AUTOMORPHIC FUNCTIONS AND THE THEORY OF REPRESENTATIONS

By I. M. GEL'FAND[1]

1. Introduction

According to F. Klein the investigation of geometrical objects may be reduced to the study of properties invariant under a group of transformations, i.e. to the study of homogeneous spaces.

By a homogeneous space X we mean any manifold acted upon by a group of transformations which is a Lie group. We can therefore study in a unified manner the whole class of the most symmetrical objects, aesthetically perfect, such as sphere, Lobachevsky space, Grassmann's manifold, the space of positively-definite matrices, etc.

It is rather remarkable that we can reach the same aesthetic perfection in the study of the set of functions $f(x)$ defined on the homogeneous space X. A transformation $x \to xg$ in X gives rise to a linear operator $T_g f(x) = f(xg)$ in the function space. So we are led to a representation of the group G, since the product of the transformations T_{g_1} and T_{g_2} corresponds to the product of the elements g_1 and g_2 of G.

Roughly speaking the problem is to decompose the function space into minimal invariant subspaces, or, which is the same, to decompose the representation into irreducible representations and to study invariant families involved.

The compact case of this problem (i.e. compact X and G) was investigated by H. Weyl and E. Cartan who took as model the rotation group. The invariant families for the rotation group consist of spherical functions. The least invariant systems of functions, arising in the general case, we shall also call the spherical functions on X. But only after one rejects the compactness condition and passes over to infinite-dimensional representations can one fully appreciate the importance and interest of this problem.

In this report I should like to tell about some results obtained by me in collaboration with my friends I. I. Pyatezki-Shapiro and M. I. Graev. Some of the results were inspired by important works of Selberg and Godement. It should be said that only through systematic employment of the theory of infinite-dimensional representations can one obtain complete understanding of these results.

Each homogeneous space X is associated with a group G and its subgroup Γ (stationary subgroup) consisting of those elements of G which leave immobile some fixed point $x_0 \in X$. The first part of the report is devoted mainly to the consideration of the case in which this subgroup Γ is a discrete group and the space X has finite volume. The functions relating to this case we shall call automorphic functions. Thus automorphic functions are spheri-

[1] The author dedicates this report to his friends, L. Schwartz, M. Morse, J. Leray, O. A. Oleinik, J. G. Petrovskiy̆, L. Bers, R. Courant, P. Lax, whose kind assistance has greatly helped him.

cal functions, associated with a discrete stationary subgroup. Automorphic functions in this sense were introduced in 1950 in a paper by S. V. Fomin and myself [3]. The ordinary automorphic functions are included among these if we take the group of real 2×2-matrices.

Permit me to cite another interesting and important example. Let us be given an arbitrary Riemann surface with a Riemann metric on it. Consider the manifold X the points of which are taken to be linear elements (i.e. points with directions) of the initial surface. Suppose that the universal covering space of our Riemann surface is the upper halfplane. Then the group G acting on X will be the group of all linear fractional transformations with real coefficients and the stationary subgroup Γ will be one of its discrete subgroups. If the original Riemann manifold is compact, i.e. if it is associated with an algebraic function, then the resulting representation of G in $L_2(X)$ can be decomposed into a countable direct sum of representations. Every irreducible representation of the group G under consideration can be characterised by a set of indices which we shall call the number of the representation. The numbers of the representations involved in the decomposition are invariants of the Riemann surface. They appear to form a complete system of invariants, and so it may happen that this question is connected with the classical problem of moduli in which such remarkable success was attained recently by Bers and Ahlfors. We can prove a weaker result: every continuous deformation of Riemann surface, leaving unchanged these invariants, is the identity.

So in the first half of the report we consider some general questions about the decomposition of a representation in a function space $L_2(X)$ on a homogeneous space X with discrete stationary subgroup. The characterization of the resulting representations and the method of decomposition lead us to very interesting functions which we call Zeta-functions of the given homogeneous space. This system of functions is closely connected with the classical Zeta-function. In the case of the group of real 2×2-matrices we can reduce these functions, using the duality theorems, to the functions, which were introduced by Selberg from particular considerations.

Now we wish to say a word about the methods. M. I. Graev and I have proposed [1] the so-called method of horyspheres for the investigation of representations in $L_2(X)$ in the case in which Γ is continuous, but not discrete as in the case discussed above. This method can be applied with suitable modifications also to the case considered here. This method leads us to a series of theorems concerning the structure of the representations and to the Zeta-function of a homogeneous space. But the general idea of the horysphere method is not sufficient. It is necessary to study the operators $\int F(g)T_g dg$ in detail, where $F(g)$ is a finite function. This part of our work reminds one strongly of the theory of the S-matrix in quantum mechanics. The analogy is intrinsic. For example the Zeta-functions of a homogeneous space are quite analogous to the Heisenberg S-matrix.

In the second part of the report we consider analogous questions concerning the structure of representations in $X = G/\Gamma$ in the case of a semisimple Lie group over a finite field (in Chevalley-Dickson sense). So it is necessary to describe the representations of Lie groups over finite fields. These questions could serve as the theme of a separate report and we are forced to give only a short account of them. We obtain here a interesting system of Zeta-functions for a homogeneous space above a finite field.

The author has chosen as the theme of his report developments that are in the very earliest stages.

We hope that the many interesting problems and relationships which arise will compensate for the unavoidable lack of a description clear and comprehensible in every detail.

We hope that a reasonable bourbakisation of all facts given in this report and of other facts and problems of the theory of representations will lead to the creation of a domain of algebraic functional analysis in which these facts will be the main examples.

2. The case of a compact homogeneous space

If G is a semisimple Lie group, Γ a discrete subgroup and X the space G/Γ of left cosets, then evidently to each element $g \in G$ there corresponds a movement in X transforming x to xg. Denote by $L_2(X)$ the set of all functions on X with integrable square. To each element $g \in G$ we make correspond the unitary shift operator T_g in $L_2(X): T_g f(x) = f(xg)$. The operators T_g form a representation of the group G.

The main problem is to decompose this representation into irreducible ones.

The operators $T_\varphi = \int_G \varphi(g) T_g dg$ where $\varphi(g)$ is a finite function on G play an important role in the theory of infinite-dimensional representations. In our situation the use of operators T_φ is based upon the fact that for every continuous finite function $\varphi(g)$ the operator T_φ is an integral operator in X with the kernel

$$K(x_1, x_2) = \sum_{\gamma \in \Gamma} \varphi(g_1^{-1} \gamma g_2), \tag{1}$$

where g_1 and g_2 are representatives of cosets x_1 and $x_2 \in X$. It is easy to verify that the series for the function $K(x_1, x_2)$ converges uniformly in every compact domain. Thus if X is compact the operator T_φ is completely continuous for every finite continuous function $\varphi(g)$.

It is easy to prove the following general proposition.

If the unitary representation $g \to T_g$ of Lie group G in a space H is such that operator T_φ is completely continuous for every finite continuous function $\varphi(g)$, then H can be decomposed into a countable sum of irreducible representations (unitary) of the group G, where the multiplicity of each irreducible representation is finite.

It follows at once from this proposition that when X is compact the representation in $L_2(X)$ is decomposable into a countable sum of irreducible unitary representations of the group G. This fact does not exhaust all information about the irreducible representations contained in $L_2(X)$ which can be obtained by the use of the operators T_φ. There is a formula which gives in a reasonable sense a complete description of all irreducible unitary representations contained in $L_2(X)$.

Let $H_1, H_2, \ldots$ be irreducible non-equivalent representations contained in $L_2(X)$; $N_1, N_2, \ldots,$ their multiplicities. Let $\pi_k(g)$ be the character of the irreducible representation H_k. The existence of such characters was proved by Godement and Harish Chandra. Then the following important formula

$$\sum_{m=1}^{\infty} \int_{F_m} \varphi(g^{-1}\gamma_m g)\, dg = \sum_{k=1}^{\infty} N_k \int \varphi(g)\, \pi_k(g)\, dg \tag{2}$$

is valid, where $\gamma_1, \gamma_2, \ldots$ is a sequence of mutually non conjugate elements of the group Γ and F_m is a fundamental domain for the centralizer Γ_m of the element γ_m. This formula makes it often feasible to find out what representations enter in $L_2(X)$, and with what multiplicities. A particular case of this formula (namely the case in which the functions $\varphi(g)$ are invariant under U—the maximal compact subgroup of the semisimple group) can be essentially reduced by the use of the so-called duality theorem [2] to the previously known formula of Selberg [4].

Formula (2) may be used to investigate the asymptotic distribution of "numbers" of the representations, occurring in $L_2(X)$. The leading term of the asymptotic expansion will be obtained if we choose the function $\varphi(g)$ to be concentrated in a decreasing sequence of neighbourhoods of the unit of the group. Now we give following example.

Consider the asymptotic behaviour not of all representations but only of the so-called representations of class I. By definition these representations are the representations entering in the decomposition of $L_2(G/U)$ where U is a maximal compact subgroup of G.

These representations are given in the following manner. Let $\mathfrak{A}$ be Cartan subalgebra of the symmetric space G/U and let $\mathfrak{A}^+$ be the cone of the positive vectors in $\mathfrak{A}$ (i.e. vectors for which $(\alpha, \alpha) \geqslant 0$ for all positive roots α). Then every representation of class I is uniquely determined by some vector in $\mathfrak{A}^+$. If now X is a compact homogeneous space G/Γ then the representations of class I entering into $L_2(X)$ form a countable set of points in $\mathfrak{A}^+$. Their asymptotic distribution is given by the following formula. Denote by α the positive roots of the symmetric space G/U and by v_α the multiplicities of the roots of G/U. Then the formula

$$N(B_n) \sim C_G C_\Gamma \int_{B_n} \prod_{\alpha > 0} (\varrho, \alpha)^{v_\alpha} d\varrho \tag{3}$$

is valid, where B_n runs through an increasing sequence of subregions in $\mathfrak{A}^+$; C_G is a constant depending only on G; C_Γ is the volume of space $X = G/\Gamma$ and $N(B_n)$ is the number of irreducible representations entering in $L_2(X)$ indices of which belong to B_n. The proof is based on some results of F. I. Karpelevich and S. G. Gindikin.

Apparently a similar formula is valid for the other types of irreducible representations.

Since for the group of 2×2-matrices as was said, the "numbers" of the representations are invariants of the Riemann surface, the terms of the asymptotic expansion are also such. It would be very interesting if a few leading terms of this expansion were to play the role of moduli. The next terms of asymptotics demand apparently more averaging than the mere evaluation of $N(B_n)$.

Classical automorphic forms enter here in natural way. Among the representations of the group G there are isolated ones, i.e. those occurring in isolated form already in $L_2(G)$. Among the isolated ones there are representations which are realizable by analytic functions [11, 12, 13]. These isolated representations, if they do occur in the decomposition of $L_2(X)$, are connected with automorphic forms in natural way. It is interesting to observe that there are still other isolated representations (Graev [6]). For every isolated representation one can deduce from the spur formula an explicit formula for the multiplicity N_k. In the case of classical automorphic forms such a formula was obtained by Hirzebruch [14] who used the Riemann–Roch theorem and by Selberg [4] who used a method close to ours.

3. Horyspheres and regular subgroups

The method of horyspheres, which has been elaborated elsewhere, mainly in the paper [1], is effective when X is not compact. In general the method is as follows. Let X be a homogeneous space acted upon by a group G. With the space X one can associate a space Ω whose elements are taken to be surfaces in X which we shall call horyspheres (the definition of horysphere will be given later). To each function defined on X we make correspond its integrals over horyspheres, then the representation in the space of functions on X maps homomorphically onto a representation in the space of functions on Ω. As a consequence the problem of decomposing the representation in the space of functions on X is reduced to the following two problems: (1) to find the kernel of the homomorphism; (2) to decompose the obtained set of functions on Ω into irreducible representations. In some important cases, discussed in [1], the kernel is zero and induced representation on Ω can be given a simple description. But in our case the kernel of homomorphism is not zero and the description is not trivial. Nevertheless the method of horyspheres is very productive in this situation.

In particular it makes it possible to separate quite effectively the series of representations occurring continuously in $L_2(X)$ from the series occurring discretely, or in other words to separate the discrete spectrum from the continuous one.

Let X be an arbitrary homogeneous space; then the set of horyspheres in X is not a homogeneous space itself and has not in general a Hausdorff structure. The structure of the space the points of which are transitive components in the space of horyspheres (i.e. the sets of horyspheres which can be carried one into another by the movements of the group) plays a fundamental role in the description of the spectrum of irreducible representations in $L_2(X)$.

Now we proceed to give the exact definitions. Let G be a real semisimple Lie group and $g(t)$ a one-parameter subgroup. The set Z consisting of all $z \in G$ such that

$$\lim_{t\to\infty} g(-t)\, z g(t) = 1$$

is called the horyspherical subgroup associated with the subgroup $g(t)$. For example if G is the group of real 2×2-matrices then every horyspheric subgroup is conjugate to the subgroup of all matrices of the form $\begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix}$.

If G is the group of real matrices of order $\mathbf{n}$, then there exist as many non-conjugate horyspherical subgroups as there are different partitions of the the number $\mathbf{n}:\mathbf{n} = k_1 + \ldots + k_s$ with positive integers k. (Partitions that are distinguished by the order are to be considered as the different ones.)

Now we proceed to define the horyspheres. Let X be a homogeneous space of a group G.

The orbits of horyspherical groups will be called horyspheres in X. The horysphere will be called compact if the set of points of which it consists, is compact.

Now we proceed to describe the class of discrete subgroups Γ which we shall be concerned with. First of all here belong these for which the volume of G/Γ is finite. It is quite possible that all such subgroups satisfy the assumptions formulated below. For the groups of second order this follows from the results of Siegel. It would be very interesting to prove the assertion in the general case.

Now we introduce the following definition. A set $Y \subset X$ will be called a cylindrical set, if it can be split into mutually non-intersecting compact horyspheres which can be carried one into another by movements. Thus in a cylindrical set all the horyspheres are of the form $x_0 Zg$ where Z is some fixed horyspherical subgroup.

Now we formulate the following fundamental definition.

A discrete subgroup Γ of the semisimple group G will be called regular if the factorspace G/Γ has a finite covering by regular bounded cylindrical sets and the intersection of each pair of them is compact (the definition of regular bounded cylindrical sets is given below).

It is easy to verify that the factor-space G/Γ for each regular discrete subgroup Γ of semisimple group G has finite volume.

We shall observe finally that except for the group of real 2×2-matrices all presently known examples of irreducible discrete subgroups of semisimple Lie groups such that the factor space has finite volume are arithmetical groups that are constructed by the well-known construction of Borel.

We shall call linear algebraic group G every group consisting of all complex $n \times n$-matrices the elements of which satisfy given polynomial relations. In the following we shall assume that the coefficients of these polynomials are rational numbers. We denote by G_z the set of all matrices in G whose elements are integers and whose determinant is equal to 1. In a similar way G_R is the set of matrices in G whose elements are real numbers. It is easy to see that G_z is a discrete subgroup of the group G_R.

A. Borel and Harish Chandra [8] have proved that the volume of the factor-space G_R/G_z is finite, if G_R is semisimple group.

Apparently it can be proved by their methods that G_z is a discrete regular subgroup of G_R.

Now we proceed to define the regular bounded cylindrical sets. It is not difficult to see that for every cylindrical set Y there exists a horyspherical subgroup Z and a set $S \subset G$, whose image in G/Γ is Y, with the following properties:

(1) For every $g \in S$ and $z \in Z$ there exists $\delta \in \Delta = \Gamma \cap Z$ such that $\delta z g \in S$.

(2) If $g_1, g_2 \in S$ and $g_1 g_2^{-1} \in \Gamma$ then $g_1 g_2^{-1} \in \Delta$.

The first condition means that Y consists of horyspheres. The second that these horyspheres do not intersect.

If the set Y is compact then one can choose S so that it will have also the following properties:

(3) There exists a neighbourhood of unity U_2 such that if $g_1^{-1} \gamma g_2 \in U$, where g_1, $g_2 \in S$, then $\gamma \in \Delta$.

(This condition is more strong than condition 2.)

(4) For every $z \in Z$ there exists a compact neighbourhood of unity U_z such that $g^{-1} z g \in U_z$ for any $g \in S$.

Condition (4) plays in this theory the fundamental role.

We shall agree to call bounded any cylindrical set Y for which there exist sets S and Z with the properties (1)–(4). Bounded cylindrical sets are in general non-compact and this seems to be the geometrical reason for the existence in semisimple groups of discrete subgroups such that the factor-space G/Γ has finite volume and at the same time is non-compact.

Let Y be a bounded cylindrical set. It can be shown that all elements $g \in S$ are representable in the form $g = zat$, where a belongs to a certain accompanying subgroup A of Z,[1] and t belongs to a certain compact set T in G. We shall agree to call a normal subgroup $\tilde{Z}$ of Z allowable, or simply allowable, if the intersection of the Lie algebra of the group $\tilde{Z}$ with every root subspace is either vacuous or contains the root subspace.

A bounded cylindrical set will be called regular if

(5) For each allowable subgroup $\tilde{Z}$ of Z (including the group Z itself) the factor-space $\tilde{Z}/(\tilde{Z} \cap \Gamma)$ is compact.

Thus finally the regular bounded cylindrical sets are characterized by the existence of the sets S, Z, A for which the conditions (1)–(5) hold.

4. The separation of the continuous spectrum of representation from the discrete one

We shall suppose as formerly that we have a homogeneous space $X = G/\Gamma$, where the discrete group Γ is regular and G is a real semisimple group.

By the use of the horysphere method we can decompose the space $L_2(X)$ into a direct orthogonal sum of two spaces. One of them, which we denote $L_2^0(X)$, is decomposable into a discrete direct sum of irreducible representations. The other is decomposable into representations of continuous spectra. However there may be representations which enter into the second space discretely, the so-called representations of the complementary series. They get into the second space because they are, so to speak, the analytical continuation of the continuous spectrum involved. One more justification of this fact is that they hit exactly at the singular points of Zeta-functions corresponding to the given continuous series. It is very much like the complementary discrete spectrum of quantum mechanics that hits exactly at the zeros of the S-matrix.

The decomposition into the direct sum is carried out in the following way. To every function $f(x) \in L_2(X)$ we make correspond its integrals over *compact* horyspheres and we denote by $L_2^0(X)$ the space of functions which have their integral over any compact horysphere equal to zero; $L_2'(X)$ is the orthogonal complement of $L_2^0(X)$.

Then *the space $L_2^0(X)$ can be decomposed into a countable number of irreducible representations*. The proof is based on a study of asymptotic properties of the kernel of the integral operator $\int \varphi(g) T_g dg$ at the regular cylindrical sets.

---

(1) A commutative subgroup A is called accompanying if each element of A is semisimple and if A is generated by oneparametrical subgroups $g_1(t), \ldots, g_n(t)$ to each of it Z is associated.

5. Functions in the space of horyspheres

We divide the set of compact horyspheres into transitive families Ω_i (i.e. sets of horyspheres which can be carried one into another by movements). When Γ is regular there will be only a finite number of such families. The families Ω_i are partially ordered in a natural way, namely $\Omega_i < \Omega_j$ if each horysphere of Ω_i can be split into horyspheres of Ω_j.

Now we take one of the transitive horysphere families Ω_i and make correspond to each function $f(x) \in L_2(X)$ its integrals over horyspheres of Ω_i. We obtain a function $\check{f}(\omega)$ defined on Ω_i. The inner product defined in $L_2(X)$ is carried in a natural way into the set of these functions. Thus in $L_2(\Omega_i)$ there is defined a quadratic form. To this form there corresponds an operator M defined by

$$\int_{\Omega_i} \check{f}\bar{\check{\varphi}}\, d\omega = [\check{f}, M\varphi],$$

where $[\check{f}_1, \check{f}_2]$ is the inner product inherited from $L_2(X)$. This operator plays a fundamental role. It is permutable with the movements in Ω_i. As a consequence this operator, by the decomposition of representation in $L_2(\Omega_i)$ into irreducible ones, will be, in every system of representations equivalent to the given one, a matrix whose order coincides with the number of equivalent representations. In the general case this number is less than or equal to the order of Weyl group. Thus after the decomposition there will arise a matrix of order equal to the order of Weyl group. (For a group of $\mathbf{n} \times \mathbf{n}$-matrices this order is equal to $\mathbf{n}!$). This matrix depending on the "number" of the representation is by definition the Zeta-function of the space X. These functions, associated with the "number" of representation, are closely connected with such functions as the Riemann Zeta-function and its generalizations. It is not excluded that a deeper development of this theory will throw light on the blank spaces in the theory of the classical Zeta-function. The results just formulated give us a chance to describe the representations involved. It could be done in detail but lack of time prevents us from doing so. We merely point out that the spectrum of the representation of the component, associated with any Ω_i, has Lebesgue type and multiplicity 1. The indices of the representations fill up several linear subspaces of Cartan algebra. The description of the allowable subspaces is carried out by induction. The dimension of these spaces is easy to calculate. It is equal to the dimension of a group which is associated with Ω_i and which consists of all the homeomorphisms permutable with the movements of G in Ω_i.

6. Example: the matrix group G and the subgroup Γ of matrices with integer elements

As an example we take the subgroup Γ of integer elements of the real $\mathbf{n} \times \mathbf{n}$-matrix group. Each transitive family of compact horyspheres is defined by a partition $\mathbf{n} = \mathbf{n}_1 + \ldots + \mathbf{n}_k$. The maximal family Ω_1 is defined by the partition $\mathbf{n} = 1 + \ldots + 1$ and consists of orbits of the triangular group. Every irreducible representation of G can be defined by an index which is a vector in a space of dimension $n-1$. The set of indices of all the irreducible representations involved in the decomposition of $L_2(X)$, where $X=G/\Gamma$, is called the spectrum of $L_2(X)$. To each Ω_i there corresponds a spectrum consisting of not more than a countable number of linear subspaces. To Ω_1 itself there corresponds a spectrum of multiplicity 1 filling up the whole space. The spectrum of $L_2(X)$ is the union of the subspaces mentioned above and of a countable number of points.

Now we shall exhibit the Zeta-function corresponding to the homogeneous space G/Γ. We shall restrict ourselves to the Zeta-function of highest dimension. The Weyl group in this case is the symmetric group of degree n. Thus the Zeta-function is a matrix of order $n!$. We shall write down the first row because the rank of the matrix is equal to 1 and all other rows are easily expressed in terms of the first. (Each representation associated with Ω_1 enters in $L_2(X)$ only one time!) So we have: if σ is a permutation of the symmetric group and $x=(x_1, \ldots, x_n)$ is the "number" of a representation $(x_1+\ldots+x_n=0)$, then

$$\zeta_\sigma(x) = \prod_{\substack{i>j \\ \sigma(i)<\sigma(j)}} B\left(\frac{1}{2}, \frac{x_i-x_j}{2}\right) \frac{\zeta(x_i-x_j+1)}{\zeta(x_i-x_j)}, \tag{5}$$

where $\sigma=(\sigma_1, \ldots, \sigma_n)$.

It is not difficult to write this formula in standard root notation. Written in this form it can be generalized.

The functional equation for the Zeta-function which is a consequence of the general theory of representations, is of the form

$$\zeta_{\sigma_1\sigma_2}(x) = \zeta_{\sigma_2}(x)\,\zeta_{\sigma_1}(\sigma_2 x). \tag{6}$$

7. The representations of finite Chevalley groups

The success of the theory of representations depends after all on a happy construction of irreducible representations. In the case of the complex semisimple Lie groups a simple construction of irreducible representations was given (in 1947–1950 by M. A. Naimark and the author [9] (later we shall speak about the construction). However, if we go to other classes of semisimple groups—the real semisimple groups, the matrix groups over finite fields etc.—the construction of Naimark and author does not yield all the representations but only a small part of them.

Now we shall discuss the classification of the irreducible representations of semisimple matrix groups over a finite field. We shall deal with the groups which were considered in the well-known papers by Dickson [15] and Chevalley [16]. Included here for example are the unimodular matrix groups over finite fields, the matrix groups leaving invariant some quadratic form etc.

Already in the last century in one of the first papers on the theory of representations Frobenius found the characters of the unimodular group of 2×2-matrices over a finite field. Later (in 1928) Hecke gave the construction for half of the irreducible representations of this group. The construction for the other groups was absent.

Now we proceed to construct the irreducible representations. In the case of a complex semisimple Lie group G the construction of the principal series of irreducible representations is as follows. We consider the homogeneous space G/Z of the right cosets of G with respect to a maximal nilpotent subgroup Z. We call this space the principal affine space associated with G. The set of homeomorphisms of G/Z permutable with the movements form a commutative group which we shall call the homothety group. A function defined on G/Γ will be called homogeneous if it is multiplied by a constant under a homothety.

The irreducible representations of the principal series are realized in spaces of homogeneous functions on G/Z; the operator of representation is defined as the shift operator. We have already noted that this construction when transferred to the simple Dickson–Chevalley groups over a finite field, will give us only a small part of all irreducible representations. Now we shall explain how to construct *all* irreducible representations. Again we consider a representation in the space of functions on G/Z where Z is a maximal nilpotent subgroup. An operator of the representation will be defined not as mere shift but as a shift operator multiplied by a fixed function depending on a point of the space and on an element of the group

$$T_\chi(g)f(z) = f(zg)\,\alpha(z,g).$$

It is easy to show that the function $\alpha(z,g)$ is defined essentially by a one-dimensional representation $\chi(z)$ of the group Z. For brevity we shall call the $\chi(z)$ simply characters. It can be shown that by the decomposition of these representations we shall obtain all the irreducible representations. This follows from the fact that in every representation of G there exists a vector which is an eigenvector with respect to the element of Z. Now we introduce a partial ordering into the set of characters: let $\chi_1 < \chi_2$ if from $\chi_2(s) = 1$ it follows $\chi_1(s) = 1$ for each $s \in Z$ such that $s = e^{E_\alpha t}$ where α is a root of G; the maximal characters $\chi(s)$ will be called the characters of general position. It turns out that the representations $T^\chi(g)$ corresponding to the characters χ of general position do contain each irreducible representation of G not more than one time. The irreducible representations entering into these $T_\chi(g)$ we shall call the principal representations of G, all others will be called degenerate. The number of all irreducible representations of G is a polynomial of k, where k is the order of field under consideration. It turns out that the number of degenerate representation is a polynomial of lower degree. In this sense we can say that the principal representations are almost all irreducible representations of G. It can be shown that the dimensions of the principal representations of G can be expressed as polynomials in k of degree N, where N is the dimension of group Z, and that the dimensions of the representations of the degenerate series are expressed as polynomial of lower degree. As an illustration we give a formula for the dimensions of the different principal representations of the unimodular matrix group of the $\mathbf{n}$th order. The principal representations are split into several series. Each series is defined by a partition $\mathbf{n} = \mathbf{n}_1 + \ldots + \mathbf{n}_l$ of $\mathbf{n}$ into sum of positive integers. The dimensions of the representations of this series are equal to(1)

(1) (Added in proof.): There is a very interesting paper of I. A. Green. *Trans. Amer. Math. Soc.* (1955).

$$\frac{(k-1)(k^2-1)\dots(k^l-1)}{(k^{n_1}-1)\dots(k^{n_e}-1)}, \tag{7}$$

where k is order of the field under consideration. There are exceptions to this formula, the so-called singular representations. The number of these divided by the total number of representations goes to zero with increasing k.

Now we shall explain how to split into series the principal representations of the arbitrary group G. We combine into one series the principal representations contained in the same $T_\chi(g)$ and with the same multiplicities. It seems that the representations belonging to same series are "contrived in the same manner"; their dimensions coincide; formulae for the characters are written in the same way; when realized they give rise to the same special functions.

Since the principal representations enter in $T_\chi(g)$ of general position with the multiplicity 1, the realization of the representation is comparatively easy. It is sufficient to consider the ring of operators, permutable with the representation $T_\chi(g)$, where χ is a character of general position. This ring is commutative (!), and so it decomposes into a direct sum of complex fields. Each summand defines one of the principal representations of G. It should be noted that the procedure leads to interesting classes of special functions on G. Thus in the case of the unimodular group of second order the one-dimensional components of the ring of operators permutable with $T_\chi(g)$ can be expressed as certain sums known as the Kloosterman sums. It is remarkable that the summation is carried sometimes over a certain "contour" in a quadratic extension of the field. Depending on the set of summation we obtain representations of one or another principal series.

8. Zeta-functions associated with homogeneous space over a finite field

The representations of the group G over a finite field being constructed we have the possibility to define the Zeta-function. Here we restrict ourselves to the Zeta-function associated with principal representations with $\chi \equiv 1$; we shall not carry out the "analytic continuation" to the other series. Let G be a Dickson-Chevalley group over a finite field, let Γ be any of its subgroups and let Z be a maximal nilpotent subgroup. In order to construct the Zeta-function we consider the functions $\varphi(g)$ constant on the cosets of G with respect to Z, i.e. the functions satisfying the equation $\varphi(\zeta g) = \varphi(g)\chi(\zeta)$. The operator M acting in the space of these functions is given by

$$M\varphi = \psi(g) = \sum_{\substack{\gamma \in \Gamma \\ \zeta \in Z}} \varphi(\gamma \zeta g)\chi'(\zeta). \tag{8}$$

Then $\psi(\zeta g) = \psi(g)\chi(\zeta)$. The operator M is permutable with the movement T_{g_0} of the group: $T_{g_0}\varphi(g) = \varphi(gg_0)$. So after the decomposition of the representation into irreducible ones it will be given by a matrix which depends on the "number" of the representation and has order equal to the order of Weyl group. By the definition the matrix is the Zeta-function of group G respect to the subgroup Γ. It depends on the index π of representation which is a multiplicative character of the Cartan subgroup. We shall give the explicit form of this function. For this purpose we construct the function $k(g)$ equal for each g to the number of ways in which g can be written in the form $g = \zeta_1 \gamma \zeta_2$, where $\gamma \in \Gamma$ and ζ_1 and ζ_2 are in Z. Each g is of the form

$g=\zeta'\delta\, s\, \zeta''$, where δ is in the Cartan subgroup and s is in the Weyl group; therefore $k(g)$ depends only on δ and s: $k(g)=k(\delta,s)$. The Fourier transform of $k(\delta,s)$ with respect to δ is now the Zeta-function:

$$\zeta_s(\pi)=\sum_{\delta} k(\delta,\, s)\, \pi(\delta).$$

We shall not write down the form of this function for the representations of other principal series.

9. Reprsentations associated with groups over other fields

We shall not consider the case of groups associated with the field of p-adic numbers which were studied by Mautner and Bruhat. We hope that the indicated constructions are applicable to this case and make it possible to describe all representations.

Very interesting is the problem of studying the representations of the Chevalley groups over the field of algebraic functions over the field of complex numbers. Here it is not evident how to define a representation. Apparently the following definition is the most natural. We require T_g to be an operator defined every where except a manifold of lower dimension. We require T_g to depend continuously on g and to satisfy the equation $T_{g_1}T_{g_2}=T_{g_1 g_2}$ everywhere except a manifold of lower dimension. These groups are examples of infinite-dimensional groups for which the problem of constructing their irreducible representations is quite real and interesting. Also in this case the constructions indicated above give many important types of representations. It is possible, however, to exhibit the examples showing that the construction does not exhaust all degenerate representations.

REFERENCES

[1]. Гельфанд, И. М. & Граев, И. М. И., *Труды Моск. Матем. общ.*, 8 (1959), 323–390.

[2]. Гельфанд, И. М. & Пятецкнй-Шациро, И. М. И., *Успехи матем. наук*, XIV 2 (1959), 172–194.

[3]. Гельфанд, И. М. & Фомин, И. С. В., *Успехи матем. наук*, 7:1 (47) (1952), 118–137.

[4]. Selberg, A., *Tata Inst. of Fund. Res., Bombay*, 1960, 147–164.

[5]. Selberg, A., *J. Ind. Math. Soc.*, 20 (1956), 47–87.

[6]. Граев, М. И. *Докл. А. Н. С.С.С.Р.*, 113 (1957), 966–969.

[7]. Selberg, A., *Analytic Functions*, vol. 2, 1960.

[8]. Borel, A. & Harish-Chandra, *Ann. Math.*, 75 3 (1962).

[9]. Гельфанд, И. М. & Наймарк, И. М. А., *Труды матем. ин-та А.Н. С.С.С.Р.*, 36 (1950), 1–288).

[10]. Harish-Chandra, *Proc. Nat. Acad. Sci. USA.*, 45 (1959), 570–573.

[11]. Bargman, *Ann. Math.*, 48 (1947), 568–640.

[12]. Гельфанд, И. М. & Граев, И. М. И., *Изв. А.Н. С.С.С.Р, сер. матем.*, 17 (1953), 189–238.

[13]. Граев, М. И., *ТрудыМоск. матем общ.*, VII (1959).

[14]. Hirzebruch, *Analytic Functions*, vol. 2, 1960.

[15]. Dickson, L., *Linear Groups*.

[16]. Chevalley, C., *Tôhoku Math. J.*, 7 (1955), 14–66.