

Mathematical Problems of Tidal Energy

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1. Tidal energy. Recent studies of alternative energy sources have embraced a number of “exotic” forms such as solar, wind, geothermal or tidal energy. Here we shall examine recent developments in the mathematical understanding of tides and tidal power, with particular reference to the Bay of Fundy in eastern Canada which has the highest tides and may be the test site for further development of this mode of electrical energy production.

From the observed increase in length of the day [21] and the observed lunar acceleration, the total rate of all tidal energy dissipation is known to be about 3×10^{19} ergs per second—a rate comparable to mankind’s present consumption of energy. Much of this energy is dissipated in certain oceanic high tide regions where shallow continental shelf areas create large amplitudes by resonance, shallowing and convergence. Thus the English Channel and Irish Sea absorb perhaps 4% of the overall total while the Bay of Fundy and Gulf of Maine account for about 1%, an energy flow equal to the present capacity of the Canadian electricity network.

Recent engineering studies have shown [2] that recommended sites at Economy Point, Nova Scotia, and Cape Maringouin, New Brunswick, would be feasible but probably not yet economical. Tidal energy is renewable, but not conservable, predictable but intermittent, and large yet limited (at these sites perhaps 8,000 megawatts capacity could be installed). An interesting recent suggestion for excess power at peak generating periods is compressed air storage in salt caverns, with subsequent coal burning.

The applied mathematical problems discussed here are of two types. First is the description and calculation of oceanic and estuarial tidal wave motion and the modifications that would be induced by the construction of a tidal barrier with

sluice gates and turbines. The second problem is the optimal control of sluice and turbine operations in which the tidal wave motions and energy generation processes interact.

2. Equations of motion. Let (θ, ϕ) be latitude and longitude, and (u, v) components of fluid velocity to the east and north, respectively. Let z denote sea level above equilibrium, and \bar{z} the formal "equilibrium tide". The equations of motion introduced by Laplace in 1775 [15] are, with vertical acceleration neglected,

$$(2.1) \quad \begin{aligned} \frac{\partial u}{\partial t} - (2\Omega \sin \theta)v &= \frac{-g}{a \cos \theta} \frac{\partial}{\partial \phi}(z - \bar{z}), \\ \frac{\partial v}{\partial t} + (2\Omega \sin \theta)u &= \frac{-g}{a} \frac{\partial}{\partial \theta}(z - \bar{z}), \\ \frac{\partial z}{\partial t} + \frac{1}{a \cos \theta} \left(\frac{\partial(Hu)}{\partial \phi} + \frac{\partial(Hv \cos \theta)}{\partial \theta} \right) &= 0, \end{aligned}$$

where also g denotes gravity, a earth's radius, Ω angular velocity of earth's daily rotation and $H = D + z$ where $D = D(\theta, \phi)$ is the ocean depth. In the late nineteenth century, these oceanographic equations were studied by Poincaré, Darwin, and others, and various particular solutions and special cases were solved. Many recent theoretical and numerical studies of these equations in various geometries and geographies have now been made, including Hendershott [14], Longuet-Higgins [17], Pekeris [23].

It is apparent that accuracy of numerical solutions is difficult to achieve because of the sensitivity of nearly resonant motions to errors of discretization.

For smaller basins or gulfs the earth's curvature can be neglected, and in Cartesian coordinates (x, y) the equations become, with quadratic friction terms ([29], [30]),

$$(2.2) \quad \begin{aligned} \frac{\partial u}{\partial t} - fv &= -g \frac{\partial(z - \bar{z})}{\partial x} + r \frac{u|u|}{H}, \\ \frac{\partial v}{\partial t} + fu &= -g \frac{\partial(z - \bar{z})}{\partial y} + r \frac{v|u|}{H}, \\ \frac{\partial z}{\partial t} + \frac{\partial(Hu)}{\partial x} + \frac{\partial(Hv)}{\partial y} &= 0. \end{aligned}$$

This is a symmetric hyperbolic system with monotone nonlinearity, of a type treated generally by Lions [16]. Boundary conditions at a coastline with normal \mathbf{n} are $\mathbf{u} \cdot \mathbf{n} = 0$ and at a sea boundary may take the form of given values for z , or $\mathbf{u} \cdot \mathbf{n}$ or of a radiation condition on z .

The tide raising forces, represented by the \bar{z} terms in (2.2), are almost periodic in time because of the various combinations (Godin [12]) of the several astronomical constants of the moon and sun. For convenience in finding analytic or numerical solutions it is customary to model the harmonic constituents separately, and thus to neglect certain nonlinear convective and frictional interactions which however are almost always very small.

When depth H and Coriolis parameter f are assumed constant, various plane

wave solutions of the Poincaré or Kelvin types are easily derived and have been extensively used in the qualitative discussion of real problems (Hendershott and Munk [14], Platzman [24], [25]). Along the western shore of Nova Scotia and the southern flank of the Bay of Fundy, the M_2 and other semidiurnal tidal components take the form of a Kelvin wave following a right-handed coastline.

3. The wave equation. If the Coriolis and frictional forces are neglected, the system (2.2) becomes equivalent to the classical wave equation

$$(3.1) \quad L\Phi \equiv \frac{\partial^2 \Phi}{\partial t^2} - \frac{\partial}{\partial x} \left(gH \frac{\partial \Phi}{\partial x} \right) - \frac{\partial}{\partial y} \left(gH \frac{\partial \Phi}{\partial y} \right) = 0 \quad \text{for a potential } \Phi = \Phi(x, y, t)$$

where $\Phi_x = u$, $\Phi_y = v$, $\Phi_t = z$. The local wave propagation velocity is c , where $c^2 = gH(x, y)$, and its variability plays a significant role in the refractive generation of topographical waves (Meyer [19]).

Actual problems will involve a nonhomogeneous boundary condition of the first or second kind, or a forcing term, which is almost periodic with respect to the time variable because of the incommensurability of the various orbital constants of the sun and moon. Thus the theory of almost periodic solutions of the wave equation of Amerio and Prouse [1], Lions and Strauss [16], Zaidman [32], and others applies directly to this model of tidal motions. The nonlinear wave equation with friction term $r\Phi_t$ treated by Amerio and Prouse is of the same type but is not directly equivalent.

At a vertical coastline the position of the boundary does not depend on sea level z . At sloping beaches in high tidal areas, the boundary position may vary as much as several kilometres depending on sea level. This is expressible as a "unilateral" condition (Brezis [4]) $\partial\Phi/\partial t = z > -D(x, y)$, and an existence theorem covering this case has been established by Brezis for the wave equation although the more general case of the Eulerian equations (2.2) remains to be treated.

4. Resonant alteration of amplitudes. Tidal power plants would be sited in bays where high amplitudes occur in part because of resonant amplification due to a near coincidence of the imposed period (usually lunar semidiurnal M_2) and the natural period of the bay or gulf. As major construction changes the geometry and dynamics of the tidal motion, the amplitude will change in response. Calculations made for barrier sites at the tip of Cape Chignecto have suggested that a substantial decrease may occur, with the new natural period well removed from the 12^h25^m forcing period (Duff [5]). To make such calculations, however, it is necessary to fix an outer sea boundary beyond which no change is assumed, and here a succession of complexities has emerged. In early work for the Bay of Fundy region the sea boundary was taken at the geographic limits of the bay, but this did not even permit an explanation of the existing resonant amplification (Yuen [30]). Subsequent extension of the sea boundary to the continental shelf edge gave an indication of resonant natural periods (Duff [5], Garrett [9]), but still did not include in the model the interaction or impedance relationship between the deep outer ocean and the shallow high tide area of energy dissipation.

Significant grounds for supposing that inclusion of extensive deep ocean areas would be necessary in a realistic tidal model arise from recent work of Garrett [8], who applied the theory of harbour resonance (Miles and Munk [20]) to the Bay of Fundy and Gulf of Maine and showed that their response to harmonic forcing at three semidiurnal frequencies indicated a fundamental natural period of approximately 13.3 hours. This suggests that tidal barrier construction at the head of the Bay of Fundy would actually increase amplitudes rather than reducing them as had earlier been supposed. A similar calculation using twelve stations at the head of the Bay of Fundy gave a period of 12.85 hours (Duff [7]).

Following a method adapted by Platzman [25], Garrett [9] has also calculated the first three natural periods (eigenvalues) and normal modes (vector eigenfunctions) of the oceanographic equations for the Bay of Fundy and Gulf of Maine. The natural period of the lowest mode is calculated in the range of 12.5 to 13 hours depending on the precise location of the lateral sea boundaries and the open or closed boundary conditions assumed at certain places. The second and third modes have periods of 9.5 and 5.7 hours, and it is apparent that the first mode carries nearly all the observed tidal oscillations in the Bay of Fundy.

The first three natural periods and modes for the North Atlantic have been calculated by Platzman [26]; the periods turn out to be 21.2, 14.0 and 11.5 hours, with some uncertainty about the third of these values. Thus it appears that the semidiurnal tidal periods M_2 of 12.42 hours, S_2 of 12.0 hours, N_2 of 12.66 hours, and others, lie between closely spaced second and third natural periods. This may help explain the unusually high semidiurnal amplitudes in the North Atlantic, but many detailed aspects of the resonant response to this array of semidiurnal frequencies remain unexplained.

The magnitude of Fundy tides may be seen as having been reached by a balance between a dissipative mechanism, with assumed quadratic frictional forces, and an energy imparting mechanism in the deep ocean where work done by the tide raising force is proportional to distance travelled and hence to the first power of amplitude. Further, it now appears that the second and third North Atlantic modes are those primarily stimulated by the Fundian resonance. To represent these processes within one model both the continental shelf shallows and oceanic areas must be included, as well as their zone of interaction across the continental shelf.

5. Numerical models of oceanic tides. Large-scale numerical calculations of global oceanic M_2 tides have been undertaken by Bogdanov and Magarik [3], Pekeris and Accad [23], Zahel [31] and Hendershott [14], the continental shelf shallows being omitted and treated as coastlines with various assumptions of permeability or impedance. Fairly good qualitative agreement for the North Atlantic has been obtained. However the substantial energy flows into the Gulf of Maine, Baffin Bay, or the English Channel-Irish Sea region suggest that accurate representation of these resonant sea motions will require detailed modelling of the dissipative regions. As deep sea tidal observations have been possible only recently and at limited numbers of stations, a detailed reconciliation of theory and observa-

tion will take many years, but it is now regarded as feasible two hundred years after Laplace [15] and nearly three hundred after Newton [22].

Such oceanic tidal models involve large-scale numerical computation (Heaps [13]), as thousands of grid points and depth measurements are needed to begin to represent the far from smooth topography of coastlines and ocean depths. To represent one harmonic tidal constituent such as M_2 a periodic solution is required, and this is found by calculating a sufficient number of tidal cycles to obtain convergence to a periodic solution for large times. Whereas in shallow waters with comparatively strong bottom friction a few cycles may suffice, a hundred cycles may be required for a deep ocean model, even with devices for acceleration of convergence. For problems of this scale the older explicit methods of numerical solution of partial differential equations are giving way to the more stable implicit and alternating direction methods that permit much longer time steps [18].

A brief description will now be given of an attempt made by the author to model the combined shallow dissipative region, in this case the Bay of Fundy and Gulf of Maine, and the deep sea region, in this case the North Atlantic. To obtain a detailed representation of the Bay of Fundy and at the same time a uniform coordinate grid suitable for implicit methods, a transformed system of spherical coordinates with pole about 3° inland from the New Brunswick coast was adopted. The system can thus be plotted on a Mercator projection with this point as North Pole. The oceanic region comprises the North Atlantic from Newfoundland to the Azores, thence to the African and South American coasts and then on a line through the West Indies to the coast of North America. The local conformal condition for a first-order square grid leads to the use of equal intervals of the longitude ϕ and of $y = \log \tan[\theta/2 + \pi/4]$, where θ represents latitude in the transformed system. Of 2,000 grid points in total, 32 lie in the Bay of Fundy and 200 in the Gulf of Maine.

The finite difference equations in this model are formulated with a splitting of z into two components $z = z_1 + z_2$ where the time rates of change of z_1 and z_2 are obtained as the x and y terms of the continuity equation. In each momentum equation, the corresponding z term is treated implicitly while the other term is included explicitly and the time steps for the two implicit systems are staggered to avoid extrapolation of explicit terms. In effect, the model treats two implicit systems of one-dimensional channels, each with explicit crossover terms. Tide raising forcing terms based on zero lunar declination and boundary data based on the oceanic tidal models described above are used.

Preliminary results from this model indicate that tidal barrier construction at the three preferred sites will increase the M_2 amplitude at Economy Point, Nova Scotia, and decrease the amplitude at the New Brunswick sites.

Further refinement of such models will be necessary if reliable forecasts for large-scale projects are ultimately required. Features such as self-gravitation, crustal reaction, and the effect of solid earth tides present themselves for consideration, and for the latter two more observational studies are needed.

6. The control problem for tidal energy generation. Let a one-dimensional channel

have headwater at a , barrier position b and open sea boundary c on the x -axis; let $Z(x, t)$ denote sea level above equilibrium at point x , $Q(x, t)$ the flow at x , $Q = A(x, t)u(x, t)$ where $A(x, t)$ is cross section and $u(x, t)$ current. Let $b(x, t)$ denote surface breadth of the channel, and $k = 0.003$ the dimensionless quadratic friction constant. The equations of motion in this channel take the form (Proudman [29], Yuen [30])

$$(6.1) \quad Z_t = -\frac{1}{b(x)} Q_x, \quad Q_t = -AgZ_x - \frac{kQ|Q|}{A(x)H},$$

where $H = D + Z$ again denotes total depth and g gravity.

Boundary conditions are $Q(a, t) = 0$ and $Z(c, t) = Z_0(c) \cos(\omega t - \gamma)$, while at the barrier position b , Z is in general discontinuous, and $Q(b, t)$ is regulated by sluice and turbine controls (Duff [6]). Thus $Q_b = Q(b, t) = \lambda q + \mu \bar{V}$ where λ ($-1 \leq \lambda \leq 1$) is the "double effect" two-way turbine control with maximum flow q , and μ ($0 \leq \mu \leq 1$) is the sluice control with maximum permitted flow \bar{V} which for simplicity may be assumed to have the Torricellian form

$$\bar{V} = g^{1/2} A |Z_+ - Z_-|^{1/2} \operatorname{sgn}(Z_+ - Z_-)$$

in terms of the limiting values Z_+, Z_- of sea level on either side of the barrier.

Let $N(q, h)$ denote the power derived from turbines operating at head $h = |Z_+ - Z_-|$ and flow q , and let $p(t)$ denote the unit value of power at time t . Assuming the conservation of total water mass rather than (6.1), Gibrat formulated the problem of maximizing returns from a tidal power plant as a problem in the calculus of variations (Gibrat [10], Godin [11]), and these results are effectively in use at the 240 megawatt Rance tidal power station at St. Malo. Integration by parts of the first variation of J yields initial terminal and boundary conditions for ϕ, Φ , a barrier condition for ϕ , as well as conditions for the variations of λ and μ , namely for λ

$$(6.2) \quad \begin{aligned} p(t) (\partial N / \partial q_1)(q, h) - [\phi/b]^\pm &> 0, & \lambda &= +1, \\ &= 0, & -1 &< \lambda < +1, \\ &< 0, & \lambda &= -1, \end{aligned}$$

and for μ ,

$$(6.3) \quad \begin{aligned} [\phi/b]^\pm \bar{V} &> 0, & \mu &= 0, \\ &< 0, & \mu &= 1. \end{aligned}$$

The dual partial differential equations for ϕ, Φ take the following forms (with certain additional minor simplifications)

$$(6.4) \quad \phi_t + \frac{\partial}{\partial x} (Ag \Phi) = 0, \quad \Phi_t + \frac{\partial}{\partial x} \left(\frac{\phi}{b} \right) - 2k \frac{|Q|\Phi}{AH} = 0.$$

The combined Hamiltonian system (6.1), (6.2), (6.3), (6.4) gives optimal (or at least, extremal) solutions to the control problem (Pontryagin [28]). Because terminal conditions for Φ, ϕ are required, the usual complexities of a time horizon appear. However for periodic solutions these difficulties can be avoided and

numerical solutions obtained for single semidiurnal periods or fortnightly or monthly cycles involving two or more frequencies and corresponding to spring and neap tidal cycles (Duff [7]).

Such a system in operation can be regarded as having limited artificial intelligence directed toward the extraction of energy from the tidal sea motions. The apparent strategy such a system will follow involves the maximizing at certain times of operating head h by the timing of internal surges in the enclosed basin, and the operation of sluice gates to maximize the extraction of energy from the sea in accordance with the change of resonance created by the barrier itself. A complete synthesis of these considerations will require a combination of the most extensive tidal models with such a control system.

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