GEOMETRY UPON AN ALGEBRAIC VARIETY

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I. Algebraic geometry — that is to say, the branch of geometry which deals with the properties of entities represented by algebraic equations — has in recent years developed in two distinct directions, which in a sense are opposed to one another. One of these directions is called *abstract* in as much as it is concerned with algebraic equations defined over commutative fields subject only to slight restrictions; here the means employed are purely algebraic, including in particular ideal theory and valuation theory. The other direction may properly be called *geometrical*; this usually deals with algebraic equations in the complex domain, and from time to time appeals to ideas and methods of analytic and projective geometry, topology, the theories of analytic functions and of differential forms.

The dualism between these two disciplines has close relationship and affinity with that which, three centuries ago, arose between *l'esprit géométrique* of Descartes and *l'esprit de finesse* of Pascal, and which, in the past century, on the one hand divided the geometers into analysts of the school of Plücker and synthesists of the school of Steiner and, on the other, the algebraists into purists à la Dedekind and arithmetizers à la Kronecker. However, this dualism, instead of proving harmful to geometry, offers undoubted advantages when the two lines of development, with their respective merits and possibilities, are regarded not as contrasting but as complementary.

We cannot fail to recognise in the abstract method and its technique a peculiar elegance, an impeccable logical coherence, and to appreciate the importance of the results so far obtained by it, particularly in the study of the foundations of geometry and the difficult questions concerning the singularities of algebraic varieties. But equally we cannot fail to recognise that the geometrical approach, with its greater concreteness, lends itself better to the formulation and initial study of new concepts and problems; and that it presents an incomparable wealth and colour of its own, due to the interweaving of many diverse strands, to the subtle and perspicuous play of geometrical intuition, and to the possibility of readily constructing examples and investigating special cases. We may also point out that, in the geometrical discipline, corresponding to a more definite notion of algebraic variety, there is a much wider range of subjects and a far greater number of orientations and contacts with other important branches of mathematics, which have found, and are finding, therein inspiration and extensions beyond the purely algebraic field.

In the present address I shall leave aside altogether the abstract method, partly for lack of time and partly because exposisions of the subject have recently been given by Hodge {46} [1]), Zariski [1] and Groebner {35}; for similar reasons I shall not speak of the relations between algebraic geometry and number theory, for which I may refer to the discourse by André Weil {100} at the 1950 Congress. Even so, the panorama, when thus delimited and restricted to the contributions of the last few years, would still be too vast, and the terrain too impenetrable, if I attempted to include significant but relatively particular results such as the recent researches of Conforto [2]), Weil {101—103}, Igusa {50—51}, Chow {19, 20}, Andreotti {3, 6} and Barsotti {13} on Abelian varieties; those of Severi {90, 94} and Conforto {87} on quasi-Abelian varieties; those of Roth {74—77} on elliptic, hyperelliptic and pseudo-Abelian varieties; the studies of Roth {67—73, 78}, Baldassarri {10—12} and Segre {86} on questions of unirationality and birationality of algebraic varieties, with special regard to threefolds; the researches of Chisini {18} and his school on the theory of braids and branch curves of multiple planes; the results of Andreotti {4, 5, 7—9} and of Conforto and Gherardelli {21} on the torsion of algebraic varieties, on the problem of uniformisation and the classification of the irregular surfaces; those of Godeaux {27—34} on multiple surfaces, of Gaeta {22—24} on families of space curves, of Scott {79, 80} and Todd {97} on correspondences between algebraic surfaces; and, finally, the investigation, by Severi {91, 92}, Néron {64, 65} and Segre {81, 83}, of various arithmetic questions which arise in algebraic geometry.

I propose instead to discuss, though only summarily, certain recent developments in the geometry on an algebraic variety, which (I hope) will be fairly clear even to non-specialists; and, at the same time, to glance at some of the still unsolved problems in the subject.

II. Let M be an algebraic variety of dimension $d \geqq 1$, defined over the complex field; and suppose that M is irreducible and non-singular (i.e. free from multiple points). By "geometry on M" we mean the study of *relative invariant* properties, that is, those properties which are unchanged by *regular* birational transformations of M; among these, special importance attaches to those which are *absolutely invariant*, that is, are unaltered by *all* possible bira-

[1]) The numbers in curly brackets refer to the bibliography at the end; those in square brackets are from the monograph {84} by B. Segre, which we do not reproduce here.

[2]) Fabio Conforto, whose premature death we all deeply regret, was to have addressed the present Congress on the subject of singular Abelian varieties. For a detailed account of his work see B. Segre {87}.

tional transformations of M (including those with exceptional elements). In this study the first salient feature that presents itself is the aggregate of the algebraic subvarieties of M; any two of these — which we suppose to be pure and of dimensions a, b — define an intersection, also algebraic, each component of which has dimension $\geqq a + b - d$. In general, but not always, the dimension of each component is exactly $a + b - d$: we then say that the two varieties present the regular case.

These notions can be extended to subvarieties which are impure, effective or virtual, namely to the aggregates

$$\sum \alpha_i A_i \tag{1}$$

formed by a finite number of subvarieties A_i of M, each taken with an assigned multiplicity α_i, where α_i is an integer (positive, zero or negative). The aggregates (1) may be associated in an obvious manner by addition and subtraction, thereby determining an Abelian group of infinite base. In this group we can define intrinsically, and in various ways, a subgroup of varieties which are to be considered as equivalent to zero, in such a manner that for the corresponding lateral classes — called *classes* (or *systems*) *of equivalence* — there subsists a multiplication, commutative and associative, geometrically related to the intersection in the regular case; this gives rise to an *equivalence ring* on M.

The useful types of equivalence, introduced by Severi [4—7] 20 years ago, are essentially four in number; in non-increasing order of restriction these are: *rational, algebraic, topological* and *enumerative* equivalence. In the case of two *hypersurfaces* (varieties of dimension $d - 1$) of M, the first reduces to the classical linear equivalence, that is, the relation which intercedes between two hypersurfaces of constant level for the same rational function of a variable point of M. In the case $d > 1$, having fixed the type of equivalence, we can similarly consider an infinity of equivalence rings, one for each of the irreducible subvarieties of M, and hence a multiplication law which (in addition to the classes determined by the factors of the product) depends on the particular subvariety of M on which the product is taken. We may complete the picture by introducing other operations of a less obvious character; it will suffice to say that these correspond geometrically to the search for a residual intersection in the non-regular case.

III. If A is a hypersurface of M, the successive powers

$$A, -A^2, \ldots, (-1)^d A^{d-1}, (-1)^{d+1} A^d,$$

effected in the rational equivalence ring on M and taken with alternating signs, form a *covariant succession of immersion of A in M*, consisting of pure subvarieties of A having the respective dimensions

$$k, k-1, \ldots, 1, 0, \tag{2}$$

where $k = d - 1$. In the case of a variety A of dimension $k < d - 1$, it is still possible to obtain an analogous succession by proceeding as follows (Cf. Segre {84, 85}).

Suppose for simplicity that A is irreducible, non-singular and of dimension k, where $0 \leqq k \leqq d - 2$. Then at each point P of A the varieties M and A have (projective, complex) tangent spaces $[d]$, $[k]$ of respective dimensions d and k, and such that $[d]$ contains $[k]$. The spaces $[k + 1]$ of $[d]$ passing through $[k]$ can be mapped by "points" of a space of dimension $d - k - 1$, there being one such "point" for each mode of tending to P from a point of M not belonging to A. By substituting, in a suitable manner, these ∞^{d-k-1} abstract points for P and repeating the operation for each of the ∞^k points of A, we may show that M can be transformed into a new algebraic variety M', defined by M and A to within a regular birational transformation, which we say is deduced from M by the *dilatation of base A* (Cf. Segre [6]). The varieties M and M' are thus related by a birational correspondence, Θ say, which has no fundamental points on M' and which admits A as locus of fundamental points on M; that is, Θ dilates A into a hypersurface A' of M', which is thus a fibred variety of base A, the fibres being the ∞^k spaces $[d - k - 1]$ which are transforms of the points P of A.

If we now consider the powers

$$A'^{d-k}, A'^{d-k+1}, \ldots, A'^{d-1}, A'^{d}, \tag{3}$$

effected in the rational equivalence ring on M', we see that they form a succession of pure subvarieties of A', with the respective dimensions (2). Now take the varieties (3) in order with the signs

$$(-1)^{d-k-1}, (-1)^{d-k}, \ldots, (-1)^{d}, (-1)^{d+1},$$

and transform the varieties so obtained by the inverse Θ^{-1} of Θ; we thereby obtain a succession of pure varieties of A, which we denote by

$$A_{M,0}, A_{M,1}, \ldots, A_{M,k-1}, A_{M,k} \tag{4}$$

with respective dimensions (2), and such that the first coincides with A. These are naturally defined to within an equivalence; we say that they form the *covariant succession of immersion of A in M.*

A detailed analysis, into which I cannot enter here, shows that the diverse geometrical problems of intersection, regular or irregular, and various other questions in which varieties possessing multiple points can also occur, can all be solved by making use of covariant successions of immersion. Moreover, the latter are closely linked with the so-called *canonical varieties*, as I now propose to indicate.

An algebraic variety M defines (to within a regular birational transformation) the variety $N = M \times M$ of its ordered point-pairs, called the product of

M by itself; this is an algebraic variety of dimension $2d$. M can then be identified with the so-called diagonal variety of N, i.e. the locus of the points which map in N the coincident point-pairs of M. It is thus permissible to consider the covariant succession of immersion of M in N; this is a relative invariant of M; and, by simple algebraic operations within the rational equivalence ring of M, we can deduce from it the *canonical succession*

$$M_0^*,\ M_1^*,\ \ldots,\ M_{d-1}^*,\ M_d^*, \tag{5}$$

consisting of pure varieties of dimensions $d, d-1, \ldots, 1, 0$ (the first of which coincides with M) such that the corresponding classes of equivalence are *invariants* of M.

In a similar manner we define the canonical succession

$$A_0^*, A_1^*, \ldots, A_{k-1}^*, A_k^* \tag{6}$$

on any pure k-dimensional subvariety A (possibly virtual) of M; and we may prove that the covariant varieties of immersion (4) are related to the canonical varieties by the formulae

$$A_i^* = \sum_{j=0}^{i} (-1)^j A_{M,j} M_{i-j}^* \quad (i = 0, 1, \ldots, k). \tag{7}$$

These equations can be solved for $A_{M,j}$ and hence, in any given problem, we can replace the covariant varieties of immersion by the canonical varieties. It follows that, in order to go deeply into the geometry upon an algebraic variety, it is essential to study the operation of *canonisation*, i.e. the association of the succession (6) with any subvariety A of M.

To begin with, we shall describe some of the results already established in this connection, omitting (for lack of time) to speak of the relations between the canonical varieties and the Jacobian varieties of linear systems of hypersurfaces. We shall then indicate some of the important questions which still remain unanswered.

In the case where A is a hypersurface of M, (7) reduces simply to

$$A_i^* = \sum_{j=0}^{i} A^{j+1} M_{i-j}^* \quad (i = 0, 1, \ldots, d-1).$$

An analogous result holds for varieties which are *complete intersections*; thus, for example, if $P = (AB)_M$ is a regular and simple intersection of two hypersurfaces A, B of M, we have

$$P_i^* = P \sum_{j=0}^{i} (A^j + A^{j-1}B + \ldots + AB^{j-1} + B^j) M_{i-j}^* \quad (i = 0, 1, \ldots, d-2).$$

Also, if A denotes any non-singular variety of M, the successive powers of A in the equivalence ring of M are expressible algebraically in terms of the canonical varieties of A and M. Further, a knowledge of the latter enables us in many cases to determine the canonical varieties of the varieties A', M' which are deduced from A, M by applying a dilatation Θ of base A. Thus, denoting by $[P]'$ an arbitrary point in the space $[d-k-1]$ of A' which corresponds to any point P of A by Θ, we have the results (Segre {85}):

$$A'^{*}_{d-1} = (-1)^{d-k-1}[(d-k)A^{*}_{k}]',$$
$$M'^{*}_{d} = \Theta(M^{*}_{d}) + (-1)^{d-k}[(d-k-1)A^{*}_{k}]',$$

where the meaning of each symbol in square brackets is deducible by addition.

More generally, it would be exceedingly important to determine the transformation law for the canonical varieties (5) of M, under any non-regular birational transformation. Among other things, this would enable us to study the algebraic combinations of the canonical varieties which are *absolutely* invariant for such transformations.

Among the algebraic combinations of (5), those which reduce to point-sets have a particular interest. These are obtained in the following manner. Denoting by $i_1, i_2, \ldots, i_l$ any partition of the number d, we see that the product $M^{*}_{i_1} M^{*}_{i_2} \ldots M^{*}_{i_l}$ defines a set of points on M (determined to within an equivalence); we represent this by the symbol $(i_1 i_2 \ldots i_l)_M$, and the number of points in the set by $[i_1, i_2 \ldots i_l]_M$. If with each partition $(i) = i_1, i_2, \ldots, i_l$ of d we associate any integer whatever, say $\lambda_{i_1 i_2 \ldots, i_l}$, we may define on M an invariant series of equivalence, consisting of sets equivalent to

$$\sum_{(i)} \lambda_{i_1 i_2 \ldots i_l} (i_1 i_2 \ldots i_l)_M. \tag{8}$$

The number of points in this set, namely

$$\sum_{(i)} \lambda_{i_1 i_2 \ldots i_l} [i_1 i_2 \ldots i_l]_M, \tag{9}$$

is a numerical invariant of M, depending on M in such a way as to reduce to zero if — and, for a generic M, only if — the numbers λ are all zero.

Presumably the only invariant point-sets of M are equivalent to a set of type (8); also any invariant variety on a generic M may presumably be expressed algebraically in terms of the canonical varieties. This question has not yet been studied; it presents serious difficulties because, as we shall soon see, for $d \geqq 2$ the situation with regard to the invariant numbers is quite different.

V. On M, the effective hypersurfaces K equivalent to a canonical hypersurface M^{*}_{1} form a complete linear system $|K|$, possibly empty, whose dimension, augmented by unity, we denote by $g_d = g_d(M)$, writing $g_d = 0$ when there is no effective K. The number g_d, called the *geometric genus* of M, is an absolute invariant of M, equal to the number of linearly independent everywhere finite

d-ple integrals on M. Similar invariants, for $d \geqq 2$, are the numbers $g_i = g_i(M)$ ($i = 1, 2, \ldots, d-1$) which give the numbers of analogous i-ple integrals on M.

For a generic M of dimension $d \geqq 2$, none of these invariants can be expressed in the form (9), while — as we shall see — the alternating sum

$$g_d - g_{d-1} + g_{d-1} - \ldots + (-1)^{d-1} g_1 \tag{10}$$

can be so expressed. For a given M, the calculation of g_i ($i < d$) in general presents difficulties which at the moment are insuperable.

Something, however, is known about the geometric genus g_d, owing to the fact that we can construct the canonical system $|K|$ algebraically by referring to a model of M in space $[d+1]$, endowed with the so-called ordinary singularities; in fact it is well known that $|K|$ is cut on such a model — of order n, say — residually to the double hypersurface, by primals of order $n-d-2$. We may thus attempt to determine the dimension of $|K|$, and hence g_d, by using the classical *postulation* (or *characteristic*) *formula*: such a formula is, however, valid only for primals of sufficiently high order [3]), and thus we cannot affirm that it is applicable for primals of order $n-d-2$. It is therefore very curious that in this case the formula should lead to an *invariant* of M, which moreover, for $d \geqq 2$, is in general distinct from g_d; this is called the *arithmetic genus*. The theory of the arithmetic genus was initiated, in the case $d = 2$, during the second half of the last century by Cayley, Zeuthen and Noether, and then completed by Italian and French geometers. The extension to varieties of higher dimension was first made by Severi [2] in a fundamental memoir of 1909, where two definitions for the arithmetic genus are proposed, which may be presented in the following manner.

Let A be any hypersurface (effective or virtual) of M, and consider two arbitrary hypersurfaces C, C' of M, subject only to suitable conditions of generality (such as those satisfied by prime sections of M). Let $\delta_e(A)$ denote the effective dimension ($\geqq -1$) of the complete linear system $|A|$ determined by A on M; then, by the postulation formula, we know that, if x and x' are sufficiently large integers, the expression

$$\delta_e(A + xC + x'C')$$

must be a polynomial of order d in x and x', with integral coefficients. The constant term in this polynomial coincides with the constant terms in the polynomials (of orders d in x and x' respectively) expressing

$$\delta_e(A + xC), \qquad \delta_e(A + x'C')$$

for sufficiently large values of x and x'. The equality of these two constant

[3]) For the limits of validity of this formula, see Gaeta {25, 26} and Severi {93}.

terms shows that they furnish an integer, $\delta_v(A)$ say, depending only on A and M, which we call the *virtual dimension* of $|A|$.

The two arithmetic genera $p_a = p_a(M)$ and $P_a = P_a(M)$ are then obtained by assuming

$$p_a = (-1)^d \delta_v(0), \qquad P_a = \delta_v(K) + 1 - (-1)^d,$$

where 0 denotes the zero of linear equivalence on M. Thus p_a and P_a are relative invariants of M [4]; and it is a very remarkable fact that they are always *equal*. This important property has been the subject of investigation by Severi [2, 9], Albanese {1}, Todd [3], Zariski {104}, Kodaira {53}, and Kodaira and Spencer {61}. It is to the last two authors that the first general and complete demonstration of this result is due; they have obtained it by appealing to the recent theory (of Leray and H. Cartan) of analytic stacks. The same authors have also established the equality between p_a and the alternating sum (10), previously conjectured by Severi [2], thereby demonstrating the invariance of the arithmetic genus [5]. Similar questions have been considered by Kodaira in a very recent and important work {60}, where other notable results are obtained, including the *Riemann-Roch theorem*, which expresses the effective dimension $\delta_e(A)$ as a function of certain invariants of A and M, on the hypothesis that the system $|A|$ is sufficiently ample (so as, among other things, to contain partially the canonical system $|K|$) [6]. Kodaira also proves the theorem concerning *the completeness of the (linear) characteristic system* defined on the generic A by the complete continuous system which totally contains A [7].

VI. The theory of the arithmetic genus can be established on a quite different basis, as Todd [3] has shown, in a notable work of 1937. In this work Todd, inspired by an idea of Severi [2], seeks to establish the invariance of P_a by showing that the expression

$$t(M) = P_a(M) + (-1)^d, \tag{11}$$

by an appropriate choice of the coefficients λ in the *rational* field, can be written in the form (9): that is to say, that there exists a relation of the form

$$\mu(d) \,.\, t(M) = \sum_{(i)} \mu_{i_1 i_2 \ldots i_l}(d). \quad [i_1 i_2 \ldots i_l]_M, \tag{12}$$

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[4]) Cf. Severi [2, 9], Albanese {2}.

[5]) For the extension of this theory to analytic varieties, cf. Spencer {95}, Kodaira {57, 58}, and Kodaira and Spencer {62}. A tentative demonstration of the equality of P_a and the sum (10) is to be found in Kodaira {54}; see also Kodaira {59}. Concerning the relations between the postulation formula and the arithmetic genus, see Muhly and Zariski {63}, Groebner {36}.

[6]) Besides the classical cases $d = 1, 2$, the Riemann-Roch theorem, under more or less restrictive conditions, had previously been established for threefolds by Severi [2], Segre [3], Kodaira {53}. For any value of d, see Hirzebruch {43}.

[7]) For the case of threefolds, cf. Segre [3].

where the coefficients μ and $\mu_{(i)}$ are relatively prime integers, the first of which is positive, and dependent only on the dimension d of M.

It cannot, however, be said that this result has been then completely established, in as much as it has been obtained by appealing to a certain postulate[8]). Nevertheless, Todd must be given the credit for calculating the coefficients in (12) in the cases $d \leqq 6$, and for verifying that, for a variety M of dimension $d \leqq 6$, which is a product

$$M = A \times B \tag{13}$$

of two varieties A, B, we have the formula

$$t(M) = t(A) \,.\, t(B). \tag{14}$$

An analogous formula for the arithmetic genus p_a of the product of any number of given varieties has been actually obtained by Gaeta [1] in 1952.

Recently Hirzebruch {40} has shown how the coefficients in (12) can be calculated explicitly for any value of d, on the hypothesis that (14) holds for the variety (13). A functional equation for $t(M)$ is thus obtained, which admits one and only one solution reducing to $(-1)^d$ for any projective space of dimension d; this solution is found by using the *multiplicative sequences*, due to Hirzebruch, in relation with the development in power series of $-x(e^{-x}-1)^{-1}$. In this way, with any algebraic variety M there is associated a rational number $T(M) = (-1)^d t(M)$, given by (12), called the *Todd genus*: but it is not thereby clear whether such a number always satisfies (11), nor even whether it is necessarily an integer. In this connection Hirzebruch {41, 42}, overcoming considerable difficulties, has in the first place shown that the product of $t(M)$ by 2^{d-1} is an integer; the same author then proceeds to obtain (11), which implies the integral character of $t(M)$, using for this purpose the theory of analytic stacks, as appears from the preliminary note {43}.

The above developments could be invested with a more geometrical character by the following procedure. With regard to (12), we can introduce the *invariant series of equivalence* $|\eta(M)|$, defined by

$$\eta(M) = \sum_{(i)} \mu_{i_1 i_2 \ldots i_l}(d) \,.\, (i_1 i_2 \ldots i_l)_M. \tag{15}$$

We should then have to show how to arrive at a direct definition of the sets (15), and to prove that the number $[\eta(M)]$ of points in such a set is equal to

$$[\eta(M)] = \mu(d) \,.\, [P_a(M) + (-1)^d]. \tag{16}$$

By virtue of (12), this would be tantamount to establishing (11), giving at the same time a geometrico-functional interpretation of the result.

[8]) In Kodaira {59}, with the use of harmonic integrals, the validity of this result is verified for varieties which are complete regular intersections of primals in any given space.

Now, in point of fact, this has already been accomplished in the particular cases $d = 2, 3$ (a circumstance which Hirzebruch seems to have overlooked); and it should also be feasible for $d > 3$. In the case $d = 2$, the sets (15) belong to the *Enriques series* (cf. Enriques [1] and Campedelli {16}):

$$\eta(M) = (2)_M + (1, 1)_M,$$

and (16) is none other than the classical relation of Noether {66}:

$$(I + 4) + (p^{(1)} - 1) = 12(p_a + 1).$$

For $d = 3$, (15) is given quite simply by

$$\eta(M) = (1, 2)_M,$$

and (16) reduces to

$$[\eta(M)] = 24(P_a - 1),$$

both results being in a paper by Segre [1] (formulae (82), (60)) dating from 1934. We may add that the authors in question have given simple constructions for the sets $\eta(M)$ and $2\eta(M)$ in the respective cases $d = 2, 3$, by considering a net of hypersurfaces on M. It would be interesting to obtain an analogous construction for $d > 3$; lacking this, we could endeavour to give a direct definition of the sets (15) by proceeding inductively in the following way.

In the first place, using some of the results described in § IV, we can easily show that all the coefficients μ in (15) must be taken to be zero if we require the result

$$\eta(M + N) = \eta(M) + \eta(N)$$

to hold for every pair M, N of hypersurfaces of a generic $(d + 1)$-dimensional variety V. We should then have to prove that, if P denotes the variety $(MN)_V$, and if $\eta(P)$ has already been defined inductively, the coefficients $\mu_{(i)}$ in (15) can be determined (and, by what has been said, uniquely) so that, for every choice of V, M, N, we have

$$\eta(M + N) = \eta(M) + \eta(N) + \nu(d) \,.\, \eta(P),$$

where $\nu(d)$ is a positive integer which is prime to the $\mu_{(i)}$'s. Since, as we know,

$$P_a(M + N) = P_a(M) + P_a(N) + P_a(P),$$

this implies that $\mu(d)$ must satisfy the relation

(17) $$\mu(d) = \nu(d) \,.\, \mu(d - 1).$$

From this method of introducing the series of equivalence $|\,\eta\,|$, we could deduce geometrical forms for the relations of Severi [2] and Albanese [1] [9]), between the arithmetic genera of the various powers $K^0 = M, K^1 = K, K^2, \ldots,$

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[9]) Relations equivalent to Albanese's were afterwards obtained independently by Todd and Maxwell [1]. For the equivalence between the two sets of relations see Todd {98}.

K^d of a canonical hypersurface K on M. Thus, with regard to the Severi relation, we have the formula

$$[(-1)^d - 1]\eta(M) = \sum_{i=1}^{d} (-1)^i \frac{\mu(d)}{\mu(d-i)} \eta(K^i), \tag{18}$$

which has already been established for the first few values of d[10]), and which it would be interesting to prove in general.

VII. The preceding results have already led to extensions in various directions, and readily suggest others. To begin with, there is no difficulty in extending the concept of dilatation to compact *complex* varieties: this has already been achieved by Hopf {49} by means of the so-called σ-process, in the special case where the base of the dilatation reduces to a point; of this, interesting applications have been made by Hopf, Behnke and Stein {14}, and Hirzebruch {39}. As regards the extension to the general case we have so far only a few results of local character (Segre {82}) and considerations of homology on dilated varieties (Guggenheimer {38}). It would be well worth while to carry these studies further, specially in relation to the covariant varieties of immersion, the canonical varieties and intersection problems in the non-regular case, as a result of which — in view of recent work by Hodge [1], Chern {17}, Vesentini {99} and Guggenheimer {37} — it would be possible to introduce Chern's characteristic classes by a new and simpler method.

It is known that there exist compact *complex* varieties which are not equivalent to algebraic varieties (Cf. for example Calabi and Eckermann {15}), that an important category intermediate between these two types is given by the *Kähler varieties* (Cf. Hodge {48}), and that — with the use of harmonic integrals — various properties of algebraic varieties can be carried over to the last-named. This has been brilliantly achieved by Kodaira {52, 55} and by Spencer {95}, as regards the Riemann-Roch theorem and the arithmetic genus. We may further point out that almost all the researches on the Todd genus have been conducted by Hirzebruch for varieties even more general than complex varieties (the so-called *almost complex* manifolds). This author {40} has also characterised the Todd genus by means of its invariance under Hopf's σ-process, and — for varieties of dimensions which are multiples of 4 — has exhibited a relation between the genus and the inertia index of certain quadratic forms {41, 42, 44}. A comparison with a known result of Hodge {45, p. 224} leads to the presumption that there exists, in the algebraic case, a relation (which it would be worth investigating) between these ideas and the notion of geometric genus.

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[10]) For $d = 3$ this is to be found in Segre [1] (formula (58)), observing that $\mu(0) = 1$, $\mu(1) = 2$, $\mu(2) = 12$, $\mu(3) = 24$ (Cf. formulae (50), (54) of Segre [1]). By virtue of (17), one could probably deduce from (18) that $\nu(d) = 2$ for all *odd* values of d.

Here too the algebraic case is revealed as a source of fruitful suggestions; and, moreover, it is not to be supposed that apparent extensions always have a greater significance. Thus it has been recently shown by Kodaira {56} that certain Kähler varieties, which Hodge {47} has called Kähler manifolds of *restricted type*, are always regularly equivalent to non-singular algebraic varieties. In addition, Kodaira {60} has found a notable case in which certain varieties which are "abstract" in the sense of Weil can be realised as algebraic varieties in the usual sense.

Even if we limit ourselves to the study of algebraic varieties, there is no lack of interesting questions which remain to be examined; among them are the specially important problems which present themselves when the series and systems of equivalence are considered in the *effective*, instead of the virtual, field. And it may be noted that the only adequate weapons of attack which we possess for such a purpose to-day are algebro-geometric in character.

In this order of ideas, we may first mention the problems concerning the dimensions of series and systems of equivalence; about these, absolutely nothing is known save in the classic case of linear systems of hypersurfaces and for an attempt made by Severi {89} to extend the Riemann-Roch theorem to series of equivalence on a surface.

Another notable type of question is that which links the construction of the canonical varieties with the use of differential forms of the first species, in the manner originally indicated by Severi [3], Todd {96} and Eger [2, 3]; another question treats of series and systems of equivalence from the topological point of view and in relation to the theory of correspondences (Cf. Severi {88}, [6]).

It would also be worth while to investigate the operation of canonisation in connection with the various compositions within the equivalence rings on a given algebraic variety, with particular regard to the iteration of this operation.

Finally, we should not neglect the problems of classification, concerning which very little is known for varieties of three or more dimensions. In particular there is Severi's conjecture — confirmed by him [6, chap. VI] solely in the case of surfaces — that the Abelian varieties of rank unity are the only algebraic varieties for which the various canonical systems of lower dimension all reduce to the zero of rational equivalence.

All these questions are undoubtedly important and difficult, even if none of them has been mentioned by von Neumann in his memorable discourse on "Unsolved problems in mathematics". To the former, as to the latter, it will be the task of the future to provide an answer!

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