ON SOME METHODS OF APPROXIMATION IN FLUID MECHANICS

S. GOLDSTEIN

1. *Introduction.*

Certain recent work leads to the belief that it is time to consider again the Navier-Stokes equations for the steady flow of an incompressible viscous fluid, with a hope that a deeper insight into the nature of the solutions may be achieved. The investigations are concerned with an understanding of the nature, and extensions, of approximations at small and at large Reynolds numbers. I shall, however, not attempt a review of the work at small Reynolds numbers; at the present time the asymptotic approach at large Reynolds numbers seems more important and interesting, and moreover I have nothing in any way new to offer about small Reynolds numbers. First, then, some recent developments of steady, laminar, boundary-layer theory will be considered. Most modern research in fluid dynamics is concerned with turbulence or the high-speed flow of gases; after the success of Prandtl's boundary-layer theory interest died down, except for special problems and (in particular) three-dimensional effects. (See Ref. 1). I shall be concerned with a different aspect, considering boundary-layer theory as a first step towards obtaining asymptotic expansions for large Reynolds numbers. The aim at this stage is simply to show what is involved in constructing such asymptotic solutions.

Approximations at large Reynolds numbers will provide flows that are unstable, and no guidance is to be expected from experiment. However, the necessity for considering such approximations remains, not only for the sake of a deeper mathematical-physical understanding (which is also needed for considering boundary layers in high-speed gas flow), but also to begin the study of asymptotic expansions that will apply at moderate, or even fairly small, Reynolds numbers.

The simplest example is still that of two-dimensional flow past a semi-infinite flat plate parallel to the stream, and it is with this example that we shall be largely concerned. The important point, however, is that the plate is semi-infinite—there is no wake. It appears that when a wake is present the limit of the steady flow as the Reynolds number tends to infinity is not known with sufficient certainty. (Even for a finite flat plate parallel to the stream it is known only approximately).

The mathematical theory involved is that of singular perturbations: when viscosity is neglected, the governing equations are non-linear; the perturbation is linear, and contains higher derivatives with a small coefficient. Fluid mechanics is full of such problems. The Navier-Stokes equations provide one of the hardest examples, for although the non-linear part, by itself, is usually integrated at once to provide the simple linear Laplace equation when viscosity is neglected, the non-linearity makes itself strongly felt as soon as the viscous perturbation is considered. Other examples in fluid mechanics may arise from problems of heat transfer in moving fluids, from entropy variations (in the initial stages) behind a shock wave of varying strength, and from the theory of long waves on shallow water.

When the viscosity tends to zero, one of the most interesting phenomena is the appearance of singular surfaces such as vortex sheets and shock waves; we want to know more details for large finite Reynolds numbers of the flows inside the thin layers which become such singular surfaces for infinite Reynolds numbers. In particular, shock-wave phenomena in one-dimensional flows are described by nonlinear wave equations perturbed by linear higher-order terms. I shall not, however, discuss the shock-wave equations directly, but shall here, for mathematical illustration, go over to results that can be proved, by using as examples (and generalizing) two equations that have been solved explicitly (one of which has some importance in chemical engineering), and which show the nature of the phenomena quite clearly.

2. *The Navier-Stokes Equations. The Boundary-Layer Equation, and Solution for Flow along a Flat Plate.*

Non-dimensional velocities and coordinates will be used in writing the Navier-Stokes equations for two-dimensional steady motion. If U is a standard velocity, ν the kinematic viscosity of the fluid, and l a typical length, the Reynolds number is defined by

$$R = Ul/\nu. \tag{1}$$

We shall be concerned with flow past a semi-infinite flat plate parallel to the stream, lying on $y = 0$, $x \geqq 0$, where U is the undisturbed velocity of the stream; there is then no obvious length in the problem. We may simply select one arbitrarily, and keep it constant as $R \to \infty$; $R \to \infty$ means $\nu \to 0$.

Then with a usual notation the Navier-Stokes equations are

$$u = \partial\psi/\partial y, \quad v = -\partial\psi/\partial x, \tag{2}$$

$$\frac{1}{R}\nabla^4\psi + \frac{\partial(\psi, \nabla^2\psi)}{\partial(x, y)} = 0. \tag{3}$$

For the unperturbed equation, $\nabla^2\psi$ is a function of ψ, and we have the classical theorem that with no vorticity in the undisturbed flow at infinity upstream, $\nabla^2\psi = 0$ in any region occupied by streamlines coming from infinity upstream. (This is practically correct for streamlines which have not entered a region where the perturbation terms are important).

According to boundary-layer theory, (cf. Ref. 2), we set

$$y_1 = R^{\frac{1}{2}}y, \quad \chi = R^{\frac{1}{2}}\psi, \tag{4}$$

in (3), and assume that the derivatives that occur are now all $O(1)$ as $R \to \infty$, so the boundary-layer equation is simply obtaining by putting $1/R = 0$ in the resulting equation, which is

$$\left.\begin{aligned} &\frac{\partial}{\partial y_1}\left\{\frac{\partial^3\chi}{\partial y_1^3} + \frac{\partial\chi}{\partial x}\frac{\partial^2\chi}{\partial y_1^2} - \frac{\partial\chi}{\partial y_1}\frac{\partial^2\chi}{\partial x\partial y_1}\right\} + \frac{1}{R}\left\{2\,\frac{\partial^4\chi}{\partial x^2\partial y_1^2} + \frac{\partial\chi}{\partial x}\frac{\partial^3\chi}{\partial x^2\partial y_1}\right. \\ &\qquad\qquad \left. - \frac{\partial\chi}{\partial y_1}\frac{\partial^3\chi}{\partial x^3}\right\} + \frac{1}{R^2}\frac{\partial^4\chi}{\partial x^4} = 0. \end{aligned}\right\} \tag{5}$$

As $y_1 \to \infty$, $\partial^3\chi/\partial y_1^3$ and $\partial^2\chi/\partial y_1^2 \to 0$, and the boundary layer equation is

$$\left.\begin{aligned} \frac{\partial\chi^3}{\partial y_1^3} + \frac{\partial\chi}{\partial x}\frac{\partial^2\chi}{\partial y_1^2} - \frac{\partial\chi}{\partial y_1}\frac{\partial^2\chi}{\partial x\partial y_1} &= P(x) = -\lim_{y_1\to\infty}\frac{\partial\chi}{\partial y_1}\frac{\partial^2\chi}{\partial x\partial y_1} \\ &= -\lim u\frac{\partial u}{\partial x}. \end{aligned}\right\} \tag{6}$$

The function $P(x)$, arising from the integration with respect to y_1, is simply the pressure gradient in the main stream outside the boundary layer.

For flow past the semi-infinite flat plate, with the equations in cartesian coordinates, as here, the boundary conditions are

$$\chi = 0,\ \partial\chi/\partial y_1 = 0 \text{ on the plate } (y_1 = 0,\ x > 0),\ \partial\chi/\partial y_1 \to 1 \quad \text{as } y_1 \to \infty \text{ or as } x \to 0 \tag{7}$$

and the known solution is

$$\chi = x^{\frac{1}{2}}f(\eta_0), \text{ where } \eta_0 = y_1/(2x^{\frac{1}{2}}), \tag{8}$$

$$f''' + ff'' = 0,\ f(0) = f'(0) = 0,\ f' \to 2 \text{ as } \eta_0 \to \infty \tag{9}$$

$$f = \frac{\alpha\eta_0^2}{2!} - \frac{\alpha^2\eta_0^5}{5!} + \dots \qquad (\alpha = 1.328), \tag{10}$$

$$\text{and as } \eta_0 \to \infty,\ f \backsim 2\eta_0 - \beta \qquad (\beta = 1.72) \tag{11}$$

and the error terms in (11) are exponentially small.

$$\left.\begin{aligned} \text{Also } u &= \tfrac{1}{2} f'(\eta_0), \; v = \frac{1}{2R^{\frac{1}{2}}x^{\frac{1}{2}}} \{\eta_0 f'(\eta_0) - f(\eta_0)\} \to V \\ &= \frac{\beta}{2R^{\frac{1}{2}}x^{\frac{1}{2}}} \text{ as } \eta_0 \to \infty \end{aligned}\right\} \tag{12}$$

If τ is the skin-friction,

$$\tau_1 = \frac{4\, R^{\frac{1}{2}}\tau}{\varrho U^2} = \frac{f''(0)}{x^{\frac{1}{2}}} = \frac{\alpha}{x^{\frac{1}{2}}} \tag{13}$$

and the (non-dimensional) displacement thickness is given by

$$\delta_1 = \int_0^\infty (1-u)dy = R^{-\frac{1}{2}} \lim_{y_1\to\infty} (y-\chi) = \frac{\beta x^{\frac{1}{2}}}{R^{\frac{1}{2}}} \tag{14}$$

so $V = d\delta_1/dx$, as it must from the conservation of mass.

The singularity in τ_1 at the leading edge, $x = 0$, is integrable. But v has a singularity all along the line $x = 0$. To order $R^{-\frac{1}{2}}$, $V \neq 0$, and the boundary-layer flow does not join smoothly on to the potential flow, for which the stream-function is $\Psi = y$ (or to any potential flow); although terms of order R^{-1} only have been neglected in (5), there is an error of order $R^{-\frac{1}{2}}$. The streamlines outside the boundary-layer are deflected through a distance δ_1, and to obtain a result correct to order $R^{-\frac{1}{2}}$ this effect must be taken into account in the external potential flow, so as to take into account the velocity V. According to usual boundary-layer theory, the boundary condition on the potential flow is applied at the plate, so the stream-function of the potential flow is taken to be $\Psi_0 + R^{-\frac{1}{2}}\Psi_1$, where $\Psi_0 = y$ in this case, and $R^{-\frac{1}{2}}\Psi_1$ must be equal to $-\delta_1 = -R^{-\frac{1}{2}}\beta x^{\frac{1}{2}}$ at $y = 0$, $x > 0$. Then we should calculate $P(x)$ in (6) for this new potential flow, and correct χ by taking $\chi = \chi_0 + R^{-\frac{1}{2}}\chi_1$, where χ_0 is given by (8).

The potential problem is immediately solved by the use of parabolic coordinates, ξ, η, for which

$$x + iy = (\xi + i\eta)^2. \tag{15}$$

We take $\eta \geqq 0$; $\eta = 0$ on the plate. $\xi = 0$ is the negative real axis, $\xi > 0$ on the upper half plane, and $\xi < 0$ on the lower half plane. The solution for Ψ_1 is $\Psi_1 = -\beta\xi$, so for the potential flow, to order $R^{-\frac{1}{2}}$,

$$\Psi = y - \frac{\beta\xi}{R^{\frac{1}{2}}} = y - \frac{\beta}{R^{\frac{1}{2}}}\left(\frac{\tau + x}{2}\right)^{\frac{1}{2}}, \tag{16}$$

(in the upper half plane) where $\tau = (x^2 + y^2)^{\frac{1}{2}}$. [The solution has a singularity at the leading edge.]

From (16) we find that on the plate $P(x) = 0$ to order $R^{-\frac{1}{2}}$, (Ref. 3, p. 88), and so $\chi_1 = 0$; away from $x = 0$ the boundary-layer solution is correct in this case to order $R^{-\frac{1}{2}}$; the potential flow only needed correction.

Away from a neighbourhood of $x = 0$, $y = 0$, the singularity on $x = 0$ is purely artificial; near $x = 0$, $y = 0$ the difficulty is fundamental.

Next, however, let us first consider the form taken by the boundary-layer equation and solution when we use parabolic coordinates throughout.

Before leaving this section, let us note again that as $\nu \to 0$ or $R \to \infty$, boundary layers become surfaces of discontinuity; that the approach to the limit is non-uniform; and that this is of practical as well as mathematical significance. This type of behaviour is of frequent occurrence in engineering science, not only in fluid mechanics, and usually for practical purposes it is not enough to have the limit; we need quantitative descriptions of the phenomena in the transition zones. In fact, if a particular quantity in which an engineer is interested changes rapidly from one value to another, and if the transition zone becomes narrower and narrower as some parameter is decreased, then often the results for the transition zone are simply shown in a graph on a bigger and bigger scale, which is practically exactly what we do in boundary-layer theory and related theories.

3. *The Boundary-layer Equation and Solution in Parabolic Coordinates.*

Parabolic coordinates have been used by a number of authors. (Refs. 4, 5, 6, 7. Ref. 7 refers to a paper by N. E. Kochin, which I have not seen).

The Navier-Stokes equation (3) is transformed to the coordinates (ξ, η), defined in (15); the substitutions

$$\eta_1 = R^{\frac{1}{2}}\eta, \quad \chi = R^{\frac{1}{2}}\psi, \tag{17}$$

are then made, and $1/R$ is put equal to zero in the equation corresponding with (5). The leading edge is now inside the boundary layer, and the potential flow is approached everywhere as $\eta_1 \to \infty$. The boundary conditions are satisfied if

$\chi = 0$, $\partial\chi/\partial\eta_1 = 0$ on $\eta_1 = 0$, and $(1/\xi)\ (\partial\chi/\partial\eta_1) \to 2$, $(1/\eta)\ (\partial\chi/\partial\xi) \to 2$ as $\eta_1 \to \infty$.

It is easily found that

$$\chi = \xi\, f(\eta_1) \tag{18}$$

is a solution of the resulting boundary-layer equation, where f is the same function as before (eqns. (9), (10), (11)). Moreover, on the plate $\xi = x^{\frac{1}{2}}$, and

$$\tau_1 = \frac{f''(0)}{\xi} = \frac{\alpha}{x^{\frac{1}{2}}} \tag{19}$$

as before (eqn. (13)). But as $\eta_1 \to \infty$,

$$\psi \backsim y - \frac{\beta\xi}{R^{\frac{1}{2}}}, \quad u \backsim 1 - \frac{\beta\eta}{2R^{\frac{1}{2}}(\xi^2+\eta^2)}, \quad v \backsim \frac{\beta\xi}{2R^{\frac{1}{2}}(\xi^2+\eta^2)}, \tag{20}$$

so in parabolic coordinates the external potential flow is included in the boundary-layer solution correctly to order $R^{-\frac{1}{2}}$. Away from the neighbourhood of the leading edge, the singularity on $x = 0$ has completely disappeared. In the expressions for u and v in the potential flow, η^2 must be retained; we must be careful not to put $\eta^2 = \eta_1^2/R$, and then drop this term; η is finite and non-zero in the potential flow.

The results are a particular case of a theorem lately proved by Kaplun (Ref. 7), for any flow without a wake, on the same assumptions that we have used — the usual assumptions of boundary-layer theory. Kaplun shows (i) that there are always coordinates for which the external potential flow is included in the boundary-layer solution correctly to order $R^{-\frac{1}{2}}$; these coordinates (not necessarily or usually orthogonal) are, in our notation, of the form $F_1(\Psi_1)$, $\Psi_0\, F_2(\Psi_1)$; (ii) that if $\chi = F(x, y_1)$ is a boundary-layer solution in *any* coordinates (x, y_1), and x and y_1 are expressed in terms of any other coordinates ξ and η_1, then the boundary-layer solution in the coordinates ξ and η_1 is $\chi = F[x(\xi, 0),\ \eta_1(\partial y_1/\partial \eta_1)_{\eta_1=0}]$; (iii) that the skinfriction τ is unaltered. In our case, $\Psi_0 = y = 2\xi\eta$, $\Psi_1 = -\beta\xi$; for $F_1(\Psi_1)$ we may take ξ, and for F_2, $1/(2\xi)$. The coordinates become ξ and η, as above. Also $x(\xi, 0) = \xi^2$, $\partial y_1/\partial \eta_1 = 2\xi$ and $\chi = x^{\frac{1}{2}} f[y_1/(2\, x^{\frac{1}{2}})]$ becomes $\chi = \xi f[2\xi\eta_1/(2\xi)] = \xi f(\eta_1)$, as above.

Note that near $\eta_1 = 0$, $\chi = \frac{1}{2}\alpha\xi\eta_1^2$, $\psi = \frac{1}{2}\alpha R^{-\frac{1}{2}}\xi\eta_1^2$. Thus for small η_1 (where the motion is slow) ψ is a biharmonic function and a solution of Stokes's equation for creeping flow. This term is the first term of an expansion near the leading edge obtained by Carrier and Lin (Ref. 5). We return to this point later.

The error of the solution given by (18) (away from the neighbourhood of the leading edge, which we shall discuss later) is $O(R^{-1})$. But for other boundaries there will usually be an error of order $R^{-\frac{1}{2}}$, arising from the curvature of the boundary.

4. *The Boundary-layer Solution to order R^{-1}, and the Potential Flow to order $R^{-3/2}$.*

To consider the next approximation to a solution for χ, we may start from the equation in parabolic coordinates analogous to (5), and seek to satisfy it to order R^{-1}. This has been done for the flow in the boundary layer away from the leading edge by Alden (Ref. 4). In our notation, he sets

$$\chi = \xi f(\eta_1) + \frac{1}{R\xi} f_2(\eta_1) + \dots \tag{21}$$

where f_2 satisfies the linear non-homogeneous equation

$$f_2'''' + f f_2''' + 3f' f_2'' + f'' f_2' - f''' f_2 = 2\eta_1 f''(\eta_1 f' - f) \tag{22}$$

and
$$f_2 = \alpha C \frac{\eta_1^2}{2!} - \beta^2 \frac{\eta_1^3}{3!} + \dots \qquad = C(\eta_1 f' - f) + \Delta f_2 \tag{23}$$

α and β are the constants of equations (10) and (11). Two of the constants of integration have been determined from the boundary conditions at $\eta_1 = 0$, and one from a condition as $\eta_1 \to \infty$, as Alden determined them, but the fourth constant C has been left for the present. Δf_2 is independent of C. As $\eta_1 \to \infty$,

$$f_2(\eta_1) \sim \beta C - a + \frac{b}{2\eta_1 - \beta)} \left\{ 1 + \frac{2}{(2\eta_1 - \beta)^2} + \dots \right\} \tag{24}$$

($a = 3.34$, $b = 1.66$). Because of the factor $1/\xi$, the solution cannot be valid near the leading edge, nor in the potential flow near $\xi = 0$.

For τ_1, Alden's result is

$$\tau_1 = \frac{\alpha}{x^{\frac{1}{2}}} + \frac{\alpha C}{R x^{3/2}} + \dots, \tag{25}$$

$$\text{and as } \eta_1 \to \infty, \ \psi = R^{-\frac{1}{2}} \chi \sim y - \frac{\beta \xi}{R^{\frac{1}{2}}} + \frac{\beta C - a}{R^{3/2}} \frac{1}{\xi}. \tag{26}$$

The first two terms give the same potential flow as before, but the third term is not harmonic, and Alden takes $\beta C - a = 0$, $C = 1.99$. The skin-friction would then have a non-integrable singularity at the leading edge, but in any case the solution is not valid there. Without some such condition as $\beta C - a = 0$, we should be one boundary condition short.

As regards the improvement we shall attempt, almost everything is achieved by replacing $f_2(\eta_1)/\xi$ by $\xi f_2(\eta_1)/(\xi^2 + \eta^2)$; but the argument may be made more convincing by attempting the use of a technique due to Lighthill (Ref. 8) for finding uniformly valid approximations. The method has mostly been used for hyperbolic equations (with a singular characteristic, for example), but Lighthill (Ref. 9) has used it for an elliptic equation, and Kuo (Ref. 3) has applied it to the boundary layer equation for a flat plate in cartesian coordinates, changing the x-coordinate only, and has thereby obtained results near the leading edge and in the potential flow that agree with the results from the use of parabolic coordinates. We shall use the method starting from parabolic coordinates. The straightforward application of the method does not, in this case, give an approximate solution of the Navier-Stokes equation uniformly valid in the whole field; we return to this point later.

We introduce new coordinates given implicitly in terms of the old; if we are, to begin with, content to change the ξ-coordinate only, it is not difficult to see that to order R^{-1} we should write

$$\xi = X + \frac{1}{R} \frac{g(Y_1) X}{X^2 + Y^2}, \ \eta = Y, \ \eta_1 = Y_1 = R^{\frac{1}{2}} Y. \tag{27}$$

We still want to include the potential flow, so to write $Y^2 = Y_1^2/R$ and to drop this term in the denominator would be wrong. We then write

$$\chi = \chi_0 + R^{-1}\chi_1,$$

where χ_0, χ_1 are functions of X and Y_1, and seek to satisfy to order R^{-1} the equation into which (3) or (5) is transformed, and to choose $g(Y_1)$ so as to annul χ_1 completely. We find that this can be done, and we find a solution

$$\chi = Xf(Y_1), \tag{28}$$

where f is the same function as before (eqns. (9), (10), (11)). The correction of order R^{-1} is entirely in the change of coordinates.

If, in the boundary layer away from the leading edge, we carry out a formal expansion in powers of R^{-1}, we should (except for a possible change in the constant C) obtain Alden's solution. For such a formal expansion

$$\chi = \left[\xi - \frac{1}{R}\frac{g(\eta_1)\xi}{\xi^2 + \eta_1^2/R}\right] f(\eta_1) = \xi f(\eta_1) - \frac{1}{R}\frac{f(\eta_1)g(\eta_1)}{\xi}, \tag{29}$$

so we expect that we shall have

$$f(\eta_1)\, g(\eta_1) = -f_2(\eta_1), \tag{30}$$

and this can be checked and agrees. So g need not be computed independently.

Also we find that

$$\tau_1 = \left(\frac{\alpha X}{\xi^2}\right)_{Y_1=0}, \tag{31}$$

and if we assume for the present that this expression is valid at the leading edge (we return to this point later), we see that τ_1 still has a non-integrable singularity (though of a lower order than in (25)) unless X vanishes with ξ on $Y_1 = 0$. This requires $g(0) = 0$, which in turn requires $C = 0$ in (23), so if this is correct, f_2 in (30) must be replaced by Δf_2. With $C = 0$, for the external flow we find that as $Y_1 \to \infty$

$$\psi = R^{-\frac{1}{2}}\chi \sim y - \frac{\beta\xi}{R^{\frac{1}{2}}} - \frac{a\xi}{R^{3/2}(\xi^2 + \eta^2)} \tag{32}$$

to order $R^{-3/2}$. [If $C \neq 0$, the coefficient in the last term is changed from $-a$ to $\beta C - a$.] The right-hand side of (32) is harmonic, and gives a potential flow which becomes the given uniform stream as $\eta \to \infty$. It seems reasonable that there should be a correction of order $R^{-3/2}$ in the potential flow (associated with a correction to the displacement thickness), so we accept (32). Then with $C = 0$,

$$\tau_1 = \frac{\alpha}{x^{\frac{1}{2}}} \tag{33}$$

and is unaltered. There is no correction to τ_1 of order R^{-1}. Further consideration shows that this result is connected mathematically with the fact that there are no negative powers in the asymptotic expansion of f (see eqn. (11)), the error terms in (11) being exponentially small. There are negative powers in the asymptotic expansion of f_2 (eqn. (24)) or of Δf_2, and there may be a change in τ_1 of order R^{-2}.

5. *The Neighbourhood of the Leading Edge.*

Near the leading edge of a flat plate the assumptions of boundary-layer theory are not valid. With parabolic coordinates, the ξ-coordinate should also be transformed by the substitution $\xi_1 = R^{\frac{1}{2}}\xi$; in fact with cartesian coordinates the correct way to consider the neighbourhood of the leading edge is to substitute

$$x_1 = Rx, \quad y_1 = Ry, \quad \psi_1 = R\psi. \tag{34}$$

When this is done all the terms in the Navier-Stokes equations are of the same order of magnitude.

Similar remarks are true near the front stagnation point of any semi-infinite cylinder. In considering the non-uniformity of approximations to solutions for large Reynolds numbers there are three regions to be considered. In the potential flow the coordinates are left unaltered; in the boundary-layer (in more general cases, in any similar vortex-layer of high vorticity) the non-dimensional length coordinate across the layer should be magnified $R^{\frac{1}{2}}$ times, and the coordinate along the layer left unaltered (cf. eqns. (4) and (17)); in a circle whose centre is at the stagnation point and whose radius is of order R^{-1}, the non-dimensional length coordinates should all be magnified R times, as in (34).

Now in § 4, to complete the solution, we applied the condition that τ should have an integrable singularity at the leading edge. Generally, when it becomes possible to discuss more fully the integration of the Navier-Stokes equation, it is to be expected that it will not be necessary to use such a boundary condition. Meanwhile, for the problem considered in § 4, and for the method used there, no other satisfactory boundary condition presents itself; and it is to be expected that this condition will lead to the correct answer, provided that it may be applied.

Further consideration shows that there is a small sector of the circle about the leading edge in which (18) and (28) are in a certain sense valid approximations. This sector is symmetrical about the radius lying along the plate ($y = 0$, $x > 0$). It has been mentioned that very near the leading edge, $\chi = \frac{1}{2}\alpha\xi\eta_1^2$, i.e. $\psi_1 = \frac{1}{2}\alpha\xi\eta_1^2$, and that this is the first term of an expansion near the leading edge obtained by Carrier and Lin. From a consideration of the next term in the expansion, and its general form, it appears that this single term is a good approximation to the full solution of the Navier-Stokes equation, not only sufficiently near to the leading edge (x_1 and y_1 sufficiently small), but also in a sector of the form mentioned ($x_1 > 0$, $|y_1|/x_1$ sufficiently small). This error cannot be estimated exactly, but if we are content to estimate it from the second term in the expansion, we find that whereas χ itself is $O(R^{-\frac{1}{2}}\theta^2)$ in the sector, the error in χ is $O(R^{-\frac{1}{2}}\theta^3)$, where θ is the angle of the sector, and $r_1 = (x_1^2 + y_1^2)^{\frac{1}{2}}$ has been taken as $O(1)$. The error in χ in (28) may be expressed as the sum of three parts, of which this is the first; the second is the difference of $\frac{1}{2}\alpha\xi\eta_1^2$ from the result in (18), and the third the difference between the results given in (18) and (28). The second is $O(R^{-\frac{1}{2}}\theta^5)$ and the third is $O(R^{-\frac{1}{2}}\theta^3)$. Hence, within the limits we have set ourselves — terms of order R^{-1} in χ and τ_1 — the fractional error in χ (or ψ_1) is $O(\theta)$; and (33) should give, to order R^{-1}, the correct limiting result for τ_1 as the leading edge is approached along the plate; without the condition $C = 0$, (31) would give this result more generally. This is what we require.

The difficulty near the leading edge is connected with the fact that the substitution (27) is not unique, for in addition to substituting for ξ by a formula of the type shown, we may also change the η_1-coordinate by writing

$$\eta_1 = Y_1 + \frac{1}{R}\frac{G(Y_1)Y_1}{X^2 + Y^2}. \tag{35}$$

Then (30) becomes

$$fg + \eta_1 f'G = -f_2. \tag{36}$$

The solution is no longer everywhere unique. But to order R^{-1} the solution turns out to be unique, and to be the same solution as before, everywhere outside the region consisting of the small circle of radius of order R^{-1} with its centre at the leading edge and with a small sector removed about the radius along the plate; in other words, to order R^{-1} the method provides a unique answer in the region in which it is correct to start from a boundary layer solution as an approximation at large Reynolds numbers. The formula for τ_1 is unaltered.

6. *Flows with Wakes and Separation.*

If we consider the limit as $\nu \to 0$, $R \to \infty$, of the steady viscous flow past a finite flat plate ($y = 0$, $0 < x < 1$), we obtain $u = 1$ everywhere except on $y = 0$, $x \geqq 0$. Vortex sheets are present on both sides of the plate, through which u drops from 1 to 0. But there is also a singular surface on $y = 0$, $x > 1$, along which u increases from 0 to 1 as x increases from 1 to ∞. This singular surface may be regarded as the confluence of two vortex sheets. According to a calculation of the flow in the wake on boundary-layer theory (Ref. 10, § 248), the limiting value of u when $\nu \to 0$ is given by

$$u = 1 - \frac{\alpha}{2\pi^{\frac{1}{2}}x^{\frac{1}{2}}} + \ldots \tag{37}$$

along this line for large x, and by

$$u = 1.23\ (x - 1)^{1/3} - 1.18\ (x - 1)^{4/3} + \ldots \tag{38}$$

for $x - 1$ small. (The singularity at $x = 1$ may well be an artificial result of the use of cartesian coordinates). Numerical values are available to a certain extent.

Thus even for this simple case, "classical" potential flow (even with vortex sheets at the solid surface) does not give the correct limit as $\nu \to 0$.

A similar singular surface will arise in the limiting flow past any cusped, streamlined cylinder (aerofoil) for which separation of the boundary layer does not take place.

Thus even for this simple case any attempt at a construction of an accurate asymptotic expansion for large R is difficult. Kuo (Ref. 3) has made an approximate calculation of the correction of order $R^{-\frac{1}{2}}$ due to the influence of the wake on the pressure, and on $P(x)$ in eqn. (6). He uses cartesian coordinates and simply takes δ_1 constant, V zero, for $x > 1$ on $y = 0$ in calculating Ψ_1. He compares the calculated drag with experiment, and obtains fairly good agreement down to $R = 15$. If this is not accidental, it would provide a little evidence that terms of order R^{-1} are absent.

To obtain the limit as $\nu \to 0$, $R \to \infty$ of steady, two-dimensional flows past cylinders with boundary-layer separation is, of course, even more difficult. Classical theory is incorrect, but it sometimes held that the "free-streamline" theory (Ref. 10, § 10) may give the correct limit. (In general, a "free-streamline" solution is not unique, but the correct solution must satisfy the condition that the position of boundary-layer separation (calculated on boundary-layer theory and independent of R, with the external flow according to free-streamline theory) coincides with the position of separation assumed for each free streamline, and there seems little doubt that the correct solution is that for which the free streamlines have finite curvature everywhere (Refs. 11, 12, 13); so that, for example, the solution obtained long ago by Brodetsky (Ref. 14) for the flow past a circular cylinder is the correct one to choose). But there are considerable difficulties in accepting the free-streamline result as the correct limit. (The question here is purely a theoretical one at this stage, and is not that of adjusting the theory to fit more closely to experiment, which is concerned with turbulent, not steady, motion). If we start from this limit for infinite R, and seek the correction for a large finite R (small but non-zero ν), we know that the vortex sheets which are the free streamlines will diffuse, but nevertheless (as Dr. Batchelor and Professor Lagerstrom have also pointed out) we shall not return to the undisturbed flow at infinity downstream, and this contradicts the usual theory of laminar wakes (Ref. 10, § 249); one or the other or both must be wrong.

Now if we consider a motion started from rest in a viscous fluid, it is known that $\lim_{t\to\infty} \lim_{\nu\to 0}$ and $\lim_{\nu\to 0} \lim_{t\to\infty}$ are quite different. Because of the known action of viscosity in diffusing the vorticity, although vortex sheets may occur in the former, diffused vorticity may not; but restricted regions of diffused vorticity may occur in the second. A very simple example is the flow between two parallel planes, when one is held stationary and the other started moving with uniform velocity in its own plane. The first limit gives zero velocity everywhere between the planes, and a vortex sheet at the moving plane; the second gives constant vorticity between the planes. (In a limiting two-dimensional steady flow, in any finite region, the diffused vorticity must be constant in the region, or in each of two or more parts of it, as Dr. Batchelor has proved.) It is suggested that in flow past a cavity in a solid body, for example (as for the usual static-pressure hole) although in the limiting steady flow there will be a vortex sheet across the mouth of the cavity, the fluid in the cavity may well be in motion with a vorticity which, for two-dimensional motion, is constant, and not at rest. Now what we require is the second limit; it appears likely that what free streamline theory gives is, in fact, the first limit.

Dr. Batchelor, in an unpublished paper, has gone much further. He suggests that the correct limiting steady flow in two-dimensional motion in the wake behind a cylinder with separation consists of two finite regions of constant, and opposite, vorticity, bounded by two free streamlines which come together at a cusp at a finite distance downstream, and separated by a singular surface from the cusp to the rear of the cylinder. There will also be a singular surface downstream from the cusp. In this picture, the drag of the cylinder in steady flow would $\to 0$ as $\nu \to 0$. (Batchelor considers further physical details, refinements of this general picture, and flow symmetrical about an axis). The cusp would appear only in the limit.

Certain plausible physical arguments from the development of vorticity, and of circulation in any circuit, in a fluid of small viscosity may be used to support Batchelor's theory, but it is difficult to decide the question with certainty. Experiment provides no guide. It appears also that the largest Reynolds number for which a numerical solution of the Navier-Stokes equation is available is $R = 40$ for flow past a circular cylinder (Ref. 15). Such numerical calculations show a vortex pair behind the cylinder with a closed streamline, but it is not known if the region enclosed by this streamline becomes infinite as $R \to \infty$ or not.

Any satisfactory asymptotic expansions for large R for separating flows with wakes appear, therefore, to be some way away.

7. *Vortex Motion without Boundaries.*

In this section we are concerned with motion that is three-dimensional. If an assumed, three-dimensional, initial distribution of vorticity is allowed to develop according to the equations of motion of an incompressible, viscous fluid, then the statistical effect of the change of length of the vortex lines is to produce an increase in the mean square vorticity, $\overline{\omega^2}$; the effect of viscosity is to decrease $\overline{\omega^2}$, which in general first increases to a maximum and then decreases. When $\overline{\omega^2}$ is at or near its maximum, regions of high vorticity appear in the fluid, the vorticity being small elsewhere. When this is the case, in a region of high vorticity the convection terms in the equations of motion (which give rise to the effect of the stretching of the vortex lines) must be of the same order of magnitude as the viscous terms when ν is small, so that these regions will be layers whose thickness is $O(\nu^{\frac{1}{2}})$ as $\nu \to 0$, rather like boundary layers, though now no solid boundaries are assumed to be present. But in such motions as those here considered, which are more usually studied in connection with turbulent motion, the rate of dissipation of energy must remain finite and non-zero as $\nu \to 0$; in layers like boundary layers the rate of dissipation for unit area of the layer is $O(\nu^{\frac{1}{2}})$. [In regions of rapid transition of the shock-wave type, lying across the direction of the stream, in compressible fluids, the thickness is $O(\nu^{-1})$ and the rate of dissipation per unit area is $O(1)$.] After a conversation with Professors Burgers and Timman, it appeared that the only way in which the convection terms and viscous terms could be of the same order, with a rate of dissipation of $O(1)$, was that there should be layers of intense vorticity of thickness of $O(\nu^{\frac{1}{2}})$, but of area of $O(\nu^{-\frac{1}{2}})$, so their volume in any finite volume of the fluid is $O(1)$; when $\nu \to 0$ these layers become surfaces, but the surfaces still occupy a finite, non-zero fraction of the volume of the fluid. For a triply periodic distribution of the initial velocity components — i.e. with u, v, and w each given by a single term of a triple Fourier series — Taylor and Green (Ref. 16) computed by series in the time t, and showed that $\overline{\omega^2}$ would rise to a maximum before falling. The computation could be carried out only at fairly low Reynolds numbers, and a solution in powers of the Reynolds number (Ref. 17) includes the solution in powers of t and converges slightly better; but neither gives any information at high Reynolds numbers, and all attempts at finding an asymptotic solution for large Reynolds numbers have been unsuccessful. In view of the nature of the singularity as $R \to \infty$ proposed above, this is not surprising.

Recently (Ref. 18) Proudman and Reid have shown that for an infinite field of homogeneous isotropic turbulence in an inviscid fluid, with the assumption that certain fourth-order correlations of the velocity components at three points of the fluid are related to second-order correlations in the same way as for a Gaussian probability distribution, then with the initial conditions $\overline{\omega^2} = \overline{\omega_0^2}$, $d\overline{\omega^2}/dt = 0$ at $t = 0$, $\overline{\omega^2} \backsim 6/(t - t_0)^2$ near $t = t_0$, where $t_0 \propto (\overline{\omega_0^2})^{\frac{1}{2}}$. $[(\overline{\omega_0^2})^{\frac{1}{2}} t_0 = 5.9$ approximately]. This result has not yet been extended to a fluid of small but finite viscosity.

8. *Viscous Gases.*

When the equations of motion for a viscous compressible fluid are considered, the situation is, naturally, one of considerably greater difficulty. I shall only mention briefly two matters closely connected with subjects referred to elsewhere in this lecture.

Boundary layers have been considered by many authors, with some considerable success for the first approximation; in particular, the boundary layer along a semi-infinite flat plate along the stream has been computed. Kuo (Ref. 19) has recently made an interesting attempt to extend his work on the interaction of the boundary layer and the external stream to calculate the flow past a plate at high Mach numbers, using a similar technique of coordinate straining in Lighthill's manner. The effects now are not small, for it is imperative to take the interaction into account; there is a shock wave of finite strength before the plate, inclined at a small inclination to the plate and curved, with rotational flow behind.

The number of known exact solutions of the full equations is very small. The simplest is the steady flow between two infinite horizontal planes in relative motion parallel to themselves (shearing motion) which has been calculated by Illingworth (Ref. 20) under quite general conditions. The difference from the incompressible case is due to the variation of the viscosity with temperature and to dissipation of energy and heat conduction. If the viscosity μ is independent of the temperature T, $u = y$ as for an incompressible fluid. The next simplest case is when both plates are at the same temperature T_1, $\mu \propto T$, and the gas is a perfect gas with constant specific heats c_p, c_p/γ, and a constant Prandil number $\sigma = \mu c_p/k$, where k is the heat conductivity. Let U_1 be the velocity of the moving plate, a_1 the velocity of sound at the temperature T_1, M_1 the Mach number U_1/a_1, and

$$b = \sigma(\gamma - 1)M_1^2. \qquad (39)$$

Then $$u + \frac{1}{4} bu^2 - \frac{1}{6} bu^3 = (1 + \tfrac{1}{12} b) y \tag{40}$$

and $$T/T_1 = 1 + \tfrac{1}{2} bu - \tfrac{1}{2} bu^2. \tag{41}$$

Unless the distance h between the plates is very large, so that gh/a_1^2 is not small, the pressure P between the plates is almost constant, since if P_0 is the pressure at the lower plate

$$P/P_0 = e^{-cu}, \quad c = \frac{\gamma g h}{a_1^2 (1 + \frac{1}{12} b)}. \tag{42}$$

Because the plates are at the same temperature, the temperature, and therefore the viscosity, are greatest in the middle; $\mu \, du/dy$ is constant, so du/dy is least in the middle. The distribution is still anti-symmetrical about $y = \frac{1}{2}$.

The unsteady problem, when the moving plate is started impulsively, is one of considerable difficulty, and has not been much studied. Without a stationary plate, when the motion is produced by a single infinite flat plate started moving impulsively in its own plane with uniform velocity in an infinite gas, the problem has been considered by a number of authors (Refs. 21 to 25). Further study is justified, perhaps by a variation of mathematical methods mentioned elsewhere in this lecture. The motion is a difficult example of mixed diffusion and wave motion; the wave from the plate must culminate in a shock wave.

9. *Singular Perturbations of the Non-Linear Wave Equation.*

I pass now to my final subject, suggested by the influence of viscosity on the formation of shock waves.

Two equations have been solved explicitly which exhibit certain typical features of shock-wave theory, and I shall discuss these rather than the approximate methods used for the actual equations (similar but harder) for the motions of gases.

As a preliminary, consider the non-linear wave equation (with x as a time-like coordinate)

$$\frac{\partial u}{\partial x} + G(u) \frac{\partial u}{\partial y} = 0. \tag{43}$$

The nature of the continuous solution is well known. Let $u_0(y) = f(y)$ be the initial distribution of u when $x = 0$. For illustrative purposes, it is sufficient here to assume that $G(u)$ and $f(y)$ are monotonic. Then if there is a continuous solution it is

$$u = f(y - G(u)x), \tag{44}$$

i.e. if $y = f_1(u)$ initially, then

$$y = f_1(u) + xG(u). \tag{45}$$

Multiple values of u for the same y will occur if $G(u)$ is increasing and $f(y)$ decreasing, or if $G(u)$ is decreasing and $f(y)$ increasing. It is assumed that the solution then becomes discontinuous. When a discontinuity occurs, as y increases through the discontinuity at a given x, u decreases discontinuously if $G(u)$ is increasing, and u increases discontinuously if $G(u)$ is decreasing. There must be a discontinuity as soon as x exceeds the least possible value of $-f_1'(u)/G'(u)$. We cannot prove that there is no discontinuity for smaller x without considering the limit of a perturbed equation, but if this is assumed the resulting course of the discontinuity can be traced. Let

$$G(u) = g'(u), \tag{46}$$

and write (43) in the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad v = g(u). \tag{47}$$

The discontinuity will be propagated along some curve in the (x, y) plane, and from the first of (47) and an application of Stokes's theorem it follows that along this curve

$$[u]dy - [v]dx = 0, \tag{48}$$

where $[u]$ and $[v]$ are the discontinuities in u and v; i.e., if u jumps from u_1 to u_2 across the discontinuity

$$\frac{dy}{dx} = \frac{g(u_2) - g(u_1)}{u_2 - u_1} = \frac{\int_{u_1}^{u_2} G(u)du}{u_2 - u_1}. \tag{49}$$

Also at the discontinuity

$$y = f_1(u_1) + xG(u_1) = f_1(u_2) + xG(u_2), \tag{50}$$

and these equations suffice to determine the position and strength of the discontinuity at any time. [They can be applied to determine the position and strength of a shock wave in a gas only in so far as the variations of entropy arising from the growth of the shock wave can be neglected].

The two equations which have been solved explicitly both represent singular perturbations of an equation of the type of (43) or (47). The first is Burgers's equation (Refs. 26):

$$\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{51}$$

with ν constant.

Given an initial distribution u_0 of u the non-linear term tends (if u_0 is a decreasing function of y) to steepen the distribution, and in the absence of the viscous term of higher order, discontinuities would result. The higher-order viscous term prevents the formation of discontinuities, causes diffusion of the momentum, and a dissipation of energy which is independent of the viscosity. According to Cole (Ref. 27) the equation was first mentioned by H. Bateman in the Monthly Weather Review in 1915, and Lagerstrom, Cole and Trilling (Ref. 21) used the equation as an approximation for a weak shock wave near the steady state with dissipation neglected; the explicit solution was given by Hopf (Ref. 28) and Cole (Ref. 27); but there is no doubt that the equation is correctly named Burgers's equation.

The explicit solution is

$$u = -2\nu F_y/F \tag{52}$$

(the subscript denotes a partial derivative), where F is a solution of the heat-conduction equation:

$$\frac{\partial F}{\partial x} = \nu \frac{\partial^2 F}{\partial y^2}. \tag{53}$$

The second set of equations is

$$\left.\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\ \frac{V}{k}\frac{\partial v}{\partial y} &= u - rv + (r-1)uv. \end{aligned}\right\} \tag{54}$$

These equations arise in a number of exchange problems when a fluid flows through the voids along a column containing matter in the solid state — for example, ion exchange between a salt or acid solution and a suitable resin, with the resin particles in a fixed column through which the liquid is flowing; the exchange is of two ions A and B (sodium and hydrogen, for example) of the same valence. If c is the concentration of ion A in the fluid, measured, say, in milliequivalents per unit volume of the fluid, q the concentration of ion A in the solid (in milliequivalents per unit volume of solid), c_0 the total concentration of ions A and B entering the column, Q the total concentration capacity of the solid phase, the equations (54) are based on the conservation equation and an assumed bilinear exchange equation

$$\frac{\partial q}{\partial t} = k\left[c(Q-q) - \frac{1}{K} q(c_0 - c)\right], \tag{55}$$

with

$$r = \frac{1}{K},\quad u = \frac{c}{c_0},\quad v = \frac{q}{Q}. \tag{56}$$

K and k are taken as constants. V is the total rate of volume flux of the fluid, equal to αRS, where R is the linear rate of flow of the fluid, $1 - \alpha$ the fraction of the volume of the column filled by the solid, and S the cross-sectional area of the column (all assumed constant). If X is the distance along the column from the entry, and t the time from the initial entry, then x is the total ion concentration (of both ions) on the resin in the length X from the entry, and y the total ion concentration in the fluid which has passed the cross-section X when the time t is reached, so x is proportional to X, and y to $Rt - X$.

$$[x = SQ(1 - \alpha)X, \; y = \alpha \, Sc_0(Rt - X).]$$

The same equations apply to other cases with different meanings of the symbols — e.g. fixed-bed adsorption, or (with $r = 1$) heat exchange between a flowing fluid and a crushed solid. When $r = 1$ the equations are linear, and the solution is the limit as $r \to 1$ of the solution for $r \neq 1$, so the case $r = 1$ will not be further considered. (See the references in Ref. 29).

We may write (54) in the form

$$u = \psi_y, \quad v = - \psi_x \tag{57}$$

and

$$\frac{V}{k} \psi_{xy} + \psi_y + r\psi_x + (1 - r) \, \psi_x \psi_y = 0; \tag{58}$$

the equation for ψ is reduced to a linear form by the substitution

$$\psi = \frac{V}{k} \frac{1}{1-r} \log \, F(x, y). \tag{59}$$

Then

$$\frac{V}{k} F_{xy} + F_y + rF_x = 0. \tag{60}$$

The boundary conditions are usually such that ψ, and therefore F, are known on the positive halves of both axes, and the solution is required for all positive x and positive y. The logarithmic substitution was used by Thomas (Ref. 30). (A slightly different substitution is a little more convenient for the usual boundary conditions, but that is irrelevant here).

If we put $\nu = 0$ in (51), or $V/k = 0$ in (54), each equation reduces to an equation of the type (47), with

$$\left.\begin{aligned} g(u) &= \tfrac{1}{2}u^2 \quad \text{in (51)} \\ &= \frac{u}{r + (1 - r)u} \quad \text{in (54)} \end{aligned}\right\} \tag{61}$$

so each equation represents a singular perturbation of a first-order non-linear wave equation.

Also, (51) may be expressed in a form similar to (57) and (58), since it may be written

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \quad v=\tfrac{1}{2}u^2-\nu\frac{\partial u}{\partial y}. \tag{62}$$

Hence u and v may be expressed as in (57), where

$$\psi_x+\tfrac{1}{2}(\psi_y)^2=\nu\psi_{yy} \tag{63}$$

and the solution is

$$\psi=-2\nu\log F \tag{64}$$

where F satisfies (53). Thus both equations which have been explicitly solved (by reduction to linear equations) have been solved by the same substitution. The result may be generalized. If α is a constant,

$$\alpha\{L\psi_{xx}+M\psi_{xy}+N\psi_{yy}\}+P\psi_x+Q\psi_y+R+L\psi_x^2+M\psi_x\psi_y+N\psi_y^2=0 \tag{65}$$

and

$$\psi=\alpha\log F, \tag{66}$$

then F satisfies the linear equation

$$\alpha\{LF_{xx}+MF_{xy}+NF_{yy}\}+PF_x+QF_y+\frac{R}{\alpha}F=0. \tag{67}$$

If F satisfies (67), then

$$u=\alpha F_y/F, \quad v=-\alpha F_x/F \tag{68}$$

is a solution of

$$\left.\begin{aligned}&\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0.\\ &Lv^2-Muv+Nu^2-Pv+Qu+R=\alpha\left\{L\frac{\partial v}{\partial x}-M\frac{\partial u}{\partial x}-N\frac{\partial u}{\partial y}\right\}.\end{aligned}\right\} \tag{69}$$

Both the two equations above, the one parabolic and the other hyperbolic, are examples of this substitution. In every case in which the equations have been solved explicitly, and the limits found as $\alpha\to 0$, the limit is the solution of the unperturbed equation for u, found as previously described. Near a discontinuity the limit is non-uniform, and does not give sufficient information, particularly for (54), which is of some practical importance in a branch of chemical engineering. Since u is given by a quotient, the process of finding the required formulae for small α is far from trivial; asymptotic values must be carefully dealt with. An example, of some practical importance, from the solution of (54) may be cited. With $r > 1$, so that $G(u) = g'(u)$ is an increasing function, let u_0 be zero for $y < 0$ and for $y > Y$, and $u_0 = 1$ for $0 < y < Y$. The discontinuities are then introduced in the initial values. With the term in V/k omitted (on the "equilibrium" theory), the discontinuity at $y = Y$ is, to begin with, propagated unaltered, and for positive x is at $y = Y + x$. The discontinuity at $y = 0$ becomes diffuse and takes the form

$$u = \frac{r - (rx/y)^{\frac{1}{2}}}{r - 1} \text{ for } x/r \leqq y \leqq rx. \tag{70}$$

But when $rx = Y + x$, the head of the diffuse trailing boundary catches up the discontinuity and, for larger values of x, it eats into it, so that the strength of the discontinuity diminishes. By the methods explained (for $V/k = 0$) it may be shown that for $x \geqq Y/(r-1)$, $u = 0$ for $y \leqq x/r$ and for $y > y^*$, and is given by the same formula as in (70) for $x/r \leqq y \leqq y^*$, where

$$(ry^*)^{\frac{1}{2}} = x^{\frac{1}{2}} + [(r-1)Y]^{\frac{1}{2}}. \tag{71}$$

The largest value of u is

$$\frac{r}{r - 1 + [(r-1)x/Y]^{\frac{1}{2}}}. \tag{72}$$

All this comes out as the limit of the exact solution. For the next approximation near y^*, u is given by the expression in (70) with an additional term in the denominator which, for fixed y and y^*, is exponentially large for $y > y^*$ and exponentially small for $y < y^*$ as $V/k \to 0$, but which is $O(1)$ when $y - y^* = O(V/k)$.

There are, of course, other ways of interpreting and ways of generalizing the result in (65), (66), and (67). We may increase the number of independent variables. We may, if we make a correct choice of the coefficients, consider ψ as a potential instead of a stream function. We may consider non-singular perturbation problems with α large, or general problems with α neither large nor small. There is some interest in other parabolic and hyperbolic equations we can discuss in this way, but the second-order non-linear elliptic equations do not seem to be of any general interest. An example is

$$\nabla^2\psi + \frac{1}{\alpha}[(\text{grad } \psi)^2 + c] = 0, \tag{73}$$

which corresponds with

$$\psi = \alpha \log F, \quad (\nabla^2 + c/\alpha^2) F = 0. \tag{74}$$

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