a calculus of variations within important subclasses of the family of univalent functions in the unit circle. Golusin[9] described a method of variations for the subclass of star-like univalent mappings and Hummel[12, 13] gave an even simpler method of this kind. Singh[34] gave a theory of variations for real univalent functions, for bounded univalent functions and other interesting subclasses. Finally, the role should be mentioned which the method of variations could play as a useful tool in the theory of quasi-conformal mappings and of extremal metrics[14].

The variational method is, of course, only one of many powerful methods in the theory of conformal mapping and complex function theory. There are many problems where other methods give the answer more easily and directly. It seems to me, however, that the method of variations is one of the most systematic and widely applicable methods which we possess in this field.

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COHOMOLOGY OPERATIONS AND SYMMETRIC PRODUCTS

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This lecture is embodied in a set of mimeographed notes entitled 'Cohomology operations and obstructions to extending continuous functions'. These can be obtained by writing to the Department of Mathematics, Fine Hall, Box 708, Princeton, N.J., U.S.A.