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Inside and Outside Manifolds

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Introduction. The classification theory of manifolds has evolved quite far. One theory fixes the homeomorphism or diffeomorphism type of a manifold in terms of the homotopy type and certain geometric invariants related to characteristic classes and the fundamental group (dimensions three and four excluded). In the simply connected case there is a further discussion which produces a purely algebraic invariant (the "homological configuration") determining the isomorphism class of the manifold and the group of automorphisms (isotopy classes) up to a finite ambiguity.

Further developments in this external theory of manifolds seems more and more algebraic. On the other hand, the study of geometrical objects inside one manifold is experiencing a resurgence which focuses attention on the classical goals and problems of "analysis situs". One organizing center for this activity is the qualitative study of dynamical systems which produces inside one manifold interesting compact subsets, families of intertwined noncompact submanifolds, geometrically defined measures and currents, with homological interpretations and relationships.

There are many problems concerning the structural stability, and a geometric description of the possible phenomena. These problems for flows generalize to higher dimensional foliations which are now known to exist abundantly.

For foliations of dimension greater than one there is a new ingredient, the Riemannian geometry of the leaves. The asymptotic properties of this geometry can be regarded as a *topological* invariant of the foliation.

Now we go into more detail. First we describe two classification theories for manifolds and then some topological problems concerning geometrical objects inside manifolds.

I. The two classification theorems. The invariants of manifolds we describe are interesting for all manifolds and classify completely for simply connected manifolds or other suitably restricted cases. Also the dimension of the manifolds must be larger than four.

The first theorem classifies the manifolds in a given homotopy type. The identification of the manifold homotopy type to a model homotopy type is part of the structure. We can picture all of our closed n-dimensional manifolds in one homotopy type as embedded in a nice domain of Euclidean space R^{2n+2} with smooth boundary. The domain will be isomorphic to a tubular neighborhood of each of these submanifolds (Figure 1). Two of these submanifolds will be considered equivalent if there is an isotopy of the domain carrying one onto another. For the first theorem we assume $\pi_1 = e$ and $n > 4$.

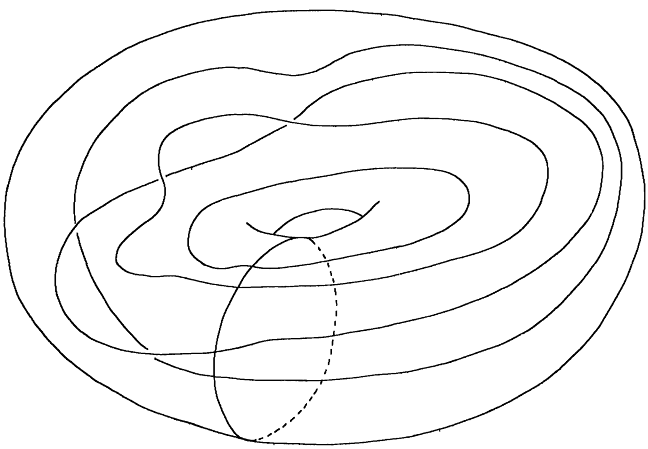


FIGURE 1. The manifolds in a homotopy type—pictured as a domain in Euclidean space.

THEOREM A. *The closed n-dimensional manifolds in a homotopy type X can be classified up to homeomorphism by the elements in a certain finitely generated abelian group h(X). h(X) is isomorphic modulo odd torsion to*

$$\bigoplus_i (H^{4i+2}(X, Z/2) \oplus H^{4i}(X, Z)), \qquad 0 < 4i,\ 4i + 2 < n.$$

The odd torsion in h(X) is the same as that in the real K-theory of X.

For more details see [S1] and [S2, Chapter 6].

We remark that the elements of $h(X)$ can be detected geometrically by spanning certain submanifolds or membranes across the domain representing X.

Each manifold in X is made transversal to these membranes, and numerical invariants are directly calculated from the intersections. The brunt of the information is carried by signatures of quadratic forms. Most of the theory for this is described in [M-S].

A nice example of this theorem is provided by complex projective n-space ($n \neq 2$). Here the homeomorphism types of manifolds having the same cohomology ring as CP^n are in one-to-one correspondence with

$$Z/2 \oplus Z \oplus Z/2 \oplus \cdots \oplus Z, \qquad n \text{ odd},$$
$$Z/2 \oplus Z \oplus Z/2 \oplus \cdots \oplus Z/2, \qquad n \text{ even},$$

where there are $(n - 1)$ summands. For any such manifold M the invariants can be read off from the sequence of submanifolds obtained by intersecting a homologically generating codimension 2 submanifold of M with itself.

To promote Theorem A to a classification up to diffeomorphism many more finite obstructions come in. For this most of the tools of algebraic topology can be utilized—K-theory, étale cohomology, localization, and specific calculations like the work of Milgram; see also [**S2**, Chapter 6]. The proof of Theorem A uses triangulations, transversality, and surgery. It depends heavily on the important work of Kirby and Siebenmann for topological manifolds. It was first proved in the piecewise linear context.

The next classification theorem will give one algebraic invariant which classifies the homeomorphism (or diffeomorphism) type up to a finite ambiguity. The new point here over Theorem A is homotopy theoretical and the homotopy problem is solved using differential forms. We will describe the "homological configuration" of a manifold. The idea is to build up a homological picture by starting with a basis of cycles in the extreme dimension (highest) and using intersections as much as possible as we work our way down through the homology. It is necessary to include

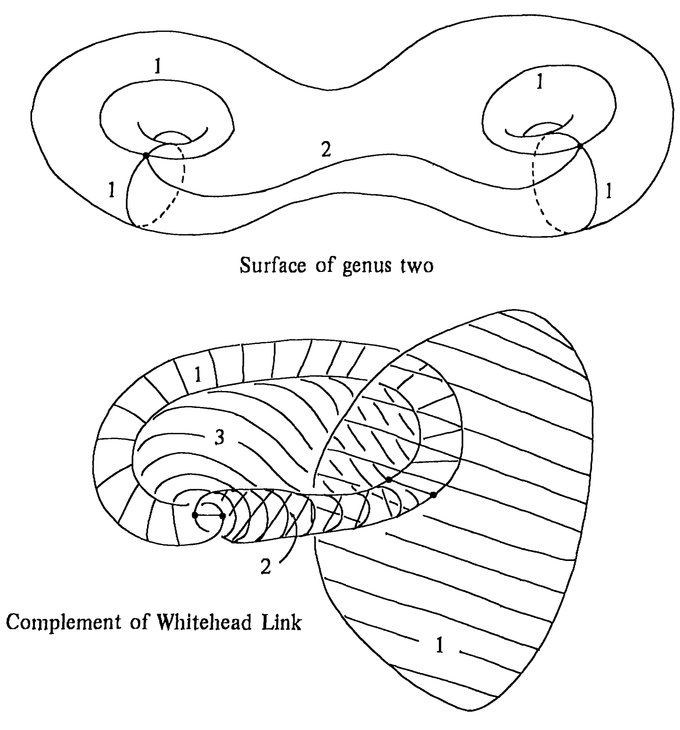


FIGURE 2. Examples of homology configurations with levels indicated.

chains or membranes to realize the homology relations among the pieces of the inductive configuration. (See Figure 2).

This construction is done rigorously using differential forms—starting in the extreme lowest dimension to build up a picture of the cohomology. One obtains a polynomial algebra tensor an exterior algebra with a differential (over Q) which determines the rational homotopy type.

The theory behind this is described in [S3] and [**DGMS**].

We can add to this Q-data

(a) the rational Pontryagin classes $p_1, p_2, \cdots$ with $p_i \in H^{4i}(M^n, Q)$,

(b) a certain lattice in the above algebra reflecting the integral structure,

(c) some information on the torsion in homology, for example, the order of the torsion subgroup.

If we refer to all this as the "integral homology configuration" of a manifold we have ($\pi_1 = e$, $n > 4$)

THEOREM B. *A manifold is determined up to a finite number of possibilities by its "integral homology configuration".*

A key step in the proof of Theorem B is the introduction of the arithmetic subgroups of Q-algebraic groups. A second part of the theorem says that the isotopy classes of automorphisms of the manifold are described up to a finite ambiguity (commensurability of groups) by automorphisms of the configuration. This shows these geometric automorphism groups are arithmetic groups [S4]. One can construct manifolds which realize any Q-homological configuration and characteristic classes subject to Poincaré duality and the Hirzebruch index theorem. Also, essentially all arithmetic groups occur as the group of components of Diff M, M simply connected.

An interesting sidelight is that the maximal normal nilpotent subgroup of all automorphisms contains those which are the identity on the spherical homology.

This theory of algebraic topology over Q based on differential forms can be used in more analytical questions, e.g., the topology of Kaehler manifolds, the study of closed geodesics, and Gel'fand-Fuks cohomology. See [S3], [**DGMS**], [H] and [S5].

II. Problems. Now we turn to more geometrical problems. The first question is the qualitative study of diffeomorphisms of manifolds under repeated iteration. One wants to describe as far as possible the orbit structure. Much has been done here but much is also unknown.

To illustrate these points consider a famous example (Figure 3) of Smale first on the solid torus and then on the 2-sphere. The solid torus is mapped into itself with degree 2, with half of it contracting into itself.

The nonwandering set $\{x\colon$ for all nghds U there exists n such that $f^n U \cap U \neq 0\}$ here is a structurally stable Cantor set plus one sink. The stable manifolds consisting of those points asymptotic (as $n \to +\infty$) to the Cantor set form a partial foliation of 2-manifolds coming out of the solid torus. The unstable manifolds of the Cantor set ($n \to -\infty$) form a dyadic solenoid running around the solid torus.

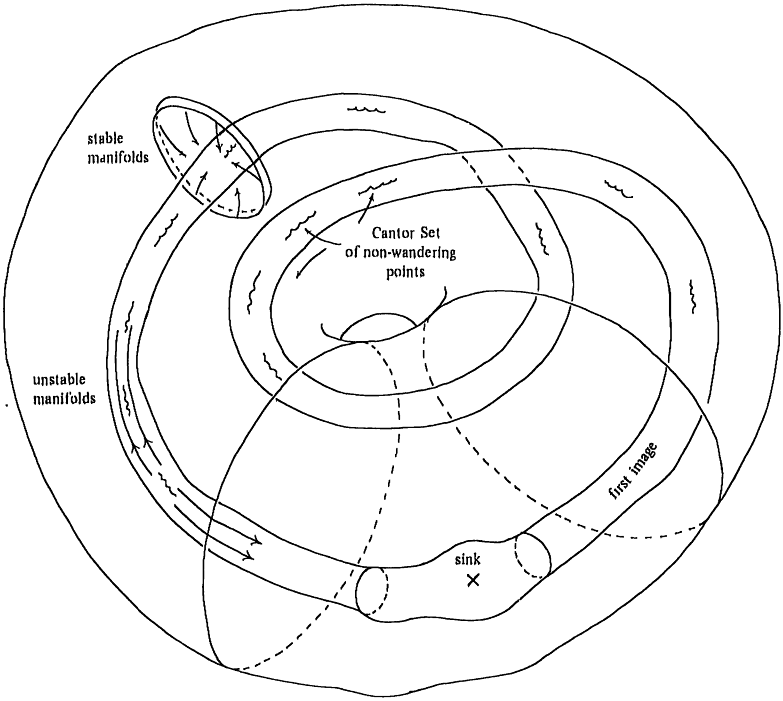


FIGURE 3. Smale's Axiom A of diffeomorphisms on the solid torus.

This picture is helpful for understanding Smale's general Axiom A diffeomorphisms. It is also not hard to see that handle preserving diffeomorphisms like these (always with zero-dimensional nonwandering set) form a C^0-dense set of all diffeomorphisms; see [Sm] and [SS].

Problem 1. Try to understand the deformations between the various Axiom A structurally stable systems. See [PN].

Problem 2. Try to construct and analyze the basic pieces of the nonwandering set having *positive* dimension. See [B] and [W] for the zero-dimensional and generalized solenoid cases respectively.

Now Smale originally studied this example on S^2. There are however many regions of Diff S^2 which are uncharted and do not contain Axiom A systems (see [N]). To begin to solve this problem one needs new notions of structurally stable descriptions. It is perhaps amusing to note that the counterexamples in this subject to the C^1 density of structurally stable *can be described*, so that their narrative description is at least structurally stable ([Sm2] and [W2]).

Problem 3. Describe more of the regions of Diff S^2 or Diff M outside the transversal Axiom A systems.

Problem 4. How much of Diff M can be described by perturbing transversal Axiom A systems to destroy carefully the transversality of stable and unstable manifolds? See [Sm2], [W], [RW].

In another direction, we might recall Arnold's theorem [A] in Diff S^1 which states that for almost all irrational rotations the probability that a smooth perturbation is C^0 equivalent to an irrational rotation approaches 1 as the size of the perturbation approaches zero. This is a kind of structural stability which is of practical importance [BK] but is not included in the topological conjugacy idea.

Problem 5. Formulate a useful mixed notion of structural stability combining continuity and probability.

For practical application, attractors—closed invariant sets with invariant neighborhoods—are important (see [T]).

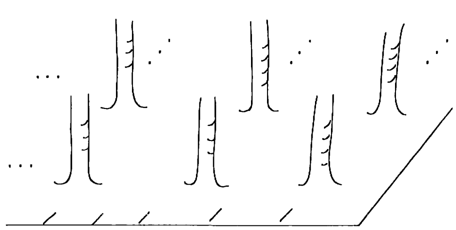
According to [BR] a measure one set of points in a no cycle Axiom A system goes to attractors. Thom asks the following:

Problem 6. Is it true, for a generic set (a countable intersection of dense opens) of Diff M, that almost all points in the manifold are asymptotic to attractors?

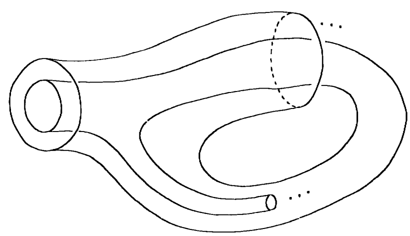
The questions of qualitative study are similar for flows. Here one uses especially the closed orbits, the Poincaré maps around them, and invariant measures. All the questions and concepts for nonsingular flows generalize to the qualitative study of foliations on a compact manifold. This generalization is quite challenging geometrically. Also understanding the qualitative behavior of foliations helps one understand the more classical problems for flows.

If we assume the ambient manifold has a Riemannian metric, each leaf of the foliation inherits a complete metric which is in a rough asymptotic sense independent of the ambient metric. For example, certain growth properties of volume $\{x \in \text{leaf}: \text{distance}(x, x_0) \leq R\}$ are topological invariants of the foliation. It is easy to see this growth rate is at most exponential, and if it is subexponential, interesting homological arguments are possible [P]. One can form a limiting cycle using the chains 1/volume times $\{x \in \text{leaf}: \text{distance}(x, x_0) \geq R\}$ and arrive at a "geometric current", roughly speaking a locally laminar submanifold with a transversal measure [RS].

More generally, we can ask what do leaves of foliations look like geometrically and topologically. See Figure 4 for examples of leaves in S^3.



A leaf in S^3 of cubic growth



Part of a leaf in S^3 of exponential growth

FIGURE 4

Problem 7. Describe the nature of the equivalence relation on leaves induced by ambient diffeomorphisms.

Problem 8. What do 2-dimensional leaves in S^3 look like?

References

[A] V. I. Arnold, *Small denominators*, I: *Mappings of the circumference onto itself*, Izv. Akad. Nauk SSSR Ser. Mat. **25** (1961), 21–86; English transl., Amer. Math. Soc. Transl. (2) **46** (1965), 213–284. MR **25** #4113.

[B] R. Bowen, *Entropy versus homology for axiom A*, Topology **13** (1974), 61.

[BR] R. Bowen and D. Ruelle, *A measure associated to Axiom A attractors*, Inst. Hautes Études Sci., Paris (preprint).

[DGMS] P. Deligne, P. Griffiths, J. Morgan and D. Sullivan, *Topology of Kaehler manifolds*, Invent. Math. (to appear).

[H] A. Haefliger (to appear).

[BK] Bruce Knight, *Various papers on nerve cells in the horseshoe crab*, Rockefeller University, New York, N. Y.

[M-S] J. Morgan and D. Sullivan, *The transversality characteristic class and linking cycles in surgery theory*, Ann. of Math. (1974).

[N] S. Newhouse *Diffeomorphisms with infinitely many sinks*, Topology **13** (1974).

[P] J. Plante, *A generalization of Poincaré-Bendixson*, Topology (1972).

[PN] S. Newhouse and J. Palis, *Bifurcations of Morse-Smale dynamical systems*, Dynamical Systems (Proc. Sympos. Salvador, 1971), Academic Press, New York and London, 1973, pp. 303–366. MR **48** #1255.

[RS] D. Ruelle and D. Sullivan, *Currents, flows, and diffeomorphisms*, Inst. Hautes Études Sci., Paris, Topology, 1975 (preprint).

[S1] D. Sullivan, *Geometric periodicity and the invariants of manifolds*, Manifolds-Amsterdam, 1970 (Proc. Nuffic Summer School), Lecture Notes in Math., vol. 197, Springer-Verlag, Berlin, 1971, pp. 44–75. MR **44** #2236.

[S2] ———, *Geometric topology*. Part I. *Localization, periodicity, and Galois symmetry*, M.I.T., Cambridge, Mass., 1970 (preprint).

[S3] ———, *Differential forms and the topology of manifolds*, Proc. Japan Conference on Manifolds, 1973.

[S4] ———, *Algebraic topology is algebraic*, Bull. Amer. Math. Soc. (to appear).

[S5] ———, *A formula for the homology of function spaces*, Bull. Amer. Math. Soc. (to appear).

[Sm] S. Smale, *Stability and isotopy in discrete dynamical systems*, Dynamical Systems. (Proc. Sympos., Salvador, 1971), Academic Press, New York and London, 1973, pp. 527–530. MR **48** #1255.

[Sm2] ———, *Notes on differentiable dynamic systems*, Proc. Sympos. Pure Math., vol. 14, Amer. Math. Soc., Providence, R. I., 1970, pp. 277–287. MR **42** #1152.

[SS] M. Shub and D. Sullivan, *Homology theory and dynamical systems*, Topology (to appear).

[W] R. F. Williams, *Expanding attractors*, Inst. Hautes Études Sci. Publ. Math. (to appear).

[W2] ———, *The "DA" maps of Smale and structural stability*, Proc. Sympos. Pure Math., vol. 14, Amer. Math. Soc., Providence, R.I., 1970, pp. 329–334. MR **41** #9296.

[RW] C. Robinson and R. F. Williams, *Finite stability is not generic*, Dynamical Systems (Proc. Sympos., Salvador, 1971), Academic Press, New York and London, 1973, pp. 451–462. MR **48** #1255.

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