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Micro-Local Analysis

Masaki Kashiwara

We mean by micro-local analysis the analysis of functions and systems of differential equations on the cotangent bundle. The role of cotangent bundles in analysis has been recognized for a long time, but the formulation which we treat here started from Sato's introduction of microfunctions around 1970. The book of Sato–Kawai–Kashiwara [10] concerned was the systematic work on micro-local analysis. One of the most remarkable results in that book is the discovery of three types of micro-differential equations: de Rham type $\partial u/\partial x_1=0$, Cauchy–Riemann type $(\partial/\partial x_1+\sqrt{-1}\,\partial/\partial x_2)u=0$ and Lewy–Mizohata type $(\partial/\partial x_1+\sqrt{-1}x_1\,\partial/\partial x_2)u=0$. Any system of differential equations (or more generally micro-differential equations) is micro-locally equivalent to the mixture of these three types at a generic point.

They also proved that the characteristic variety of any system of differential equations is involutive. (See also [9].)

Let $P(x, D)=\sum_\alpha a_\alpha(x)D^\alpha$ be a differential operator defined on an open subset X of C^n. Here, $\alpha=(\alpha_1, \ldots, \alpha_n)$ is an n-tuple of non-negative integers, $|\alpha|=\alpha_1+\ldots+\alpha_n$ and $D^\alpha=\partial^{|\alpha|}/\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}$. The largest m such that $a_\alpha(x)\neq 0$ for some α with $|\alpha|=m$ is called the order of $P(x, D)$. The function $\sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ is called the principal symbol of $P(x, D)$. Here, $(x, \xi)=(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ is the coordinate system of the cotangent bundle T^*X of X and ξ^α means $\xi_1^{\alpha_1}, \ldots, \xi_n^{\alpha_n}$.

Let us consider a system of differential equations

$$\mathfrak{M} : P_1(x, D)u = \ldots = P_N(x, D)u = 0.$$

The common zeroes of the principal symbols of linear combinations $\sum_j A_jP_j$ with differential operators A_j as coefficients is called the characteristic variety of $\mathfrak{M}$.

Let c denote the codimension of the characteristic variety in the cotangent bundle. The characteristic variety being always involutive, the codimension c is equal to or greater than n. The number $n-c$ indicates, roughly speaking, the number of variables on which solutions of $\mathfrak{M}$ depend. For example, in the case of the system of differential equations $\partial u/\partial x_1=\ldots=\partial u/\partial x_c=0$, a solution $u(x)$ depends on the $(n-c)$ variables $x_{c+1}, \ldots, x_n$. Hence, when $n=c$, we can expect that the space of solutions of $\mathfrak{M}$ is of finite dimension. In fact, this is a case [3]. We say that $\mathfrak{M}$ is holonomic if $c=n$. This notion is a generalization of the notion of ordinary differential equations into several variables. As we succeeded to study the properties of functions with one variable through their ordinary differential equations, we can expect that the study of holonomic systems of differential equations gives many properties of their solutions.

Let P be an ordinary differential operator of the form $\sum_{j=0}^{m} a_j(t)(t\,d/dt)^j$ with $a_m(0)\neq 0$. In this case, $t=0$ is called the regular singularity of $Pu=0$. Then, any solution of $Pu=0$ has the form:

$$u(t)=\sum_{j,\nu}\varphi_{j,\nu}(t)\,t^{\lambda_j}(\log t)^{\nu}$$

with holomorphic functions $\varphi_{j,\nu}(t)$ defined on a neighborhood of $t=0$. Moreover, λ_j are the solutions of the equation

$$\sum a_j(0)\lambda^j=0.$$

This phenomenon can be generalized to the case of holonomic systems of partial differential equations.

First, we can introduce the notion of regular singularities for holonomic systems (see § 4). Let $\mathfrak{M}: P_1(x,D)u=\ldots=P_N(x,D)u=0$ be a holonomic system of differential equations with regular singularities. Then, we can find a nonzero polynomial $b(s)$ of degree m and a linear combination $Q=\sum A_jP_j$ such that Q is written in the form $b(x_1D_1)+x_1\sum_{j+|\alpha|\leq m} g_{j,\alpha}(x)(x_1D_1)^jD'^{\alpha}$, where $\alpha=(\alpha_2, \ldots, \alpha_n)$ is an $(n-1)$-tuple of nonnegative integers, $|\alpha|=\alpha_2+\ldots+\alpha_n$ and $D'^{\alpha}=\partial^{|\alpha|}/\partial x_2^{\alpha_2}\ldots\partial x_n^{\alpha_n}$. This implies that any solution of $\mathfrak{M}$ satisfies $Qu=0$. In the case of ordinary differential equation with regular singularities,

$$\sum_j a_j(t)\left(t\frac{d}{dt}\right)^j=\sum_j a_j(0)\left(t\frac{d}{dt}\right)^j$$

$$+t\sum_j t^{-1}(a_j(t)-a_j(0))\left(t\frac{d}{dt}\right)^j.$$

The hyperfunction solution $u(x)$ of the system $\mathfrak{M}$ has an "asymptotic expansion"

$$u(x)=\sum_{j=1}^{r}\sum_{\nu=0}^{N}\sum_{k=0}^{\infty}v_{j,\nu,k}(x')x_1^{\lambda_j+k}(\log x_1)^{\nu}.$$

Here, $v_{j,\nu,k}(x')$ are hyperfunctions on the variables $x'=(x_2, \ldots, x_n)$. λ_j satisfies

$b(\lambda_j)=0$. Moreover, the hyperfunctions $v_{j,\nu,k}(x')$ satisfy holonomic systems of differential equations with regular singularities.

1. Hyperfunctions and microfunctions [10]. Hyperfunctions are generalized functions obtained by sum of "ideal boundary values" of holomorphic functions, and microfunctions are "singular parts" of hyperfunctions. First, we shall remember the definitions of hyperfunctions and microfunctions.

1.1. *Tangent cone.* Let A and B be two subsets of a differential manifold X. For a point p of X, choosing a local coordinate system around p, we define the subset $C_p(A; B)$ of the tangent vector space T_pX as the totality of $\lim a_n(x_n-y_n)$, where $\{x_n\}$ (resp. $\{y_n\}$) is a sequence of points in A (resp. B) which converges p and $\{a_n\}$ is a sequence of positive numbers.

We define $C(A; B)$ as the union of $C_p(A; B)$.

If B is a closed submanifold of X, then $C(A; B)$ is invariant by translations of tangent vectors of B; we denote by $C_B(A)$ the subset $C(A; B)/TB$ of the normal bundle T_BX of B.

1.2. Let M be an open subset of $\boldsymbol{R}^n$ and let X be a neighborhood of M in $\boldsymbol{C}^n$. The tangent bundle TX (resp. TM) of X (resp. M) is identified with $X\times\boldsymbol{C}^n$ (resp. $M\times\boldsymbol{R}^n$), and the normal bundle T_MX of M is identified with $\sqrt{-1}TM= M\times\sqrt{-1}\boldsymbol{R}^n$. The conormal bundle T_M^*X of M is identified with $\sqrt{-1}\,T^*M$. We shall denote by τ and π the projection from $\sqrt{-1}\,TM$ and $\sqrt{-1}\,T^*M$ onto M, respectively. A subset Ω of $\sqrt{-1}\,TM$ (resp. $\sqrt{-1}\,T^*M$) is called convex or cone if $\Omega\cap\tau^{-1}(p)$ (resp. $\Omega\cap\pi^{-1}(p)$) is convex or cone. An open set U of X is called an *infinitesimal neighborhood* of a subset Ω of $\sqrt{-1}\,TM$ if $C_M(X-U)\cap\Omega=\emptyset$.

We denote by $\tilde{\mathfrak{A}}(\Omega)$ the inductive limit of the space $\mathcal{O}(U)$ of holomorphic functions defined on U, where U runs over the set of infinitesimal neighborhoods of Ω.

Let V be an open subset of M. Let $\mathscr{F}(V)$ be the set of families $\{\Omega_i, \varphi_i\}_{i\in I}$, where Ω_i is a convex open cone of $\sqrt{-1}\,TM$ such that $\tau(\Omega_i)=V$, φ_i is an element of $\tilde{\mathfrak{A}}(\Omega_i)$ and I is a finite index set. We say that two members $\{\Omega_i, \varphi_i\}_{i\in I}$ and $\{\Omega'_j, \varphi'_j\}_{j\in J}$ of $\mathscr{F}(V)$ are equivalent if there are open convex cones $\Omega_{i,j}$ and $\varphi_{i,j}\in\tilde{\mathfrak{A}}(\Omega_{i,j})$ $(i\in I, j\in J)$ satisfying the conditions: $\Omega_{i,j}\supset\Omega_i\cup\Omega'_j$, $\varphi_i=\sum_{j\in J}\varphi_{i,j}|_{\Omega_i}$ and $\varphi'_j=\sum_{i\in I}\varphi_{i,j}|\Omega'_j$.

We denote by $\mathscr{B}(V)$ the set of equivalence classes of $\mathscr{F}(V)$, and the element of $\mathscr{B}(V)$ is called a *hyperfunction* defined on V. For an open convex cone Ω and $\varphi\in\tilde{\mathfrak{A}}(\Omega)$, the hyperfunction corresponding to $\{\Omega, \varphi\}$ is denoted by $b_\Omega(\varphi)$ (or, if φ is defined on an infinitesimal neighborhood U of Ω, denoted by $b_U(\varphi)$, or simply $b(\varphi)$), and called the *boundary value* of φ. If $b_\Omega(\varphi)=0$, then $\varphi=0$. $\mathscr{B}(V)$ has clearly a structure of vector space, and the equivalence class of $\{\Omega_i, \varphi_i\}_{i\in I}$ equals $\sum_i b_{\Omega_i}(\varphi_i)$.

1.3. Let $(x_0, \sqrt{-1}\,\xi_0)$ be a point of the conormal bundle $\sqrt{-1}\,T^*M=M\times \sqrt{-1}\,\boldsymbol{R}^n$ of M, and let u be a hyperfunction defined in a neighborhood of x.

We say that u is *micro-analytic* if u can be expressed in the form: $u=\sum_i b_{\Omega_i}(\varphi_i)$ with open convex cones Ω_i of $\sqrt{-1}\,TM$ and $\varphi_i\in\tilde{\mathfrak{A}}(\Omega_i)$ such that $\tau^{-1}(p)\cap\Omega_i\subset\{\sqrt{-1}\,v\in\tau^{-1}(p)=\sqrt{-1}\,\boldsymbol{R}^n;\ \langle\sqrt{-1}\,v, \sqrt{-1}\,\xi_0\rangle=-\langle v, \xi_0\rangle>0\}$. We denote by $\mathrm{SS}(u)$ the set of points of $\sqrt{-1}\,T^*M$ where u is not micro-analytic, and call it the *singular spectrum* of u.

A real analytic function u defined on V is considered as hyperfunction on V; in fact, u is a restriction of a holomorphic function φ defined on a neighborhood U of V. u corresponds to the hyperfunction $b_U(\varphi)$. A hyperfunction u is real analytic if and only if $\mathrm{SS}(u)$ is contained in the zero section $\{(x, \sqrt{-1}\,\xi)\in\sqrt{-1}\,T^*M;\ \xi=0\}$.

1.4. For any open subset W of $\sqrt{-1}\,T^*M$, we define

$$\mathscr{C}'(W) = \mathscr{B}(V)/\{u\in\mathscr{B}(V);\ \mathrm{SS}(u)\cap W=\emptyset\},$$

where V is an open subset of M which contains $\pi(W)$. This definition does not depend on the choice of V. Let $\mathscr{C}$ be the sheaf on $\sqrt{-1}\,T^*M$ associated with the presheaf $W\mapsto\mathscr{C}'(W)$. If W is an open cone, we have $\mathscr{C}(W)=\mathscr{C}'(W)$. A section of $\mathscr{C}$ is called a *microfunction*, and we denote by sp the homomorphism from $\mathscr{B}(V)$ to $\mathscr{C}(W)$.

A differential operator $P(x, D)$ with real analytic functions as coefficients operates on the sheaf of hyperfunctions and microfunctions in the following way:

$$P\Big(\sum b_{\Omega_j}(\varphi_j)\Big) = \sum b_{\Omega_j}(P\varphi_j) \quad\text{and}\quad P\,\mathrm{sp}(u) = \mathrm{sp}(Pu).$$

2. Micro-differential operators. We can construct a class of operators, wider than the class of differential operators, which operate on the sheaf of microfunctions.

2.1. Let X be an open subset of $\boldsymbol{C}^n$. The cotangent bundle T^*X of X is identified with $X\times\boldsymbol{C}^n$. For a complex number λ and an open subset Ω of T^*X, we denote by $\mathscr{E}^{\infty(\lambda)}(\Omega)$ the set of sequences $\{P_{\lambda+j}(z,\zeta)\}_{j\in Z}$ of holomorphic functions satisfying the following conditions:

(2.1.1) $P_{\lambda+j}(z,\zeta)$ is a holomorphic function defined on Ω, homogeneous of degree $\lambda+j$ with respect to ζ; i.e.

$$\Big(\sum_{\nu=1}^{n}\zeta_\nu\partial/\partial\zeta_\nu\Big)P_{\lambda+j} = (\lambda+j)P_{\lambda+j}.$$

(2.1.2) $P_{\lambda+j}(z,\zeta)$ satisfies the following growth conditions:

(2.1.2.1) For any compact subset K of Ω and a positive number ε, there is a positive number $C_{K,\varepsilon}$ such that

$$|P_{\lambda+j}(z,\zeta)| \leqslant \frac{C_{K,\varepsilon}}{j!}\varepsilon^j \quad\text{for any}\quad j\geqslant 0 \text{ and } (z,\zeta) \text{ in } K.$$

(2.1.2.2) For any compact subset K of Ω, there is a positive number R_K such that

$$|P_{\lambda+j}(z,\zeta)| \leqslant (-j)!\,R_K^{-j} \quad\text{for any}\quad j<0 \text{ and } (z,\zeta) \text{ in } K.$$

We denote by $\mathscr{E}(\lambda)(\Omega)$ the set of $\{P_{\lambda+j}\}$ of $\mathscr{E}^{\infty(\lambda)}(\Omega)$ such that $P_{\lambda+j}=0$ for $j>0$. $\mathscr{E}^{\infty(\lambda)}$ and $\mathscr{E}(\lambda)$ are clearly sheaves on T^*X. We define $\mathscr{E}^{(\lambda)}$ as the union of $\mathscr{E}(\lambda+j)$, and set $\mathscr{E}^{\infty}=\mathscr{E}^{\infty(0)}$, $\mathscr{E}=\mathscr{E}^{(0)}$.

$\{P_{\lambda+j}(z,\zeta)\}_{j\in Z}$ will be denoted by $\sum P_{\lambda+j}(z,D)$. The section of $\mathscr{E}^{\infty(\lambda)}$ is called *micro-differential operator*, and the section of $\mathscr{E}(\lambda)$ will be called *micro-differential operator of order* λ.

For a section $\sum P_{\lambda+j}(z,D)$ of $\mathscr{E}(\lambda)$, $P_{\lambda}(z,\zeta)$ is called *principal symbol* and denoted by $\sigma_{\lambda}(P)$. Hence, σ_{λ} is a sheaf homomorphism from $\mathscr{E}(\lambda)$ onto the sheaf $\mathcal{O}(\lambda)$ of holomorphic functions on T^*X, homogeneous of degree λ with respect to ζ.

2.2. We define a product $R=\sum R_{\lambda+\mu+j}(z,D)$ of micro-differential operators $P=\sum P_{\lambda+j}(z,D)$ and $Q=\sum Q_{\mu+j}(z,D)$ by

$$R_{\lambda+\mu+l}(z,\zeta)=\sum_{l=j+k-|\alpha|}\frac{1}{\alpha!}\big(D_{\zeta}^{\alpha}P_{\lambda+j}(z,\zeta)\big)\big(D_{z}^{\alpha}Q_{\mu+k}(z,\zeta)\big),$$

where $\alpha=(\alpha_1,\dots,\alpha_n)$ is a set of non negative integers, $|\alpha|=\alpha_1+\dots+\alpha_n$ and $D_z^{\alpha}=\partial^{|\alpha|}/\partial z_1^{\alpha_1}\dots\partial z_n^{\alpha_n}$. By this law of multiplication, we have $(PQ)R=P(QR)$ for $P\in\mathscr{E}^{\infty(\lambda)}$, $Q\in\mathscr{E}^{\infty(\mu)}$ and $R\in\mathscr{E}^{\infty(\varrho)}$. We have $\mathscr{E}(\lambda)\cdot\mathscr{E}(\mu)\subset\mathscr{E}(\lambda+\mu)$. $\mathscr{E}^{\infty}$, $\mathscr{E}$ and $\mathscr{E}(0)$ are sheaves of rings with the identity $1=\sum P_j(z,D)$ ($P_j=1$ for $j=0$ and $P_j=0$ for $j\neq 0$). For $P\in\mathscr{E}(\lambda)$ and $Q\in\mathscr{E}(\mu)$, we have $\sigma_{\lambda+\mu}(PQ)=\sigma_{\lambda}(P)\sigma_{\mu}(Q)$.

A differential operator $P=\sum_{\alpha}a_{\alpha}(z)D^{\alpha}$ is identified with the micro-differential operator $\sum P_j(z,D)$ with $P_j(z,\zeta)=\sum_{j=|\alpha|}a_{\alpha}(z)\zeta^{\alpha}$.

The following lemma shows an importance of principal symbols.

LEMMA 2.2.1. *If the principal symbol* $\sigma_{\lambda}(P)$ *of* $P\in\mathscr{E}(\lambda)$ *does not vanish at a point* p *of* T^*X, *then* P *has an inverse* R *in* $\mathscr{E}(-\lambda)$; *i.e.* $RP=PR=1$.

The rings $\mathscr{E}$ and $\mathscr{E}(0)$ have nice ring theoretic properties; for example, they are coherent rings, any stalk of them is a noetherian ring, etc.

2.3. Let M be an intersection of $\boldsymbol{R}^n$ and an open subset X of $\boldsymbol{C}^n$. Then, the conormal bundle $T_M^*X=\sqrt{-1}\,T^*M$ is a closed submanifold of T^*X. Let V be an open subset of $\sqrt{-1}\,T^*M$ and let Ω be an open subset of T^*X containing V. Then, $\mathscr{E}^{\infty(\lambda)}(\Omega)$ operates on $\mathscr{C}(V)$, and we have $1\cdot u=u$, $(PQ)u=P(Qu)$ for $u\in\mathscr{C}(V)$, $P\in\mathscr{E}^{\infty(\lambda)}(\Omega)$, $Q\in\mathscr{E}^{\infty(\mu)}(\Omega)$.

By using this operation of micro-differential operators, the following proposition is easily obtained from Lemma 2.2.1.

PROPOSITION 2.3.1. *For a differential operator* P *and a hyperfunction* u, *we have*

$$\mathrm{SS}(u)\subset\sigma(P)^{-1}(0)\cup\mathrm{SS}(Pu).$$

In fact, outside $\sigma(P)^{-1}(0)\cup\mathrm{SS}(u)$, P is invertible as micro-differential operator by Lemma 2.2.1, and hence $P\,\mathrm{sp}(u)=0$ implies $\mathrm{sp}(u)=0$.

2.4. Let $p_0=(z_0,\zeta_0)$ and $p_1=(z_1,\zeta_1)$ be two points of $T^*\boldsymbol{C}^n$ and F a holomorphic map from a neighborhood of p_0 to a neighborhood of p_1 such that

$F(p_0)=p_1$. Assume that F is a homogeneous symplectic transformation, i.e. $F^*(\sum\zeta_j dz_j)=\sum\zeta_j dz_j$. Then, we can construct a sheaf isomorphism $\Phi: F^{-1}(\mathscr{E}^{\infty(\lambda)})\tilde{\to}\mathscr{E}^{\infty(\lambda)}$ on a neighborhood of p_1 such that $\Phi(PQ)=\Phi(P)\Phi(Q)$ for $P\in\mathscr{E}^{\infty(\lambda)}$ and $Q\in\mathscr{E}^{\infty(\mu)}$, and that $\sigma_\lambda(\Phi(P))$ $\sigma_\lambda(P)\circ F$ for any $P\in\mathscr{E}(\lambda)$. Moreover if p_0 and p_1 are contained in $\sqrt{-1}\,T^*M$ and if F maps $\sqrt{-1}\,T^*M$ into $\sqrt{-1}\,T^*M$, then we can construct a sheaf isomorphism $\Psi: F^{-1}(\mathscr{C})\tilde{\to}\mathscr{C}$ such that $\Psi(Pu)=\Phi(P)\cdot\Psi(u)$ for any $P\in\mathscr{E}^{\infty(\lambda)}$ and $u\in\mathscr{C}$. We shall call (Φ, F) or (Φ, Ψ, F) the *quantized contact transformation.*

Quantized contact transformations are effectively used in order to transform micro-differential operators into the normal forms. For example, let us consider a micro-differential operator P of order 1 such that $\sigma_1(P)\wedge(\sum\zeta_j dz_j)\neq 0$ and that $\sigma_1(P)|_{\sqrt{-1}\,T^*R^n}$ is real-valued. Then, there is a quantized contact transformation which transforms P into $\sqrt{-1}\,\partial/\partial z_1$. Hence, we can deduce the properties of P from those of $\partial/\partial z_1$, which is easy to analyze. Thus, we obtain the following

PROPOSITION 2.4.1. *Let P be a micro-differential operator such that the restriction of the principal symbol p of P to $\sqrt{-1}\,T^*\mathbf{R}^n$ is real valued and that the differential of the principal symbol is not parallel to $\sum\zeta_j dz_j$. Then, $P: \mathscr{C}\to\mathscr{C}$ is surjective and the support of microfunction solution of $Pu=0$ is the union of bicharacteristics of P, i.e. integral curves of the Hamiltonian*

$$\sum_j \frac{\partial p}{\partial\zeta_j}\frac{\partial}{\partial z_j}-\frac{\partial p}{\partial z_j}\frac{\partial}{\partial\zeta_j}.$$

3. System of micro-differential equations.

3.1. In this talk, a *system of micro-differential equations* is, by definition, a coherent left $\mathscr{E}$-Module. Let $\mathfrak{M}$ be a coherent $\mathscr{E}$-Module. Then, $\mathfrak{M}$ has locally a free resolution

$$0\leftarrow\mathfrak{M}\leftarrow\mathscr{E}^{N_0}\overset{P}{\longleftarrow}\mathscr{E}^{N_1},$$

where $P=(P_{ij})$ is an $N_1\times N_0$ matrix of micro-differential operators and $P: \mathscr{E}^{N_1}\to\mathscr{E}^{N}$ is given by $(Q_1, \dots, Q_{N_1})\mapsto(Q_1, \dots, Q_{N_1})P=(\sum Q_i P_{i1}, \dots, \sum Q_i P_{iN_0})$.

We have $\mathscr{H}om_{\mathscr{E}}(\mathfrak{M}, \mathscr{C})=\operatorname{Ker}(\mathscr{C}^{N_0}\overset{P}{\to}\mathscr{C}^{N_1})$, and hence $\mathscr{H}om_{\mathscr{E}}(\mathfrak{M}, \mathscr{C})$ is a sheaf of microfunction solutions of a system of micro-differential equations:

$$\sum_{j=1}^{N_0} P_{ij}u_j = 0\ \ (i=1, \dots, N_1).$$

3.2. The support of a coherent $\mathscr{E}$-module is not an arbitrary subset of T^*M.

THEOREM 3.2.1 ([9], [10]). *The support of a coherent $\mathscr{E}$-module is an involutive analytic subset of T^*X.*

Remember that an analytic subset V of T^*X is called *involutive* if the Poisson bracket

$$\{f, g\} = \sum_j\left(\frac{\partial f}{\partial\zeta_j}\frac{\partial g}{\partial z_j}-\frac{\partial g}{\partial\zeta_j}\frac{\partial f}{\partial z_j}\right)$$

vanishes on V for any holomorphic functions f and g vanishing on V. An involutive analytic subset has always codimension equal or greater than $n=\dim X$. We say an involutive analytic subset V is *Lagrangian* if codim $V=\dim X$, and a system of micro-differential equations is *holonomic* if its support is Lagrangian.

THEOREM 3.2.2 ([6], [8]). *Let* $\mathfrak{M}$ *be a holonomic system of micro-differential equations. Then, we have*

(1) *For any point* p *of* $\sqrt{-1}\,T^*\mathbf{R}^n$, $\mathcal{H}om_{\mathscr{E}}(\mathfrak{M}, \mathscr{C})_p$ *is a finite-dimensional vector space. More generally, so is* $\mathcal{E}xt^j_{\mathscr{E}}(\mathfrak{M}, \mathscr{C})_p$ *for any* j.

(2) *There is a stratification* $\sqrt{-1}\,T^*\mathbf{R}^n=\amalg V_\alpha$ *of* $\sqrt{-1}\,T^*\mathbf{R}^n$ *into subanalytic submanifolds* V_α *such that* $\mathcal{E}xt^j_{\mathscr{E}}(\mathfrak{M}, \mathscr{C})|_{V_\alpha}$ *is a locally constant sheaf for any* α *and* j.

A microfunction which satisfies a holonomic system of micro-differential equations is called *holonomic microfunction*. Theorem 3.2.2 suggests that we have a great chance to analyze holonomic microfunctions through their holonomic systems of micro-differential equations.

4. Holonomic system with regular singularities.

4.1. Let $\mathfrak{M}$ be a holonomic system of micro-differential equations and let Λ be the support of $\mathfrak{M}$. Λ is therefore a Lagrangian analytic subset of T^*X. At a generic point p of Λ, Λ is a conormal bundle of a complex submanifold Y of X. Let us take a local coordinate system $x=(x_1, \dots, x_n)$, such that Y is given by $x_1=\dots=x_l=0$ and $p=(0, dx_1)$. Let $\mathscr{L}_{\lambda,m}$ be the holonomic system defined by

$$\mathscr{L}_{\lambda,m}=\mathscr{E}/\mathscr{E}(x_1D_1-\lambda)^m+\mathscr{E}x_2+\dots+\mathscr{E}x_l+\mathscr{E}D_{l+1}+\dots+\mathscr{E}D_n=\mathscr{E}u_{\lambda,m}.$$

Here, $u_{\lambda,m}$ is the modulo class of 1.

THEOREM 4.1.1. *There are complex numbers* λ_j *and integers* m_j $(j=1, \dots, N)$ *such that* $\mathscr{E}^\infty\otimes_{\mathscr{E}}\mathfrak{M}$ *is isomorphic to* $\oplus\mathscr{E}^\infty\otimes_{\mathscr{E}}\mathscr{L}_{\lambda_j,m_j}$. $(\lambda_j \bmod \mathbf{Z}, m_j)$ *are uniquely determined by* $\mathfrak{M}$ *up to permutation.*

The integer $\sum m_j$ is called the *multiplicity* of $\mathfrak{M}$.

We say that $\mathfrak{M}$ has *regular singularities* at p if $\mathfrak{M}$ is isomorphic to $\bigoplus_j\mathscr{L}_{\lambda_j,m_j}$. In general, we say that $\mathfrak{M}$ has *regular singularities,* if $\mathfrak{M}$ has regular singularities at a generic point of any irreducible component of the support of $\mathfrak{M}$.

Any holonomic system can be transformed into a holonomic system with regular singularities by using micro-differential operators of infinite order. More precisely, we have the following

THEOREM 4.1.2 [7]. *For any holonomic system* $\mathfrak{M}$, *there exists a holonomic system* $\mathfrak{M}'$ *with regular singularities such that* $\mathscr{E}^\infty\otimes_{\mathscr{E}}\mathfrak{M}$ *is isomorphic to* $\mathscr{E}^\infty\otimes_{\mathscr{E}}\mathfrak{M}'$. *Moreover,* $\mathfrak{M}'$ *is unique up to isomorphism.*

4.2. Let Y be a complexification of a real analytic submanifold of $\boldsymbol{R}^n$, and let $\mathscr{L}_{\lambda,m}$ be as in § 4.1. Then, any homomorphism from $\mathscr{L}_{\lambda,1}$ into the sheaf of micro-functions $\mathscr{C}$ is given by $u_{\lambda,1} \mapsto c(D_1/\sqrt{-1})^{-\lambda-1}\delta(x_1, \ldots, x_l)$ for some complex number c. Let $\mathfrak{M}$ be a holonomic system with multiplicity 1 (this implies that $\mathfrak{M}$ has regular singularities), and suppose that $\mathfrak{M}$ is generated by a section u. Then, $\mathfrak{M}$ is isomorphic to $\mathscr{L}_{\lambda,1}$ for some complex number λ, and the isomorphism from $\mathfrak{M}$ onto $\mathscr{L}_{\lambda,1}$ is given by $u \mapsto Pu_{\lambda,1}$ with a micro-differential operator $P \in \mathscr{E}(k)$ (for some integer k) such that $\sigma_k(P)|_\Lambda \not\equiv 0$. Hence, for any $F \in \mathscr{H}om_{\mathscr{E}}(\mathfrak{M}, \mathscr{C})$, $F(u)$ equals $cP(D_1/\sqrt{-1})^{-\lambda-1}\delta(x_1, \ldots, x_l)$ for a complex number c. We define the *principal symbol* $\sigma(F(u))$ of $F(u)$ by

$$\sigma(F(u)) = \frac{1}{(2\pi)^{l/2}} c(\sigma_k(P)|_\Lambda)\xi_1^{-\lambda-1}\sqrt{|d\xi_1 \ldots d\xi_l\, dx_{l+1} \ldots dx_n|/|dx_1 \ldots dx_n|}$$

regarded as a section of $\Omega_\Lambda^{1/2} \otimes (\Omega_M^{1/2})^{\otimes(-1)}$. Here $\Omega_\Lambda^{1/2}$ is the sheaf of half densities on Λ. For a holonomic microfunction which satisfies a holonomic system with regular singularities with multiplicity 1, we can define its principal symbol in this manner (see [2]).

The above observation shows that a solution $F: \mathfrak{M} \mapsto \mathscr{C}$ is uniquely determined by the principal symbol of $F(u)$.

5. Asymptotic expansion of holonomic microfunctions.

5.1. Let us consider the situation in § 4 with $l=1$. Then, $F(u)$ is given by $P(x, D)(D_1/\sqrt{-1})^{-\lambda-1}\delta(x_1)$. For the sake of simplicity let us assume that λ is not an integer. Then, we can show that $F(u)$ equals $\mathrm{sp}(\varphi(x)(x_1+\sqrt{-1}\,0)^\mu)$ with $\mu=\lambda-k$ and a real analytic function $\varphi(x)$. We have

$$\sigma(F(u)) = \frac{\sqrt{2\pi}\xi_1^{-\mu-1}}{\Gamma(-\mu)\exp(-\pi i\mu/2)}\varphi(0, x_2, \ldots, x_n)\sqrt{|d\xi_1\, dx_2 \ldots dx_n|/|dx_1 \ldots dx_n|}.$$

Since

$$\varphi(x)(x_1+i0)^\mu = \varphi(0, x_2, \ldots, x_n)(x_1+i0)^\mu + \frac{\partial\varphi}{\partial x_1}(0, x_2, \ldots, x_n)(x_1+i0)^{\mu+1} + \ldots,$$

$\sigma(F(u))$ gives the first coefficient of the power series expansion of $F(u)$ with respect to x_1.

5.2. Let us consider another example. Let $f(x)$ be a real-valued real analytic function. Then, it is known that $\delta(t-f(x))$ has the asymptotic expansion

$$\delta(t-f(x)) \sim \sum_{\nu=0}^{n-1}\sum_{j=1}^{N}\sum_{k=0}^{\infty} a_{j,\nu,k}(x)t^{\lambda_j+k}(\log t)^\nu \quad (t \searrow 0).$$

Here, $a_{j,\nu,k}(x)$ is a distribution related with the residues of the analytic continuation of $f(x)_+^s$ with respect to s.

The above asymptotic expansion means the following: for any compactly supported C^∞-function $\varphi(x)$, we have the asymptotic expansion

$$\int \delta(t-f(x))\varphi(x)\,dx \sim \sum_{\nu}\sum_{j}\sum_{k}\Big(\int a_{j,\nu,k}(x)\varphi(x)\,dx\Big)t^{\lambda_j+k}(\log t)^{\nu}.$$

5.3. Let us consider a more general case. Let X be an open subset of C^{1+n}. We shall denote by $(t,x)=(t,x_1,\dots,x_n)$ the point of C^{1+n}. Let $\mathscr{D}$ be the sheaf of differential operators on X. A left coherent $\mathscr{D}$-module is called a *system of differential equations.* Let us denote by π the projection from T^*X onto X. Then, $\mathscr{E}$ contains $\pi^{-1}\mathscr{D}$ as its subring. For a system $\mathfrak{M}$ of differential equations, the support of $\mathscr{E}\otimes_{\pi^{-1}\mathscr{D}}\pi^{-1}\mathfrak{M}$ is called the *characteristic variety* of $\mathfrak{M}$.

We say that $\mathfrak{M}$ is *holonomic* if so is $\mathscr{E}\otimes_{\pi^{-1}\mathscr{D}}\pi^{-1}\mathfrak{M}$.

Let $\mathfrak{M}$ be a holonomic system of differential equations, u a section of $\mathfrak{M}$, and let F be a $\mathscr{D}$-linear homomorphism from $\mathfrak{M}$ into the sheaf $\mathscr{B}$ of hyperfunctions on $M=X\cap R^{1+n}$.

Suppose that $F(u)$ has an asymptotic expansion

$$F(u)\sim\sum_{j}\sum_{\nu}\sum_{k}a_{j,\nu,k}(x)t^{\lambda_j+k}(\log t)^{\nu}. \tag{5.3.1}$$

We may assume that $\lambda_j-\lambda_{j'}$ is not an integer for $j\neq j'$ and that for any j there is ν satisfying $a_{j,\nu,0}\neq 0$.

First, let us determine λ_j.

THEOREM 5.3.1. *There exist a non zero polynomial $b(s)$ and a differential operator $P(t,x,tD_t,D_x)$ satisfying the following conditions*

(1) $b(tD_t)=tP(t,x,tD_t,D_x)u$,

(2) $P(t,x,tD_t,D_x)$ *has the form* $\sum_{j,\alpha}g_{j,\alpha}(t,x)(tD_t)^jD_x^\alpha$.

This theorem gives λ_j. We have

$$b(\lambda_j)=0 \quad \text{for any } j. \tag{5.3.2}$$

In fact, by (1) of Theorem 5.3.1, we have

$$\big(b(tD_t)-tP(t,x,tD_t,D_x)\big)\sum_{j}\sum_{\nu}\sum_{k}a_{j,\nu,k}(x)t^{\lambda_j+k}(\log t)^{\nu}=0. \tag{5.3.3}$$

If $a_{j,\nu_0,0}=0$ for $\nu>\nu_0$ and $a_{j,\nu_0,0}\neq 0$, then the coefficient of $t^{\lambda_j}(\log t)^{\nu_0}$ of the expansion of the left hand side of (5.3.3) is $b(\lambda_j)a_{j,\nu_0,0}(x)$. Hence, we obtain $b(\lambda_j)=0$.

We shall call $b(s)$ the *b-function* of u with respect to the hypersurface $t=0$.

EXAMPLE 5.3.2. $u=(t^2-x^3)^\alpha$, i.e.

$$\Big(\frac{1}{2}tD_t+\frac{1}{3}xD_x-\alpha\Big)u=(3x^2D_t+2tD_x)u=0.$$

We have

$$(tD_t)\left(tD_t-2\alpha-\frac{2}{3}\right)\left(tD_t-2\alpha-\frac{4}{3}\right)u=-\frac{8}{27}t^2D_x^3u,$$

and hence $b(s)=s(s-2\alpha-2/3)(s-2\alpha-4/3)$.

EXAMPLE 5.3.3. $u=e^{-1/t}$; i.e. $(t^2D_t-1)u=0$. In this example, $b(s)=1$. In fact, $u=t^2D_tu$. See § 5.4.

EXAMPLE 5.3.4. Let $f(x)$ be a holomorphic function. Let $b(s)$ be the b-function of $u=\delta(t-f(x))$. Then, we have

$$b(tD_t)\delta(t-f(x))=tP(x,tD_t,D_x)\delta(t-f(x)).$$

By multiplying t^s, we obtain

$$b(tD_t-s)t^s\delta(t-f(x))=P(x,tD_t-s-1,D_x)t^{s+1}\delta(t-f(x)).$$

This implies, with the change of variables $(t,x)\mapsto(t+f(x),x)$, the following:

$$b((t+f(x))D_t-s)f(x)^s\delta(t)=P(x,(t+f(x))D_t-s-1,D_x-(df)D_t)f(x)^{s+1}\delta(t).$$

Comparing the coefficients of $\delta(t)$, we obtain

$$b(-s-1)f(x)^s=P(x,-s-2,D_x)f(x)^{s+1}.$$

Hence, $b(-s-1)$ is the b-function of $f(x)$, i.e. $b(-s-1)f(x)^s\in\mathscr{D}[s]f(x)^{s+1}$. See [1], [4].

5.4. We shall micro-localize the situation of § 5.3. Let Λ be a Lagrangian submanifold of T^*X and J_Λ the sheaf of holomorphic functions which vanish on Λ. There is $f\in J_\Lambda\cap\mathcal{O}(1)$ such that $df\equiv\omega\left(=\sum\zeta_j dz_j\right)$ modulo J_Λ, i.e. $df-\omega\in J_\Lambda\Omega^1$. The function f is determined modulo J_Λ^2. Let $\mathscr{J}_\Lambda$ be the subsheaf of $\mathscr{E}(1)$ defined by $\sigma_1^{-1}(J_\Lambda)$ and $\mathscr{E}_\Lambda$ the subring of $\mathscr{E}$ generated by $\mathscr{J}_\Lambda$. We denote by $\mathscr{E}_\Lambda(m)$ the $\mathscr{E}_\Lambda$-submodule $\mathscr{E}(m)\cdot\mathscr{E}_\Lambda=\mathscr{E}_\Lambda\cdot\mathscr{E}(m)$ of $\mathscr{E}$ for $m\in\boldsymbol{Z}$ and we define $\mathscr{E}_{\Lambda,m}=\mathscr{E}_\Lambda\cap\mathscr{E}(m)=\mathscr{J}_\Lambda^m$. Let $\Phi=\sum\Phi_j(x,D)$ be a micro-differential operator in $\mathscr{J}_\Lambda$ such that $\Phi_1\equiv f \bmod J_\Lambda^2$ and $\Phi_0-\frac{1}{2}\sum_\nu\partial^2\Phi_1/\partial z_\nu\partial\zeta_\nu\equiv0 \bmod J_\Lambda$. Then, Φ is determined modulo $\mathscr{E}_{\Lambda,2}(-1)=\mathscr{J}_\Lambda^2(-1)$. Note that $[\Phi,P]\in\mathscr{E}_\Lambda(-1)$ for any $P\in\mathscr{E}_\Lambda$.

THEOREM 5.4.1. *Let $\mathfrak{M}$ be a holonomic $\mathscr{E}$-module with regular singularities, and let u be a section of $\mathfrak{M}$. Then, there are a nonzero polynomial $b(s)$ and $P\in\mathscr{E}_\Lambda(-1)$ such that $b(\Phi)u=Pu$ and that* $\operatorname{ord}P\leqq\deg b$.

Consider the quotient ring $\mathscr{A}=\mathscr{E}_\Lambda/\mathscr{E}_\Lambda(-1)$. Then this ring is locally isomorphic to the ring of differential operators on Λ homogeneous of degree 0 with respect to ζ. Let ϑ be the modulo class of Φ. Then, the center of $\mathscr{A}$ is the polynomial ring generated by Φ. $\mathscr{A}$ is embedded (locally) into the ring $\mathscr{D}_\Lambda$ of differential operators on Λ.

THEOREM 5.4.2. *Let* $\mathfrak{M}$ *be a holonomic* $\mathscr{E}$*-module with regular singularities and* $\mathfrak{N}$ *a coherent* $\mathscr{E}_\Lambda$*-sub-module of* $\mathfrak{M}$. *Then, the* $\mathscr{A}$*-module* $\mathfrak{N}/\mathscr{E}(-1)\mathfrak{N}$ *has the following properties.*

(1) *There is a nonzero polynomial* $b(s)$ *such that* $b(\vartheta)(\mathfrak{N}/\mathscr{E}(-1)\mathfrak{N})=0$.

(2) *The* $\mathscr{D}_\Lambda$*-module* $\mathscr{D}_\Lambda\otimes_{\mathscr{A}}(\mathfrak{N}/\mathscr{E}(-1)\mathfrak{N})$ *is a holonomic system of differential equations on* Λ *with regular singularities.*

5.5. Let X be an open subset of $C^{1+n}=\{(t,x);\ t\in C, x\in C^n\}$ and Λ the conormal bundle of the hypersurface $t=0$. Let $\mathfrak{M}$ be a holonomic system of micro-differential equations with regular singularities defined on a neighborhood of $p_0=(0, \sqrt{-1}\,dt)$, and u a section of $\mathfrak{M}$. Let F be an $\mathscr{E}$-linear homomorphism from $\mathfrak{M}$ into the sheaf $\mathscr{C}$ of microfunctions. Then, $v(t,x)=F(u)$ is a holonomic microfunction.

Let $b(s)$ be the polynomial given in Theorem 5.4.1, $\{\lambda_j\}_{j=1,\ldots,N}$ the set of the distinct roots of $b(s)$, and let m_j be the number of the set $\{j';\ j'\neq j,\ \lambda_{j'}-\lambda_j$ is a nonnegative integer$\}$. Then, $v(t,x)$ has the "asymptotic expansion" at p_0:

$$v(t,x)\sim\sum_{j=1}^{N}\sum_{\nu=0}^{m_j}\sum_{k=0}^{\infty}a_{j,\nu,k}(x')(D_t/\sqrt{-1})^{\lambda_j-1/2-k}(\log D_t/\sqrt{-1})^{\nu}\delta(t).$$

Moreover, it is easy to see that $a_{j,\nu,k}(x)$ can be calculated from $a_{j,\nu,k}(x)$'s with $k<\deg b$ by using the equation $b(\Phi)v=Pv$. Each hyperfunction $a_{j,\nu,k}(x)$ satisfies the system of differential equations derived from $\mathfrak{N}/\mathscr{E}(-1)\mathfrak{N}$ and hence $a_{j,\nu,k}(x)$ satisfies a holonomic system of differential equations with regular singularities by Theorem 5.4.2.

EXAMPLE 5.5.1. Let us consider the hyperfunction $u=(t^2-x^3)^\alpha_+\ldots$ as in Example 5.3.2. This hyperfunction has the meromorphic continuation on α and has poles $\alpha=-5/6-n,\ -1-n,\ -7/6-n$ $(n=0,1,2,\ldots)$. This u has the asymptotic expansion:

$$u=c_0\sum_{n=0}^{\infty}(4/27)^n\frac{[\alpha+5/6]_n}{n![1/3]_n}D_x^{3n}\delta(x)(D_t/\sqrt{-1})^{-2\alpha-5/3-2n}\delta(t)$$

$$+c_1\sum_{n=0}^{\infty}(4/27)^n\frac{[\alpha+7/6]_n}{n![4/3]_n}D_x^{3n+1}\delta(x)(D_t/\sqrt{-1})^{-2\alpha-7/3-2n}\delta(t).$$

Here,

$$c_0=-\frac{2^{2\alpha+2/3}}{\sqrt{3\pi}}\Gamma(1/3)\Gamma(\alpha+1)\Gamma(\alpha+5/6)\sin\pi(\alpha+1/6)$$

and

$$c_1=-\frac{2^{2\alpha+4/3}}{\sqrt{3\pi}}\Gamma(2/3)\Gamma(\alpha+1)\Gamma(\alpha+7/6)\sin\pi(\alpha-1/6).$$

$[s]_n$ means $s(s+1)(s+2)\ldots(s+n-1)$.

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