Proceedings of the International Congress of Mathematicians
Helsinki, 1978

Quasiconformal Mappings, Teichmüller Spaces, and Kleinian Groups

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I am extremely grateful to the Committee to select hour speakers for the great honor they have bestowed on me, and above all for this opportunity to address the mathematicians of the whole world from the city of my birth. The city has changed a great deal since my childhood, but I still get a thrill each time I return to this place that holds so many memories for me. I assure you that today is even a more special event for me.

I have interpreted the invitation as a mandate to report on the state of knowledge in the fields most directly dominated by the theory and methods of quasiconformal mappings. I was privileged to speak on the same topic once before, at the Congress in Stockholm 1962, and it has been suggested that I could perhaps limit myself to the developments after that date. But I feel that this talk should be directed to a much wider audience. I shall therefore speak strictly to the non-specialists and let the experts converse among themselves at other occasions.

The whole field has grown so rapidly in the last years that I could not possibly do justice to all recent achievements. A mere list of the results would be very dull and would not convey any sense of perspective. What I shall try to do, in the limited time at my disposal, is to draw your attention to the rather dramatic changes that have taken place in the theory of functions as a direct result of the inception and development of quasiconformal mappings. I should also like to make it clear that I am not reporting on my own work; I have done my share in the early stages, and I shall refer to it only when needed for background.

* This work has been supported by the National Science Foundation of the United States under Grant number MCS77 07782

1. Historical remarks. In classical analysis the theory of analytic functions of complex variables, and more particularly functions of one variable, have played a dominant role ever since the middle of the nineteenth century. There was an obvious peak around the turn of the century, centering about names like Poincaré, Klein, Picard, Borel, Hadamard. Another blossoming took place in the 1920s with the arrival of Nevanlinna theory. The next decade seemed at the time as a slackening of the pace, but this was deceptive; many of the ideas that were later to be fruitful were conceived at that time.

The war and the first post-war years were of course periods of stagnation. The first areas of mathematics to pick up momentum after the war were topology and functions of several complex variables. Big strides were taken in these fields, and under the leadership of Henri Cartan, Behnke, and many others, the more-dimensional theory of analytic functions and manifolds acquired an almost entirely new structure affiliated with algebra and topology. As a result of this development the gap between the conservative analysts who were still doing conformal mapping and the more radical ones involved with sheaf-theory became even wider, and for some time it looked as if the one-dimensional theory had lost out and was in danger of becoming a rehash of old ideas. The gap is still there, but I shall try to convince you that in the long run the old-fashioned theory has recovered and is doing quite well.

The theory of quasiconformal mappings is almost exactly fifty years old. They were introduced in 1928 by Herbert Grötzsch in order to formulate and prove a generalization of Picard's theorem. More important is his paper of 1932 in which he discusses the most elementary but at the same time most typical cases of extremal quasiconformal mappings, for instance the most nearly conformal mapping of one doubly connected region on another. Grötzsch's contribution is twofold: (1) to have been the first to introduce non-conformal mappings in a discipline that was so exclusively dominated by analytic functions, (2) to have recognized the importance of measuring the degree of quasiconformality by the maximum of the dilatation rather than by some integral mean (this was recently pointed out by Lipman Bers).

Grötzsch's papers remained practically unnoticed for a long time. In 1935 essentially the same class of mappings was introduced by M. A. Lavrentiev in the Soviet Union whose work was connected more closely with partial differential equations than with function theory proper. In any case, the theory of quasiconformal mappings, which at that time had also acquired its name, slowly gained recognition, originally as a useful and flexible tool, but inevitably also as an interesting piece of mathematics in its own right.

Nevertheless, quasiconformal mappings might have remained a rather obscure and peripheral object of study if it had not been for Oswald Teichmüller, an exceptionally gifted and intense young mathematician and political fanatic, who suddenly made a fascinating and unexpected discovery. At that time, many special extremal problems in quasiconformal mapping had already been solved, but these were isolated results without a connecting general idea. In 1939 he presented to the

Prussian Academy a now famous paper which marks the rebirth of quasiconformal mappings as a new discipline which completely overshadows the rather modest beginnings of the theory. With remarkable intuition he made a synthesis of what was known and proceeded to announce a bold outline of a new program which he presents, rather dramatically, as the result of a sudden revelation that occurred to him at night. His main discovery was that the extremal problem of quasiconformal mapping, when applied to Riemann surfaces, leads automatically to an intimate connection with the holomorphic quadratic differentials on the surface. With this connection the whole theory takes on a completely different complexion: A problem concerned with non-conformal mappings turns out to have a solution which is expressed in terms of holomorphic differentials, so that in reality the problem belongs to classical function theory. Even if some of the proofs were only heuristic, it was clear from the start that this paper would have a tremendous impact, although actually its influence was delayed due to the poor communications during the war. In the same paper Teichmüller lays the foundations for what later has become known as the theory of Teichmüller spaces.

2. Beltrami coefficients. It is time to become more specific, and I shall start by recalling the definition and main properties of quasiconformal (q.c.) mappings. To begin with I shall talk only about the two-dimensional case. There is a corresponding theory in several dimensions, necessarily less developed, but full of interesting problems. One of the reasons for considering q.c. mappings, although not the most compelling one, is precisely that the theory does not fall apart when passing to more than two dimensions. I shall return to this at the end of the talk.

Today it can be assumed that even a non-specialist knows roughly what is meant by a q.c. mapping. Intuitively, a homeomorphism is q.c. if small circles are carried into small ellipses with a bounded ratio of the axes; more precisely, it is K-q.c. if the ratio is $\leq K$. For a diffeomorphism f this means that the complex derivatives $f_z=\frac{1}{2}(f_x-if_y)$ and $f_{\bar z}=\frac{1}{2}(f_x+if_y)$ satisfy $|f_{\bar z}|\leq k|f_z|$ with $k=(K-1)/(K+1)$.

Already at an early stage it became clear that it would not do to consider only diffeomorphisms, for the class of diffeomorphisms lacks compactness. In the beginning rather arbitrary restrictions were introduced, but in time they narrowed down to two conditions, one geometric and one analytic, which eventually were found to be equivalent. The easiest to formulate is the analytic condition which says that f is K-q.c. if it is a weak L^2-solution of a *Beltrami equation*

$$f_{\bar z} = \mu f_z \tag{1}$$

where $\mu=\mu_f$, known as a Beltrami coefficient, is a complex-valued measurable function with $\|\mu\|_\infty\leq k$.

The equation is classical for smooth μ, but there is in fact a remarkably strong existence and uniqueness theorem without additional conditions. If μ is defined in the whole complex plane, with $|\mu|\leq k<1$ a.e., then (1) has a homeomorphic solution which maps the plane on itself, and the solution is unique up to conformal mappings. Simple uniform estimates, depending only on k, show that the class of K-q.c. mappings is compact.

It must be clear that I am condensing years of research into minutes. The fact is that the post-Teichmüller era of quasiconformal mappings did not start seriously until 1954. In 1957 I. N. Vekua in the Soviet Union proved the existence and uniqueness theorem for the Beltrami equation, and in the same year L. Bers discovered that the theorem had been proved already in 1938 by C. Morrey. The great difference in language and emphasis had obscured the relevance of Morrey's paper for the theory of q.c. mappings. The simplest version of the proof is due to B. V. Boyarski who made it a fairly straightforward application of the Calderón-Zygmund theory of singular integral transforms.

As a consequence of the chain rule the Beltrami coefficients obey a simple composition law:

$$\mu_{g\circ f^{-1}} = \left[\frac{\mu_g - \mu_f}{1-\bar{\mu}_f \mu_g} (f_z/|f_z|)^2\right] \circ f^{-1}.$$

The interesting thing about this formula is that for any fixed z and f the dependence on $\mu_g(z)$ is complex analytic, and a conformal mapping of the unit disk on itself. This simple fact turns out to be crucial for the study of Teichmüller space.

3. Extremal length. The geometric definition is conceptually even more important than the analytic definition. It makes important use of the theory of extremal length, first developed by A. Beurling for conformal mappings. Let me recall this concept very briefly. If L is a set of locally rectifiable arcs in R^2, then a Borel measurable function ϱ: $R^2 \to R^+$ is said to be *admissible* for L if $\int_\gamma \varrho\, ds \geqslant 1$ for all $\gamma \in L$. The *module* $M(L)$ is defined as $\inf \int \varrho^2 dx$ for all admissible ϱ; its reciprocal is the *extremal length* of L. It is connected with q.c. mappings in the following way: If f is a K-q.c. mapping (according to the analytic definition), then $M(fL) \leqslant KM(L)$. Conversely, this property may be used as a geometric definition of K-q.c. mappings, and it is sufficient that the inequality hold for a rather restrictive class of families L that can be chosen in various ways. This definition has the advantage of having an obvious generalization to several dimensions.

Inasmuch as extremal length was first introduced for conformal mappings, its connection with q.c. mappings, even in more than two dimensions, is another indication of the close relationship between q.c. mappings and classical function theory.

4. Teichmüller's theorem. The problem of extremal q. c. mappings has dominated the subject from the start. Given a family of homeomorphisms, usually defined by some specific geometric or topological conditions, it is required to find a mapping f in the family such that the maximal dilatation, and hence the norm $\|\mu_f\|_\infty$ is a minimum. Because of compactness the existence is usually no problem, but the solution may or may not be unique, and if it is there remains the problem of describing and analyzing the solution.

It is quite obvious that the notion of q.c. mappings generalizes at once to mappings from one Riemann surface to another, each with its own conformal structure, and that the problem of extremal mapping continues to make sense. The Beltrami coefficient becomes a Beltrami differential $\mu(z)d\bar{z}/dz$ of type $(-1, 1)$. Note that $\mu(z)$ does not depend on the local parameter on the target surface.

Teichmüller considers topological maps $f: S_0 \to S$ from one compact Riemann surface to another. In addition he requires f to belong to a prescribed homotopy class, and he wishes to solve the extremal problem separately for each such class. Teichmüller asserted that there is always an extremal mapping, and that it is unique. Moreover, either there is a unique conformal mapping in the given homotopy class, or there is a constant $k, 0<k<1$, and a *holomorphic quadratic differential* $\varphi(z)dz^2$ on S_0 such that the Beltrami coefficient of the extremal mapping is $\mu_f = k\bar{\varphi}/|\varphi|$. It is thus a mapping with constant dilatation $K=(1+k)/(1-k)$. The inverse f^{-1} is simultaneously extremal for the mappings $S\to S_0$, and it determines an associated quadratic differential $\psi(w)dw^2$ on S. In local coordinates the mapping can be expressed through

$$\sqrt{\psi(w)}\,dw = \sqrt{\varphi(z)}\,dz + k\sqrt{\bar{\varphi}}(z)\,d\bar{z}.$$

Naturally, there are singularities at the zeros of φ, which are mapped on zeros of ψ of the same order, but these singularities are of a simple explicit nature. The integral curves along which $\sqrt{\varphi}\,dz$ is respectively real or purely imaginary are called horizontal and vertical trajectories, and the extremal mapping maps the horizontal and vertical trajectories on S_0 on corresponding trajectories on S. At each point the stretching is maximal in the direction of the horizontal trajectory and minimal along the vertical trajectory.

This is a beautiful and absolutely fundamental result which, as I have already tried to emphasize, throws a completely new light on the theory of q.c. mappings. In his 1939 paper Teichmüller gives a complete proof of the uniqueness part of his theorem, and it is still essentially the only known proof. His existence proof, which appeared later, is not so transparent, but it was put in good shape by Bers; the result itself was never in doubt. Today, the existence can be proved more quickly than the uniqueness, thanks to a fruitful idea of Hamilton. Unfortunately, time does not permit me to indicate how and why these proofs work, except for saying that the proofs are variational and make strong use of the chain rule for Beltrami coefficients.

5. Teichmüller spaces. Teichmüller goes on to consider the slightly more general case of compact surfaces with a finite number of punctures. Specifically, we say that S is of finite type (p, m) if it is an oriented topological surface of genus p with m points removed. It becomes a Riemann surface by giving it a conformal structure. Following Bers we shall define a conformal structure as a sense-preserving topological mapping σ on a Riemann surface. Two conformal structures σ_1 and σ_2 are equivalent if there is a conformal mapping g of $\sigma_1(S)$ on $\sigma_2(S)$ such that $\sigma_2^{-1} \circ g \circ \sigma_1$ is homotopic to the identity. The equivalence classes $[\sigma]$ are the points of the Teichmüller space $T(p, m)$, and the distance between $[\sigma_1]$ and $[\sigma_2]$ is defined to be

$$d([\sigma_1], [\sigma_2]) = \log \inf K(f)$$

where $K(f)$ is the maximal dilatation of f, and f ranges over all mappings homotopic to $\sigma_2 \circ \sigma_1^{-1}$. It is readily seen that the infimum is actually a minimum, and that the extremal mapping from $\sigma_1(S)$ to $\sigma_2(S)$ is as previously described, except that the quadratic differentials are now allowed to have simple poles at the punctures.

With this metric $T(p, m)$ is a complete metric space, and already Teichmüller showed that it is homeomorphic to $R^{6p-6+2m}$ (provided that $2p-2+m>0$).

Let f be a self-mapping of S. It defines an isometry f^* of $T(p, m)$ which takes $[\sigma]$ to $[\sigma \circ f]$. This isometry depends only on the homotopy class of f and is regarded as an element of the *modular group* Mod (p, m). It follows from the definition that two Riemann surfaces $\sigma_1(S)$ and $\sigma_2(S)$ are conformally equivalent if and only if $[\sigma_2]$ is the image of $[\sigma_1]$ under an element of the modular group. The quotient space $T(p, m)/\text{Mod}(p, m)$ is the Riemann space of algebraic curves or moduli. The Riemann surfaces that allow conformal self-mappings are branch-points of the covering.

6. Fuchsian and quasifuchsian groups. The universal covering of any Riemann surface S, with a few obvious exceptions, is conformally equivalent to the unit disk U. The self-mappings of the covering surface correspond to a group G of fractional linear transformations, also referred to as Möbius transformations, which map U conformally on itself. More generally one can allow coverings with a signature, that is to say regular covering surfaces which are branched to a prescribed order over certain isolated points. In this case G includes elliptic transformations of finite order. It is always discrete.

Any discrete group of Möbius transformations that preserves a disk or a half-plane, for instance U, is called a Fuchsian group. It is a recent theorem, due to Jørgensen, that a nonelementary group which maps U on itself is discrete, and hence Fuchsian, if and only if every elliptic transformation in the group is of finite order. As soon as this condition is fulfilled the quotient U/G is a Riemann surface S, and U appears as a covering of S with a signature determined by the orders of the elliptic transformations. The group acts simultaneously on the exterior U^* of U, and $S^* = U^*/G$ is a mirror image of S. G is determined by S up to conjugation.

A point is a limit point if it is an accumulation point of an orbit. For Fuchsian groups all limit points are on the unit circle; the set of limit points will be referred to as the *limit set* $\Lambda(G)$. Except for some trivial cases there are only two alternatives: either Λ is the whole unit circle, or it is a perfect nowhere dense subset. With an unimaginative, but classical, terminology Fuchsian groups are accordingly classified as being of the first kind or second kind.

If S is of finite type, then G is always of the first kind; what is more, G has a fundamental region with finite noneuclidean area. Consider a q.c. mapping $f\colon S_0 \to S$ with corresponding groups G_0 and G. Then f lifts to a mapping $f\colon U \to U$ (which we continue to denote by the same letter), and if $g_0 \in G_0$ there is a $g \in G$ such that $f \circ g_0 = g \circ f$. This defines an isomorphism $\theta\colon G_0 \to G$ which is uniquely determined, up to conjugation, by the homotopy class of f. Moreover, f extends to a homeomorphism of the closed disks, and the boundary correspondence is again determined uniquely up to normalization. The Teichmüller problem becomes that of finding f with given boundary correspondence and smallest maximal dilatation. The extremal mapping has a Beltrami coefficient $\mu = k\bar{\varphi}/|\varphi|$ where φ is an invariant quadratic differential with respect to G_0.

Incidentally, the problem of extremal q.c. mappings with given boundary values makes sense even when there is no group, but the solution need not be unique. The questions that arise in this connection have been very successfully treated by Hamilton, K. Strebel, and E. Reich.

For a more general situation, let $\mu d\bar{z}/dz$ be any Beltrami differential, defined in the whole plane and invariant under G_0 in the sense that $(\mu \circ g_0)\bar{g}_0'/g_0' = \mu$ a.e. for all $g_0 \in G$. Suppose f is a solution of the Beltrami equation $f_{\bar{z}} = \mu f_z$. It follows from the chain rule that $f \circ g_0$ is another solution of the same equation. Therefore $f \circ g_0 \circ f^{-1}$ is conformal everywhere, and hence a Möbius transformation g. In this way μ determines an isomorphic mapping of G_0 on another group G, but this time G will in general not leave U invariant. For this reason G is a Kleinian group rather than a Fuchsian group. It has two invariant regions $f(U)$ and $f(U^*)$, separated by a Jordan curve $f(\delta U)$. The surfaces $f(U)/G$ and $f(U^*)/G$ are in general not conformal mirror images.

The group $G = fG_0f^{-1}$ is said to be obtained from G_0 by q.c. deformation, and it is called a quasifuchsian group. Evidently, quasifuchsian groups have much the same structure as fuchsian groups, except for the lack of symmetry. The curve that separates the invariant Jordan regions is the image of the unit circle under a q.c. homeomorphism of the whole plane. Such curves are called quasicircles. It follows by a well-known property of q.c. mappings that every quasicircle has zero area, and consequently the limit set $\Lambda(G)$ has zero two-dimensional measure.

Strangely enough, quasicircles have a very simple geometric characterization: A Jordan curve is a quasicircle if and only if for any two points on the curve at least one of the subarcs between them has a diameter at most equal to a fixed multiple of the distance between the points. It means, among other things, that there are no cusps.

7. The Bers representation. There are two special cases of the construction that I have described: (1) If μ satisfies the symmetry condition $\mu(1/\bar{z})\bar{z}^2/z^2 = \bar{\mu}(z)$, then G is again a Fuchsian group and f preserves symmetry with respect to the unit circle. (2) If μ is identically zero in U and arbitrary in U^*, except for being invariant with respect to G_0, then f is conformal in U, and $f(U)/G$ is conformally equivalent to $S = U/G$, while $f(U^*)/G$ is a q.c. mirror image of S.

I shall refer to the second construction as the Bers mapping. Two Beltrami differentials μ_1 and μ_2 will lead to the same group G and to homotopic maps f_1, f_2 if and only if $f_1 = f_2$ on ∂U (up to normalizations). When that is the case we say that μ_1 and μ_2 are equivalent, and that they represent the same point in the Teichmüller space $T(G_0)$ based on the Fuchsian group G_0.

In other words the equivalence classes are determined by the values of f on the unit circle. These values obviously determine $f(U)$, and hence f, at least up to a normalization. One obtains strict uniqueness by passing to the Schwarzian derivative $\varphi = S_f$ defined in U (recall that $S_f = (f''/f')' - \frac{1}{2}(f''/f')^2$). From the properties of the Schwarzian it follows that $\varphi(gz)g'(z)^2 = \varphi(z)$ for all $g \in G_0$. Furthermore, by a theorem of Nehari $|\varphi(z)|(1-|z|^2)^2$ is bounded (actually ≤ 6). Thus φ belongs to the Bers class $B(G_0)$ of bounded quadratic differentials with respect to the group G_0. The Bers map is an injection $T(G_0) \to B(G_0)$.

It is known that the image of $T(G_0)$ under the Bers map is *open*, and as a vector space $B(G_0)$ has a natural complex structure. The mapping identifies $T(G_0)$ with a certain open subset of $B(G_0)$ which in turn endows $T(G_0)$ with its own complex structure. If S is of type (p, m) the complex dimension is $3p-3+m$. The nature of the subset that represents $T(p, m)$ in C^{3p-3+m} is not well known. For instance it seems to be an open problem whether $T(1,1)$ is a Jordan region in C.

The case where $G = I$, the identity group, is of special interest because it is so closely connected with classical problems in function theory. An analytic function φ, defined on U, will belong to $T(I)$ if and only if it is the Schwarzian S_f of a schlicht (injective) function on U with a q.c. extension to the whole plane. The study of such functions has added new interest to the classical problems of schlicht functions.

To illustrate the point I would like to take a minute to tell about a recent beautiful result due to F. Gehring. Let $\boldsymbol{S}$ denote the space of all $\varphi = Sf$, f analytic and schlicht in U, with the norm $\|\varphi\| = \sup\,(1-|z|^2)^2|\varphi(z)|$, and let $\boldsymbol{T} = T(I)$ be the subset for which f has a q.c. extension. Gehring has shown (i) that $\boldsymbol{T} = \operatorname{Int} \boldsymbol{S}$, (ii) the closure of $\boldsymbol{T}$ is a proper subset of $\boldsymbol{S}$. To prove the second point, which gives a negative answer to a question raised by Bers maybe a dozen years ago, he constructs, quite explicitly, a region with the property that no small deformation, measured by the norm of the Schwarzian, changes it to a Jordan region, much less to one whose boundary is a quasicircle. I mention this particular result because it is recent and because it is typical for the way q.c. mappings are giving new impulses to the classical theory of conformal mappings.

In the finite dimensional case $T(p, m)$ has a compact boundary in $B(G_0)$. It is an interesting and difficult problem to find out what exactly happens when φ approaches the boundary. The pioneering research was carried out by Bers and Maskit. They showed, first of all, that when φ approaches a boundary point the holomorphic function f will tend to a limit which is still schlicht, and the groups G tend to a limit group which is Kleinian with a single, simply connected invariant region. Such groups were called B-groups (B stands either for Bers or for boundary) in the belief that any such group can be obtained in this manner. It can happen that the invariant simply connected region is the whole set of discontinuity; such groups are said to be *degenerate*. Classically, degenerate groups were not known, but Bers proved that they must exist, and more recently Jørgensen has been able to construct many explicit examples of such groups.

Intuitively, it is clear what should happen when φ goes to the boundary. We are interested to follow the q.c. images $f(U^*)$. In the degenerate case the image disappears completely. In the nondegenerate case the fact that one approaches the boundary must be visible in some way, and the obvious guess is that one or more of the closed geodesics on the surface is being pinched to a point. In the limit $f(S^*)$ would either be of lower genus or would disintegrate to several pieces, and one would end up with a more general configuration consisting of a "surface with nodes", each pinching giving rise to two nodes.

A lot of research has been going on with the intent of making all this completely rigorous, and if I am correctly informed these attempts have been successful, but much remains to be done. This is the general trend of much of the recent investigations of Bers, Maskit, Kra, Marden, Earle, Jørgensen, Abikoff and others; I hope they will understand that I cannot report in any detail on these theories which are still in *status nascendi.*

In a slightly different direction the theory of Teichmüller spaces has been extended to a study of the so-called universal Teichmüller curve, which for every type (p, m) is a fiber-space whose fibers are the Riemann surfaces of that type. A special problem is the existence, or rather non-existence, of holomorphic sections.

The Bers mapping is not concerned with extremal q.c. mappings, and it is rather curious that one again ends up with holomorphic quadratic differentials. The Bers model has a Kählerian structure obtained from an invariant metric, the Petersson-Weil metric, on the space of quadratic differentials. The relation between the Petersson-Weil metric and the Teichmüller metric has not been fully explored and is still rather mystifying.

8. Kleinian groups. I would have preferred to speak about Kleinian groups in a section all by itself, but they are so intimately tied up with Teichmüller spaces that I was forced to introduce Kleinian groups somewhat prematurely. I shall now go back and clear up some of the terminology.

It was Poincaré who made the distinction between Fuchsian and Kleinian groups and who also coined the names, much to the displeasure of Klein. He also pointed out that the action of any Möbius transformation extends to the upper half space, or, equivalently, to the unit ball in three-space. Any discrete group of Möbius transformations is discontinuous on the open ball. Limit points are defined as in the Fuchsian case; they are all on the unit sphere, and the limit set Λ may be regarded either as a set on the Riemann sphere or in the complex plane. The elementary groups with at most two limit points are usually excluded, and in modern terminology a Kleinian group is one whose limit set is nowhere dense and perfect. A Kleinian group may be looked upon as a Fuchsian group of the second kind in three dimensions. As such it cannot have a fundamental set with finite non-euclidean volume. Therefore, the relatively well developed methods of Lie group theory which require finite Haar measure are mostly not available for Kleinian groups. However, the important method of Poincaré series continues to make sense.

Let G be a Kleinian group, Λ its limit set, and Ω the set of discontinuity, that is to say the complement of Λ in the plane or on the sphere. The quotient manifold Ω/G inherits the complex structure of the plane and is thus a disjoint union of Riemann surfaces. It forms the boundary of a three-dimensional manifold $M(G) = B(1) \cup \Omega/G$.

What is the role of q.c. mappings for Kleinian groups? For one thing one would like to classify all Kleinian groups. It is evident that two groups that are conjugate to each other in the full group of Möbius transformations should be regarded as essentially the same. But as in the case of quasifuchsian groups two groups can also be conjugate in the sense of q.c. mappings, namely if $G' = fGf^{-1}$ for some q.c. mapping of the sphere. In that case G' is a q.c. deformation of G, and such groups should be in the same class.

But this is not enough to explain the sudden blossoming of the theory under the influence of q.c. mappings. As usual, linearization pays off, and it has turned out that infinitesimal q.c. mappings are relatively easy to handle. An infinitesimal q.c. mapping is a solution of $f_{\bar{z}} = \nu$ where the right-hand member is a function of class L^∞. This is a non-homogeneous Cauchy-Riemann equation, and it can be solved quite explicitly by the Pompeiu formula, which is nothing else than a generalized Cauchy integral formula. In order that f induce a deformation of the group ν must be a Beltrami differential, $\nu \in \mathrm{Bel}\, G$, this time with arbitrary finite bound. There is a subclass N of trivial differentials that induce only a conformal conjugation of G, and the main theorem asserts that the dual space of $\mathrm{Bel}\, G/N$ can be identified with the space of quadratic differentials on $\Omega(G)/G$ which are of class L^1.

This technique is particularly successful if one looks only at finitely generated groups. In that case the deformation space is finite dimensional, so that there are only a finite number of linearly independent integrable quadratic differentials. This result led me to announce, somewhat prematurely, the so-called *finiteness theorem:* If G is finitely generated, then $S = \Omega(G)/G$ is a finite union of Riemann surfaces of finite type. I had overlooked the fact that a triply punctured square carries no quadratic differentials. Fortunately, the gap was later filled by L. Greenberg, and again by L. Bers who extended the original method to include differentials of higher order. With this method Bers obtained not only an upper bound for the number of surfaces in terms of the number of generators, but even a bound on the total Poincaré area of S.

It was not unreasonable to expect that finitely generated Kleinian groups would have other simple properties. For instance, since a finitely generated Fuchsian group has a fundamental polygon with a finite number of sides one could hope that every finitely generated Kleinian group would have a finite fundamental polyhedron. All such hopes were shattered when L. Greenberg proved that a degenerate group in the sense of Bers and Maskit can never have a finite fundamental polyhedron. Groups with a finite fundamental polyhedron are called *geometrically finite*, and it has been suggested that one should perhaps be content to study only geometrically finite groups. With his constructive methods that go back to Klein, Maskit has been able to give a complete classification of all geometrically finite groups, and Marden has used three-dimensional topology to study the geometry of the three-manifold. These are very farreaching and complicated results, and it would be impossible for me to try to summarize them even if I had the competence to do so.

9. The zero area problem. An interesting problem that remains unsolved is the following: Is it true that every finitely generated Kleinian group has a limit set with twodimensional measure zero?

The most immediate reason for raising the question is that it is easy to prove the corresponding property for Fuchsian groups of the second kind, two-dimensional measure being replaced by one-dimensional. How does one prove it? If the limit set of a Fuchsian group has positive measure one can use the Poisson integral to construct a harmonic function on the unit disk with boundary values 1 a.e. on the limit set and 0 elsewhere. If the group is finitely generated the surface must have a finitely generated fundamental group, and it is therefore of finite genus and connectivity. The ideal boundary components are then representable as points or curves. If they are all points the group would be of the first kind, and if there is at least one curve the existence of a nonconstant harmonic function which is zero on the boundary violates the maxium principle. Therefore the limit set must have zero linear measure. The proof is thus quite trivial, but it is trivial only because one has a complete classification of surfaces with finitely generated fundamental group.

For Kleinian groups it is easy enough to imitate the construction of the harmonic function, which this time has to be harmonic with respect to the hyperbolic metric of the unit ball. If the group is geometrically finite this leads rather easily to a proof of measure zero. For the general case it seems that one would need a better topological classification of three-manifolds with constant negative curvature. It is therefore not suprising that the problem has come to the attention of the topologists, and I am happy to report that at least two leading topologists are actively engaged in research on this problem. I believe that this pooling of resources will be very fruitful, and it would of course not be the first time that analysis inspires topology, and vice versa.

Some time ago W. Thurston became interested in a topological problem concerning foliations of surfaces, and he proved a theorem which is closely related to Teichmüller theory. I have not seen Thurston's work, but I have seen Bers' interpretation of it as a new extremal problem for self-mappings of a surface. It is fascinating, and I could and perhaps should have talked about it in connection with the Teichmüller extremal problem, but I am a little hesitant to speak about things that are not yet in print, and therefore not quite in the public domain. Nevertheless, since many exciting things have happened quite recently in this particular subject, I am taking upon myself to report very informally on some of the newest developments, including some where I have to rely on faith rather than proofs.

Thurston has now begun to apply his remarkable geometric and topological intuition and skill to the problem of zero measure. I certainly do not want to preempt him in case he is planning to talk about it in his own lecture, and I have seen only glimpses of his reasoning, but it would seem that he can prove zero area for all groups that are limits, in one sense or another, of geometrically finite groups. This would be highly significant, for it would show that all groups on the boundary of Teichmüller space have limit sets with zero measure. It would neither prove nor disprove the original conjecture, but it would be a very big step. Personally, I feel that a definitive solution is almost imminent.

Very recently there was a highly specialized conference on Riemann surfaces in the United States, and there was an air of excitement caused not only by what Thurston had done and was doing, but also by the presence of D. Sullivan who had equally fascinating stories to tell. Sullivan, too, has worked hard on the area problem, and he has come up with a by-product that does not solve the problem, but is extremely interesting in itself. He applies the powerful tool of what has been called topological dynamics. If a transformation group acts on a measure space, the space splits into two parts, a dissipative part with a measurable fundamental set, and a recurrent part whose every measurable subset meets infinitely many of its images in a set of positive measure. This powerful theorem, which goes back to E. Hopf, does not seem to have been familiar to those who have approached Kleinian group from the point of view of q.c. mappings. The dissipative part of a finitely generated group is the set of discontinuity, and nothing more; this is a known theorem. The recurrent part is the limit set, and it is of interest only if it has positive measure. But even if the area conjecture is true Sullivan's work remains significant for groups whose limit set is the whole sphere.

Sullivan has several theorems, but the one that has captured my special interest because I understand it best asserts that there is no invariant vector field supported on the limit set. If the limit set is the whole sphere there is no invariant vector field, period. In an equivalent formulation, the limit set carries no Beltrami differential. It was known before that there are only a finite number of linearly independent Beltrami differentials on the limit set of a finitely generated Kleinian group, but that there are none was a surprise to me, and Sullivan's approach gives results even for groups that are not finitely generated. Sullivan's results, taken as a whole, give a new outlook on the ergodic theory of Kleinian groups. They are related to, but go beyond the results of E. Hopf which were already considered deep and difficult, and as a corollary Sullivan obtains a strengthening of Mostow's rigidity theorem. I cannot explain the proofs beyond saying that they are very clever and show that Sullivan is not only a leading topologist, but also a strong analyst.

10. Several dimensions. In the remaining time I shall speak briefly about the generalizations to more than two dimensions. There are two aspects: q.c. mappings *per se*, and Kleinian groups in several dimensions.

The foundations for q.c. mappings in space are essentially due to Gehring and J. Väisälä, but very important work has also been done in the Soviet Union and Roumania. I have already mentioned, in passing, that correct definitions can be based on modules of curve families, and the modules give the only known workable technique. Otherwise, the difficulties are enormous. It is reasonably clear that the Beltrami coefficient should be replaced by a matrix-valued function, but this function is subject to conditions that were known already to H. Weyl, but which are so complicated that nobody has been able to put them to any use. Very little is known about when a region in n-space is q.c. equivalent to a ball, and there is not even an educated guess what Teichmüller's theorem should be replaced by. On the positive side one knows a little bit about boundary correspondence.

In two dimensions there is not much use for mappings that are locally q.c. but not homeomorphic, for by passing to Riemann surfaces they can be replaced by homeomorphisms. In several dimensions the situation is quite different, and there has been rapid growth of the theory of so-called quasiregular mappings from one n-dimensional space to another. It has been developed mostly in the Soviet Union and Finland, and this is perhaps a good opportunity to congratulate the young Finnish mathematicians to their success in this area. In the spirit of Rolf Nevanlinna they have even been able to carry over parts of the value distribution theory to quasiregular functions. In fact, less than a month ago I learned that Rickman has succeeded in proving a generalization of Picard's theorem that I know they have been looking for for a long time. It is so simple that I cannot resist quoting the result: There exists $q=q(n, K)$ such that any K-q.c. mapping $f: R^n \to R^n - \{a_1, \ldots, a_q\}$ is constant. (They believe that the theorem is true with $q=2$.)

As for Kleinian groups, they generalize trivially to any number of dimensions, and the distinction between Fuchsian and Kleinian groups disappears. Some properties that depend purely on hyperbolic geometry will carry over, but they are not the ones that use q.c. mappings. However, infinitesimal q.c. mappings have an interesting counterpart for several variables. There is a linear differential operator that takes the place of $f_{\bar{z}}$, namely $Sf=\frac{1}{2}(Df+Df') - (1/n)\operatorname{tr} Df \cdot 1_n$ which is a symmetric matrix with zero trace. It has the right invariance, and the conditions under which the Beltrami equation $Sf=\nu$ has a solution can be expressed as a linear integral equation. The formal theory is there, but it will take time before it leads to tangible results.

My survey ends here. I regret that there are so many topics that I could not even mention, and that my report has been so conspicuously insufficient as far as research in the Soviet Union is concerned. I know that I have not given a full picture, but I hope that I have given you an idea of the extent to which q.c. mappings have penetrated function theory.

References

The following surveys have been extremely helpful to the author:

Lipman Bers, ***Quasiconformal mappings, with applications to differential equations, function theory and topology***, Bull. Amer. Math. Soc. (6) **83** (1977), 1083—1100.

L. Bers and I. Kra (Editors), ***A crash course on Kleinian groups***, Lecture Notes in Math., vol. 400, Springer-Verlag, Berlin and New York, 1974.

W. J. Harvey (Editor), ***Discrete groups and automorphic functions***, Proc. an Instructional Conf. organized by the London Math. Soc. and the Univ. of Cambridge, Academic Press, London, New York, San Francisco, 1977.

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