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THE COHOMOLOGY OF INFINITE DIMENSIONAL LIE ALGEBRAS; SOME QUESTIONS OF INTEGRAL GEOMETRY

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This report is concerned with certain results and problems arising in the theory of the representation of groups. In the last twenty years much has been achieved in this field and—most important—its almost boundless possibilities have become apparent.

Indeed, its problems, touching on the interests of algebraic geometry, on many questions of the algebraic number theory, analysis, quantum field theory and geometry, as well as its inner symmetry and beauty have resulted in the growing popularity of the theory of representations.

It is impossible to list even briefly its main achievements, and this is not the aim of this communication. Nevertheless, one cannot omit mentioning the outstanding papers by Harish-Chandra, Selberg, Langlands, Kostant, A. Weil, which considerably advanced the development of the theory of representations and opned up new relationships; and, since we do not go into these questions, we will not be able to touch upon many of the deep notions and results of the theory of representations.

We feel that the methods which have arisen in the theory of representation of groups may be used in a considerably more general non-homogeneous situation. We will give some examples:

1. The proof of the fact that the spectrum of a flow on symmetric spaces of constant negative curvature is a Lebesgue spectrum [1] was based on methods of the theory of representations, namely the decomposition of representations into irreducible ones. One of the most useful methods of decomposing representations into irreducible representations is the orisphere method [5]. In the works of Sinai, Anosov, Margulis [2], [3], [4], only the orispheres are considered and groups symmetries are left out. This rendered possible the study of the spectrum of dynamic systems in a considerably more general situation.

2. The theorem of Plancherel and the method of orispheres gives rise to the consideration of more general problems of integral geometry, taking place in a non-homogeneous situation [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17].

3. If we have a manifold and its mapping, the study of distributions " constant on the inverse image of each point " of this mapping is an extremely interesting problem, special examples of which were studied in the homogeneous situation (functions in

four-dimensional space, invariant relative to the Lorenz group, functions constant on classes of conjugate elements of a semi-simple Lie group [18], etc.). There are various aspects of this problem which are considerably more interesting and important than may seem at first glance. Of course, the main interest of the problem is the study of these distributions at singularies of the mapping. To be more precise, suppose X is a manifold (C^∞-analytical, algebraic) and $\mathscr{G}$ is some (perhaps infinitely dimensional) Lie algebra of smooth vector fields. One wishes to describe the space of unvariant distributions.

A more natural statement of the problem is obtained by replacing the distributions by generalised sections of a vector bundle which vary according to a given finite dimensional representation. Unfortunately consideration of length prevent me from giving a series of existing examples. Those examples are particularly interesting when X has only a finite number of orbits relative to $\mathscr{G}$. For interesting example in the non-homogeneous situation see [34].

4. The theory of representation of groups makes the consideration of interesting examples possible and shows the importance of studying the ring of all the regular differential operators on those algebraic manifolds which are homogeneous spaces. It is quite natural to wish to describe the structure of the ring R of regular differential operators on any algebraic manifold. Perhaps, as in [19], [20], it would be helpful to consider the quotient skew-field of the ring R. Another interesting problem is the description of the involutions of this ring R.

In this report I would like to tell about certain problems which were studied by my friends and myself while thinking about questions connected with representation theory.

I. Representations of semisimple Lie algebras.

0. Suppose $\mathscr{G}$-is a semisimple Lie algebra. The study of representations is essentially the study of a category of $\mathscr{G}$-modules. The choice of the particular category of $\mathscr{G}$-modules considered in the algebraic problems of the theory of representations is essential. Suppose $\mathscr{f}$ is a fixed subalgebra of $\mathscr{G}$. $\mathscr{G}$-the module will be called $(\mathscr{g}, \mathscr{f})$ finite iff 1° it is a finitely generated $\mathscr{U}(\mathscr{G})$-module and 2° as an $\mathscr{U}(\mathscr{G})$-module it is the algebraic direct sum of finite dimensional irreducible representations of $\mathscr{f}$ and in this decomposition each of the irreducible representation appears only a finite number of times.

The following two cases are very interesting:

1° $\mathscr{G}$ is a real semisimple algebra, $\mathscr{f}$ is the subalgebra corresponding to the maximum compact subgroup. The corresponding $(\mathscr{g}, \mathscr{f})$-modules were considered by V. A. Ponamaryov and the author and were called by them " Harish-Chandra modules ".

2° $\mathscr{G}$ is a real Lie algebra, $\mathscr{f}$ is a Cartan subalgebra or, more generally, the semisimple part of the parabolic subalgebra.

1. Let us consider in more detail the category of Harish-Chandra modules in the case when $\mathscr{G}$ is the algebra of a complex semisimple Lie group.

Further, each module is the direct sum of modules on each of which the Laplace operators have only one eigen-value.

Consider an example. Suppose G is a simply connected Lie group over the algebra $\mathscr{G}$, B – its Borel subgroup, $\mathscr{N}$ is a unipotent radical of B, H – a Cartan subgroup. Consider the indecomposable finite dimensional representation ρ of the group H. Note that since $H = C^* \times C^* \times \ldots \times C^*$ the question of the finite dimensional representations of H is reduced to the determination of a finite number of pairwise commutative matrices. Let us extend this representation ρ of the group H to a representation of the group B and consider, further, the representation of the group G induced by this representation B. The representation thus obtained will be called a Jordan representation. In the case when ρ is of dimension one, we obtain the well-known representation of the principal series. Thus we have constructed, using the representation of the group H, a representation of the group G. Note that the description of the canonical form of the representation of H is in some sense an unsolvable problem if the rang of H is greater than 1 [21].

If we consider the representation of the algebra $\mathscr{G}$ thus constructed only on the space of vectors which vary over the finite dimensional representation of the maximum compact subgroup, we will obtain Harish-Chandra modules. Apparently the following hypothesis holds: at the points of general position all the indecomposable Harish-Chandra modules are all Jordan representations (*).

For $SL(2, \mathbb{C})$ this statement follows from work of Zhelobenko. The most interesting is the study of Harish-Chandra modules at singular points. Of course, the problem of listing all the Jordan modules is already a badly stated (unsolvable) problem, since it is based on the classification of systems of pairwise commutative matrices. However, it is not clear whether it is possible to solve this problem at a singular point, considering the Jordan modules as given. If such a solution were possible, it would have exceptional interest.

The problem of describing Harish-Chandra modules was completely solved by V. A. Ponomaryov and the author for the Lie algebra of the group $SL(2, \mathbb{C})$ [22], [23], [24]. Then these representations were constructed as a group representation (and not only as an algebra representation) by M. I. Graev and the authors cited above [25].

The classification of indecomposable Harish-Chandra modules is carried out in two stages.

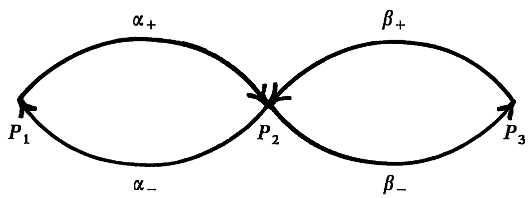
1. The problem is reduced to a problem in linear algebra.
2. The linear algebra problem obtained for $SL(2, \mathbb{C})$ generalises the problem of describing the canonical form of pairs of matrices A, B such that $AB = BA = 0$. To solve this problem we apply the Maclane relation theory, which allows us to use the relations $A^\#$ and $B^\#$, inverse to the degenerate operators A and B, as well as the monomials $A^{\# k_1} B^{\# k_2} A^{\# k_3} \ldots$.

The Harish-Chandra modules at a singular point may be divided into two classes.

(*) To be more precise, each HARISH-CHANDRA module is decomposed into direct sum of submodules on which the Laplace operators have precisely one eigen-value. The set of eigenvalues thus obtained is called singular if the representation of the fundamental series with the same eigen-values of the Laplace operator are reducible. The points of general position will be exactly the non-singular points.

The modules of first class are uniquely defined by any set of natural numbers, the modules of the second class are determined by any set of natural numbers together with one complex number λ. It is thus interesting to note that at singular points the module space is not discrete. The most convenient canonical form of Harish-Chandra modules are given in [25].

In the case of $SL(2, \mathbb{R})$ the problem of classifying Harish-Chandra modules is easily reduced to a problem in linear algebra; explicitely the category of Harish-Chandra modules at a given singular point is isomorphic to the following category of diagrams in the category of finite dimensional linear spaces:



with the condition $\alpha_+\alpha_- = \beta_+\beta_- = \gamma$, where γ is nilpotent. The question of the classification of the objects of this category is aparantly solvable but leads to considerable difficulties.

CONJECTURE. — The category of Harish-Chandra modules for any semisimple group with given eigen-values of Laplace operators is equivalent to a certain category of diagrams in the category of finite dimensional linear spaces.

2. This and the following section of the report summarise some results of I. N. Bernstein, S. I. Gel'fand and the author.

Suppose $\mathscr{G}$ is a semisimple Lie algebra over $\mathbb{C}$, b is its Borel subalgebra, $\mathfrak{u}$ is a radical and $\mathscr{f}$ is a Cartan subalgebra. Consider the following category $\mathcal{O}$. Its objects are $(\mathscr{G}, \mathscr{f})$ – finite modules M, satisfying the following condition: for every vector $\xi \in M$ the space $\mathscr{U}(u)\xi$ is finite dimensional. This category is most useful for the application of the theory of highest weights. In this category, let us chose a class of objects which will be called elementary. All the others will be constructed from them and their factor modules by step by step extensions.

Suppose χ is a linear functional over $\mathscr{f}$. Denote by $M_\chi \mathscr{U}(\mathscr{G})$-module, generated by f_χ, with the relations $nf_\chi = 0$ and $hf_\chi = (\chi - \rho, h).f_\chi$ for all $h \in \mathfrak{f}$ and $n \in \mathfrak{u}$. Here ρ denotes the half-sum of the positive roots. By studying the modules M_χ we get extensive information on the representation of the algebras $\mathscr{G}$, including finite dimensional ones. We now state a few theorems on M_χ modules and their morphisms.

THEOREM 1 (Verma). — Let the modules M_{χ_1} and M_{χ_2} be given. Two cases are possible:

1° $$\mathrm{Hom}\,(M_{\chi_1}, M_{\chi_2}) = 0;$$

and

2° $$\mathrm{Hom}\,(M_{\chi_1}, M_{\chi_2}) \approx \mathbb{C},$$

then any non-trivial homomorphism M_{χ_1} into M_{χ_2} is an embedding.

To state the next theorem we must introduce a partial ordering in the Weyl group W. Suppose $s_1, s_2 \in W$. We shall say that $s_1 > s_2$ iff there exist reflexions $\sigma_1, \ldots, \sigma_r$ in W such that $s_1 = \sigma_1 \ldots \sigma_r s_2$ and $l(\sigma_{i+1} \ldots \sigma_r s_2) = l(\sigma_{i-1} \ldots \sigma_r s_2) + 1, i = 1, \ldots, r$, where $l(s)$ is the length of the element $s \in W$.

THEOREM 2. — Let M_{χ_1} and M_{χ_2} be given. M_{χ_1} imbeds into M_{χ_2} if and only if,

1. There exists such an χ that Re χ lies in the positive Weyl chamber and such a pair of elements $s_1, s_2 \in W$, $s_1 > s_2$ that $\chi_1 = s_1\chi$, $\chi_2 = s_2\chi$.

2. $\chi_1 - \chi_2 = \Sigma n_i\alpha_i$, where n_i are integers, α_i are simple roots.

The module M_{χ_0} is richest in submodules for integer values of χ_0 from the positive Weyl chamber. It follows from theorem 2 that M_{χ_0} contains a submodule $M_{s\chi_0}$ for all $s \in W$. In this case the embedding of $M_{s\chi_0}$ into M_{χ_0} is determined in the following way. Suppose s_{α_i} is the reflection with respect to the simple roots α_i, $s = s_{\alpha_i} \ldots s_{\alpha_k}$ is the decomposition of minimum length. Let

$$\chi_i = s_{\alpha_i}s_{\alpha_{i+1}} \ldots s_{\alpha_k}\chi_0.$$

Then

$$f_{s\chi_0} = af_{\chi_0},$$

where

$$a = E_{-\alpha_1}^{\frac{(\chi_2-\chi_1,\alpha_1)}{(\alpha_1,\alpha_1)}} . E_{-\alpha_2}^{\frac{(\chi_3-\chi_2,\alpha_2)}{(\alpha_2,\alpha_2)}} \ldots E_{-\alpha_k}^{\frac{(\chi_0-\chi_k,\alpha_k)}{(\alpha_k,\alpha_k)}}$$

Since the minimum representation s in the form of the product of s_{α_i} is not unique, whereas the injection $M_{s\chi_0}$ into M_{χ_0} is uniquely determined, the theorem gives relations between " chains " of the type described. In the general case the embedding is more complicated.

The relations between $M_{s\chi_0}$ may easily be shown by the following commutative diagram. The vertices of the diagram are numbered by the elements s of the Weyl group and correspond to the modules $M_{s\chi_0}$. If $s_1 < s_2$, then an arrow going from s_2 to s_1 is drawn. The mapping is defined by the embedding of $M_{s_2\chi_0}$ into $M_{s_1\chi_0}$. We obtain a commutative diagram. It is not difficult, using this diagram, to get in particular, a resolution of the finite dimensional representation by free $\mathscr{U}(u)$-modules.

The finite dimensional representation with highest weight $\chi_0 - \rho$ is of the form

$$M = M_{\chi_0}/\sum_{s \neq l} M_{s\chi_0}.$$

The theorems stated above and this diagram contain, in this case, the formulas of Kostant, Weyl's formulas for characters, the Borel-Weil theorem and the Harish-Chandra theorem concerning the left ideals of enveloping algebras.

3. *The ring of differential operators on the principal affine space and the generalisation of the Segal-Bargman representation to any compact group.*

Suppose G is a complex semisimple Lie group, $\mathscr{N}$ is the maximum unipotent subgroup, H — a Cartan subgroup. The manifold $A = \mathscr{N}\backslash G$ is called the principal affine space of the group G. It is an algebraic quasi affine manifold. It is interesting to consider the ring $\mathscr{D}$ of regular differential operators on A. Suppose $f(g)$ ranges over all the regular algebraic functions (polynomials) on the group G. We will give a method allowing to construct for any such function a differential operator on A. Since H normalises $\mathscr{N}$, the transformation $g \to hg$ may be carried over to A (left translations [5]). Using these left translations we can assign to every element of the Lie algebra $\not{h}$ of the group H a differential operator on A. The commutative ring of differential operators on A generated by these operators will be denoted, following [20], by W_u. Suppose π is the natural map of G into A. Denote by π^* extension of the functions over A to functions over G induced by π. The operation π_*, mapping the functions on G into functions on A is less obvious and supplements, in our case, the operation of averaging the function over the subgroup. The construction of π_* is carried out in the following way.

Suppose $f(g)$ is a regular algebraic function on G. Consider it as the linear combination of matrix elements of finite dimensional irreducable representations in the basis of weight vectors H. Threw out all the elements of this sum except the summands corresponding to those matrix elements whose first index is the highest weight of the corresponding representations. Denote by $\pi_* f$ the function thus obtrained.

Suppose f is a fixed function on G. Define the operator f in the functions by the formula

$$\bar{f}(\varphi) = \pi_*(f\pi_*(\varphi))$$

THEOREM 1. — There exists an element $w \in W_u$ such that $w_0 f$ is a regular differential operator on A. Conversely, every regular differential operator on A may be represented in the form $\Sigma w_i . \bar{f_i}$, $w_i \in W_u$ where f_i are functions on G.

Suppose $\mathscr{K}$ is the quotient field of the W_u ring, $\mathscr{F}(G)$ is the ring of regular algebraic functions on G. The map constructed in theorem 1 may be expanded to the map

$$i : \mathscr{D} \bigotimes_{W_u} \mathscr{K} \to \mathscr{F}(G) \bigotimes_{\mathbb{C}} \mathscr{K}$$

THEOREM 2. — i is a linear space isomorphism over $\mathscr{K}$, compatible with the right translations by elements of G.

Note that the fact of the existence of an isomorphism of the spaces above was obtained earlier in a joint paper of A. A. Kirillov and the author [20].

For the group $SU(2)$ there exists an extremely useful realisation of the whole series of representations of this group due to Segal and Bargmann. This realisation is in the Hilbert space of analytic functions of two complex variables, square, integrable with weight $e^{-|z_1|^2-|z_2|^2}$. We will point out a generalisation of this construction for any compact Lie group.

Suppose K is a simply connected compact Lie group of rang r, G — its complexification, A — the principal affine space of the group G. Introduce the weight function $e^{-H(a)}$, $a \in A$. Suppose ρ_i is the i'-th fundamental representation of G, let ξ_i denote the vector of highest weight in ρ_i. Put

$$H_i(g) = (\rho_i(g)\xi_i, \rho_i(g)\xi_i),$$

where (,) is the scalar product in the space of the representation ρ_i invariant relative to K. It is clear that $H_i(g)$ is a function on A and we can then put

$$H(a) = \sum_{i=1}^{r} H_i(a).$$

Now consider the analytic functions on A which are square integrable with weight $e^{-H(a)}$. Call the Hilbert space of all these functions a " generalised Segal-Bargmann space ". The group K thus obtained acts on it in a natural way and the unitary representation thus obtained contains every irreducible one exactly once. Let us call any operator with polynomial regular algebraic coefficients a " differential operator on A ".

CONJECTURE. — The operator conjugate (in the generalised Segal-Bargmann space) with a regular differential operator is again a regular differential operator.

The involutions which arise in the ring of regular differential operators are far from trivial. Thus, for the case of $SU(n)$ the operator, say, conjugate with multiplication by a simple first order function, is a differential operator of the $(n - 1)$-st order. The techniques developed in the previous section apparently will turn out to be very useful in the study of the ring of differential operators on A, in particular, for the proof of the conjecture stated aboce. The fact of the matter is that the construction of the involution itself is most conveniently carried out in the terms developed there. Using this method the conjecture was checked for $SU(3)$.

We state another problem. Let the real form of the group G be given. Its unitary representation naturally gives rise to an involution in the enveloping algebra $\mathcal{U}(\mathcal{G})$. We must find all the possible extensions of this involution from $\mathcal{U}(\mathcal{G})$ to the ring of all the regular differential operators on A. In the simpliest examples these extended involutions correspond to series of unitary representations (of real groups) contained in the regular one. It would be interesting to list the involutions in the ring of regular differential operators on any quasiaffine algebraic manifold.

It would also be interesting to consider the factor space of the group G, not only over the maximal unipotent group, but also over any orispherical subgroup.

II. Integral geometry.

In this paragraph I will only consider one elementary example [17]. The derivation of the Plancherel formula for $G = GL(n, \mathbb{C})$ is based on the following problem in integral geometry. Denote by $\mathcal{N} \in G$ the set of all the upper triangular matrices with units on the diagonal. Suppose the function $f(x)$, $x \in G$ is given. Let

$$\varphi(x_1, x_2) = \int_{\mathcal{N}} f(x_1^{-1} z x_2) dz,$$

where x_1 and x_2 are any matrices. The problem is: given $\varphi(x_1, x_2)$ find $f(x)$. It suffices to solve the problem when $x = e$ is the unit matrix. We can assume that the fonction f is given on $\mathbb{C}^{n^2}$ and the equation $y = x_1^{-1} z x_2$ for fixed x_1 and x_2 defines in $\mathbb{C}^{n^2}$ a plane of dimension $\dfrac{n(n-1)}{2}$.

Now replace our problem with the following, at first glance meaningless, problem. Consider the space $H_{n^2,k}\left(k = \dfrac{n(n-1)}{2}\right)$ of all the k-dimensional planes in $\mathbb{C}^{n^2}$. For all $h \in H_{n^2,k}$ consider the function

$$\varphi(h) = \int_h f(x)dx.$$

We must now recover $f(x)$. In the paper [10] this problem is solved in the following manner. Using the function φ and its derivatives construct a differential (k, h) form $\mathscr{H}\varphi$ on the Grassman manifold $G_{n^2,k}$ of k-dimensional planes containing the point x. This form $\mathscr{H}\varphi$ is closed and the value of $f(x)$ is equal to $\int_{\gamma_0} \mathscr{H}\varphi$, where γ_0 is any cycle homologic to the set of all k-dimensional planes containing the point x and lying in a fixed $k+1$-dimensional plane passing through the point (Euler's cycle (*)). As to the integral over the other k-dimensional cycles in the basis of Schubert cells in $G_{n^2,k}$, it is equal to zero.

In our case the function $\varphi(x_1, x_2)$ is known not on the whole manifold $H_{n^2,k}$ but only on a certain submanifold. The submanifold of $H_{n^2,k}$ will from now on be called the " complex of k-dimensional planes ". The complex is called permissible if the form $\mathscr{H}\varphi$ on this complex is determined by the values of the function φ on this complex only. In the case when φ is given on a permissible complex we can recover $f(x)$ by using the formula

$$f(x) = C_\gamma \int_\gamma \mathscr{H}\varphi,$$

where γ is a cycle lying in the complex; thus to find C_γ it suffices to decompose the cycle γ over the Schubert cell basis. In our case the complex will consist of planes of the form $h_{x_1,x_2} = \{ y/y = x_1^{-1}zx_2 \}$ and has dimension n^2. It turns out to be permissible. The set of these planes of this complex which contain the point e has the necessary dimension k and forms a cycle. The coefficient of the Euler cycle is equal to $n!$ Considering the form $\mathscr{H}\varphi$ only on the complex, we will obtain the classical inversion formula

$$f(e) = [(2i)^k\pi^{2k}n!]^{-1}\int\prod_{q<p}\left(\frac{\partial}{\partial\delta_p}-\frac{\partial}{\partial\delta_q}\right)\left(\frac{\partial}{\partial\bar{\delta}_p}-\frac{\partial}{\partial\bar{\delta}_q}\right)\times\varphi(\mathscr{C}^{-1}\delta\mathscr{C})\,|_{\delta=e}\bigwedge_{q<p}d\mathscr{C}_{qp}\bigwedge_{q<p}d\overline{\mathscr{C}}qp.$$

Apparently one can obtain the Paley-Wiener theorem for $GL(n, \mathbb{C})$, in a similar manner; in other words, obtain conditions on φ, which imply the decrease of f at infinity. To do this we embed $GL(n, \mathbb{C})$ not into $\mathbb{C}^{n^2}$ but into $\mathbb{CP}^{n^2}$ and consider the problem as a projective problem of integral geometry (see [15]). Since in this case we can recover $f(x)$ in the points at infinity as well, the Paley-Wiener conditions will consist in the

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(*) Note that other problems of integral geometry give rise to integration over other cycles in $G_{n^2,k}$; see, for example [16].

following: the function f' and its derivatives (recovered by using φ) must be equal to zero at all the points of infinity.

III. **Cohomology of infinite algebras.**

0. *This part of the report contains results obtained jointly by D. B. Fuks and the author.*

We know how difficult it is to describe any reasonable category of representations. On the other hand, the problem of determining cohomology groups is a sumpler one. Here we list results about the cohomology of Lie algebras of vector spaces, which show that these cohomologies are reasonable, are not equal to zero and are not infinite dimensional.

Recall that the cohomology $H^*(\mathcal{G}; M) = \sum_q H^q(\mathcal{G}; M)$ of the topological algebra $\mathcal{G}$ with coefficients in the $\mathcal{G}$-module is defined as the cohomology of the complex $C(\mathcal{G}; M) = \{ c^q(\mathcal{G}; M), d^q(\mathcal{G}; M)\}$ where $c^q(\mathcal{G}; M)$ is the space of continuous skew-symetric q-linear functionals on $\mathcal{G}$ ranging over M, and the differential $d^q = d^r(\mathcal{G}; M)$ is defined by the formula

$$(d^qL)(\xi_1,\ldots,\xi_{q+1}) = \sum_{1\leqslant s<t\leqslant q+1} (-1)^{s+t-1}L([\xi_s, \xi_t], \xi_1,\ldots,\hat{\xi}_s,\ldots,\hat{\xi}_t,\ldots,\xi_{q+1}) - \sum_{1\leqslant s\leqslant q} (-1)^s\xi_s L(\xi_1,\ldots,\hat{\xi}_s,\ldots,\xi_{q+1}).$$

If M is a ring, and the operators on $\mathcal{G}$ are its differentials, then the complex $C(\mathcal{G}; M)$ has a natural multiplicative structure.

1. *Problems and examples.*

The main example of an infinitely dimensional Lie algebra will be the algebra of smooth vector fields on a smooth manifold.

Suppose M is a closed orientable connected smooth (*) manifold. Denote by $\mathfrak{A}(M)$ the Lie algebra of smooth tangent vector fields on M with Poisson brackets for commuting. The first of the problems considered is a follows. Define the cohomology ring $\mathfrak{H}^*(M) = H^*(\mathfrak{A}(M); \mathbb{R})$ of the algebra $\mathfrak{A}(M)$ with coefficients in the unit representation, i. e., in the field $\mathbb{R}$ of real numbers with a trivial $\mathfrak{A}(M)$-module structure. This ring obviously is a differential invariant of the manifold M. Looking ahead we shall say that the space $H^q(\mathfrak{A}(M); \mathbb{R})$ will turn out to be finite dimensional for any q (see [28]). The problem of computing the ring $\mathfrak{H}^*(M)$ is not as of yet completely solved.

We would like to point out the difference between the method of constructing invarients of manifolds by using objects of differential geometry (the Lie algebra of vector fields) and the usual method of constructing differential invariants. Whereas usually the differential form representing a Pontryagin of Chern class on the manifold X is built up from the individual object (by using the metric) on the manifold,

(*) By smooth we always mean of class C^∞.

in our case the invariants are constructed using the infinite dimensional set of all smooth vector fields on the manifold.

As an example consider the case when M is the circle S^1. We can show that the ring $H^*(S^1)$ is generated by a two-dimensional generator a and a three-dimensional generator, the two being related only by the skewsymetry condition.

Further the generators $a \in \mathfrak{H}^2(S^1)$, $\mathscr{C} \in \mathfrak{H}^3(S^1)$ are represented by cocycles $A \in C^2(\mathfrak{A}(S^1); \mathbb{R})$, $B \in C^3(\mathfrak{A}(S^1); \mathbb{R})$ given by the formulas

$$A(f, g) = \int_{S^1} \begin{vmatrix} f'(x) & f''(x) \\ g'(x) & g''(x) \end{vmatrix} dx$$

$$B(f, g, h) = \int_{S^a} \begin{vmatrix} f(x) & f'(x) & f''(x) \\ g(x) & g'(x) & g''(x) \\ h(x) & h'(x) & h''(x) \end{vmatrix} dx$$

When the dimension of the manifold M increases the ring $\mathfrak{H}^*(M)$ becomes considerably richer; thus the ring $\mathfrak{H}^*(S^2)$ has 10 generators, and the ring $\mathfrak{H}^*(S^1 \times S^2)$, 20 generators (see [29]).

The cohomology of the Lie algebra of smooth vector fields is intimately connected with the cohomology of Lie algebras of formal vector fields. By a formal vector field at the point O of the space R^n we mean a linear combination of the form $\Sigma p_i(x_1, \ldots, x_n)e_i$ where $e_1, \ldots, e_n$ are the standard basis vectors of the space R^n and $p_i(x_1, \ldots, x_n)$, the formal power series with real coefficients in the coordinates $x_1, \ldots, x_n$ of the space. The set of formal vector fields is, in an obvious sense, a linear topological space, and a natural commutation operation transforms it into a topological Lie algebra. This algebra is denoted by W_n.

2. *The algebra of formal vector fields. The cohomology of the algebra W_n with coefficients in,R.*

In order to state the final result it is necessary to describe a certain auxilliary topological space X_n $(n = 1, 2, \ldots)$. Suppose $\mathscr{N} \geqslant 2n$ and let $p_i E(N, n) \to G(N, n)$ be the canonical $U(n)$ bundle over the (complex) Grasman manifold $G(\mathscr{N}, n)$. The usual (W-complex of the manifold $G(\mathscr{N}, n)$ has the following property: the $2n$-th skeleton $[G(\mathscr{N}, n)]_{2n}$ does not depend on $\mathscr{N}$ when $\mathscr{N} \geqslant 2n$. The inverse image of the set $[G(\mathscr{N}, n)]$ under the map p will be denoted by X_n.

The space X_1 is a three-dimensional sphere, the other spaces do not have such a simply visualised description. We have the following.

THEOREM 2.1. — For all q, n there is an isomorphism

$$H^q(W_n ; R) = H^q(X_n ; R).$$

Multiplication in the ring $H^*(W_n ; R)$ (as well as in the ring $H^*(X_n ; R)$) is trivial, i. e., the product of any two elements of positive dimension is equal to zero.

The cohomology of the space X_n may be computed by using standard topological methods. For example, it is trivial for $0 < q \leq 2n$ and for $q > n(n + 2)$.

Theorem 2.1 is the central result of the article [30]. Its proof uses a somewhat modified version of the Serre-Hoschild spectral sequence [31] corresponding to the subalgebra of the algebra W_n, generated by the elements $x_i e_j$ (*); this subalgebra is isomorphic to $\mathscr{G}l(n, R)$. Beginning with the second member, this spectral sequence turns out to be isomorphic to the Leray-Serre spectral sequence of the bundle $X_n \to [G(\mathscr{N}, n)]_{2n}$ with fibre $U(n)$.

It turns out also that each element $\alpha \in H^q(W_n; \mathbb{R})$ is represented by such a cocycle $A \in C^q(W_n; \mathbb{R})$, that $A(\xi_1, \ldots, \xi_q)$ depends only on the 2-jets of formal vector fields $\xi_1, \ldots, \xi_q$ (see [30]).

To study the cohomology of W_n with coefficients in other modules (and to describe those modules) it is important to know the structure of the subalgebras

$$\ldots \subset L_k \subset \ldots \subset L_0 \subset W_n$$

where L_k consists of vector spaces whose components are series without terms of power less than or equal to K.

The relation between the cohomology of the algebras W_n and L_0. The following general fact is easily generalised to the case of the cohomology of infinite dimensional Lie algebras.

Suppose B is an subalgebra of Lie algebra A; M – some B-module; $\hat{M}$ – an induced A-module (i. e. $\hat{M} = \mathrm{Hom}_{[B]}(M, [A])$ where $[A]$, $[B]$ are enveloping algebras for A, B). Then

$$H^*(A; \hat{M}) = H^*(B; M).$$

We will apply this statement in the case when M is a tensor representation of the algebra L_0 (i. e. a finite dimensional representation obtained from the representation of the algebra $\mathscr{G}l(n; R)$ by means of the projection $L_0 \to L_0/L_1 = \mathscr{G}l(n; R)$). At the same time the induced representation $\hat{M}$ of the algebras W_n is none other than the space of the corresponding formal tensor fields. For example, if $M = R$ is the unit representation of the algebra L_0, then $\hat{M}$ is the space $F(R^n)$ of formal power series in n variables with the natural action of the algebra W_n; if M is the space $\Lambda^r(R^n)'$ of skewsymetric r-linear forms in R^n, then $\hat{M}$ is the space Ω^r of formal exterior differential forms of r order in R^n.

The cohomology of the algebra W_n with coefficients in the spaces of formal exterior differential forms. The space

$$H^*(W_n, \Omega^*) = \sum_{q,\mathfrak{A}} H^q(W_n; \Omega^r)$$

is obviously a bigraduated algebra (over R), isomorphic, as we just found out, to $H^*(L_0; \Lambda^*(R^n)')$.

THEOREM 2.2. — The bigraduated ring $H^*(W_n; \Omega^*) = H^*(L_0; \Lambda^*(R^n)')$ is multiplicatively generated by $2n$ generators

$$\begin{array}{lll} \rho_i \in H^{2i-1}(L_0; \Lambda^0(R^n)') & & (i = 1, \ldots, n) \\ \tau_i \in H^i \quad (L_0; \Lambda^i(R_n)') & & (i = 1, \ldots, n) \end{array}$$

These generators are connected only by the following relations $\rho_i \rho_k = -\rho_k \rho_i$; $\rho_k \tau_i = \tau_i \rho_k$; $\tau_i \tau_k = \tau_k \tau_i$; $\tau_1^{i_1} \tau_2^{i_2} \ldots \tau_n^{i_n} = 0$ if $i_1 + 2i_2 + \ldots + ni_n > n$.

(*) $i, j = 1, \ldots, n$.

In particular, the ring $H^*(L_0; R) = H^*(L_0; \Lambda^0(R^n)') = H^*(W_n; F(R^n))$ is an exterior algebra in generators of dimension $1, 3, 5, \ldots, 2n-1$. i. e.

$$H^*(W_n; F(R^n)) = H^*(\mathfrak{gl}(n, R); R).$$

Moreover,

$$H^q(L_0; \Lambda^r(R^n)') = \begin{cases} 0 & \text{where} \quad q < r \\ H^r(L_0; \Lambda^r(R^n)') \otimes H^{q-r}(\mathfrak{gl}(n, R); R) & \text{where} \quad q \geqslant r \end{cases}$$

while the dimension of the space $H^r(L_0, \Lambda^r(R^n)')$ is equal to the number of ways in which the number r may be represented as the sum of natural numbers.

The computation of the cohomology of L_0 with coefficients in the tensor representation reduces to the computation of the cohomology of the algebra L_1 with coefficients in $\mathbb{R}$. In a similar way for jets, to the cohomology of L_k with coefficients in $\mathbb{R}$.

Apparently the following statement holds.

CONJECTURE. — For any n the spaces $H^q(L_k; R)$ are finite dimensional.

For $n=1$ the dimension of the space $H^q(L_k; R)$ equals $C_q^{k-1} + C_{q+1}^{k-1}$, $q, k = 0, 1, \ldots$).

Using previously mentioned results to compute the cohomology of the algebra L_0 with tensor coefficients we can deduce that the classes of cohomology of the algebra W_n (even W_1) with coefficients in tensor fields is not always representable by cocycles depending only on 2-jets of their arguments (in contrast with the cases of constant and skewsymetric coefficients).

We have been unsuccessful, so far, in computing the cohomology $H^*(A, \mathbb{R})$ for other Cartan algebras. Note that all these cohomologies are connected with very important standard complexes. For this complex consists of the polynomials $P(\alpha_1, \ldots, \alpha_q); (\beta_1, \ldots, \beta_q)$, $\alpha_i \in \mathbb{R}^n$, $\beta_i \in (\mathbb{R}^n)'$; the polynomial P is skewsymetric under the simultaneous interchange of α_i, β_i with α_j, β_j. The differential is given by the formula

$$\begin{aligned} dP(\alpha_1, \ldots, \alpha_{q+1}; \beta_1, \ldots, \beta_{q+1}) \\ = \Sigma(-1)^{s+t}(\alpha_s, \beta_t) - (\alpha_t, \beta_s))P(\alpha_f + \alpha_t, \alpha_f, \ldots, \hat{\alpha}_f, \ldots, \hat{\alpha}_t, \ldots; \\ \beta_f + \beta_t, \beta_1, \ldots, \hat{\beta}_f, \ldots, \hat{\beta}_t, \ldots, \beta_{q+1}). \end{aligned}$$

Usually, the infinite dimensional Lie algebras which arise in the formal theory are factor subcomplexes of this complex.

3. *The algebra of smooth vector fields. Cohomology with coefficients in R.*

Suppose M is a compact connected orientable smooth n-dimensional manifold without boundary, $\mathfrak{A}(M)$ – the Lie algebra of smooth tengent yest fields on M. In the standard complex $C(M) = \{ C^a(M) = C(\mathfrak{A}(M); R)d^a \}$ we introduce a filtration $0 = C_0(M) \subset C_1(M) \subset \ldots \subset C(M)$ where $C_k(M) = \{ C_k^a(M) \}$ is a subcomplex of the complex $C(M)$, defined in the following way. A cochain $L \in C^a(M)$ belongs to $C_k^a(M)$ if it equals zero on any C^a the vector fields $\xi_1, \ldots, \xi_q$ such that for any k points of the manifold M one of the fields $\xi_1, \xi_2, \ldots, \xi_q$ equals zero in the neighbourhood of each of these points. For example, $C_0^a(M) = 0$; $C_1^q(M)$ consists of such cochains L that $L(\xi_1, \ldots, \xi_q) = 0$ when the supports of the fields $\xi_1, \ldots, \xi_q$ are pairwise nonintersecting; $C^{a}_{q-1}(M)$ consists of such cochains L that $L(\xi_1,\ldots,\xi_q)=0$ when the supports of the fields $\xi_1,\ldots,\xi_q$ have no common intersection to all of them; $C^{a}_{k}(M)=C^{a}(M)$ when $k\geqslant q$. It is clear that $C_k(M)$ for all k is a subcomplex of the complex $C(M)$ and that $C^{q}_{k}(M)C^{r}_{l}(M)\subset C^{q+r}_{k+l}(M)$.

To compute the cohomology of the factor complex $C_k(M)/C_{k-1}(M)$ we have defined a spectral sequence, the first term of which may be expressed by using the cohomology of the manifold M and the algebra W_n. A special role is played by the complex $C_1(M)$. This complex we shall call a diagonal complex.

CONJECTURE. — The image of the cohomology of the diagonal complex $C_1(M)$ in $\mathfrak{H}^*(M)$ under the embedding $C_1(M)\to C(M)$ multiplicatively generates all of the ring $\mathfrak{H}^*(M)$. In particular the ring $\mathfrak{H}^*(M)$ is always finitely generated.

Remark. — This is true for the second term of the spectral sequence.

Let us describe a spectral sequence which converges to the cohomology of the diagonal complex. It arises in connection with two different filtrations of the diagonal complex of the manifold. In order to describe the first filtration, note that the q-cochains of the diagonal complex $C_1(M)$ are determined by distributions (more precisely, by the generalized sections of a certain fibre bundle) on M^q which are supported by the diagonal. The m-th term $C^{q}_{1,m}$ of the first filtration consists of those distributions which have an order (relative to Δ) less than or equal to m.

To define the second filtration fix a triangulation of the manifold

$$M=M_n\supset M_{n-1}\supset\ldots\supset M_0$$

where M_i is the i-dimensional skeleton, and the m-th term $C_{1,m}$ of the filtration consider those q-cochains which are realised by distributions whose support is $M_m\subset\Delta$.

Knowing the cohomology of W_n can construct a spectral sequence which allows us to compute the cohomology of the diagonal complex.

THEOREM 3.1. — There exists a spectral sequence $\mathscr{E}=\{E^{p,q}_r, d^{p,q}_r\}$ which converges to the cohomology of the diagonal complex $\mathfrak{H}^*(M)$ such that

$$E^{p,q}_r=H^{p+n}(M)\otimes H^q(W_n\,;R);$$

$E^{p,q}_r$, in particular, can be different from zero only when $-n\leqslant p\leqslant 0$.

Let us clarify the operation of " globalizing " the formal cohomology: construct a mapping of the space $E^{-r,q+r}_r=H^{n-r}(M)\otimes H^{q+r}(W_n, R)$ into $C^{q}_{1}(M)$. This mapping is not uniqual determined: it depends on the choice of the system of local coordinates on M. Suppose $\Gamma=\{U_1,\ldots,U_{\mathscr{N}}\}$ is a coordinate covering of M with coordinates $y_{k_1},\ldots,y_{k_n}$ on U_i and $\{\rho_i\}$ is a decomposition of unity consistent with this covering. In order to construct the element $\mathscr{I}(a\otimes\Psi)(a\in H^{n+r}(W_n, R), \Psi\in H^{n-v}(M))$ find a cochain $\alpha\in C^{n+v}(W_n\,;R)$ representing the closed form ω from the class Ψ. Set

$$\mathscr{I}(a_n\otimes\Psi)(\xi_1,\ldots,\xi_q)=\int_M \omega\Lambda[\sum_{k=1}^{N}\rho_k\varphi(\alpha, U_k\,;\xi_1,\ldots,\xi_q)]$$

where $\varphi(\alpha, U_i; \xi_1,\dots, \xi_q)$ is a form on U_k, which equals

$$\sum_{1 \leqslant i < \dots < i_r \leqslant n} \alpha(\xi_1(u, U_k), \dots, \xi_q(u, U_i), e_{k,i_1 \dots} e_{k,i_r}) \times dy_i \wedge \dots \wedge dy_{i_v}$$

at the point $u \in U_i$, where the ξ_i are considered as a formal field in the neighbourhood of the point u under the coordinates y_{k_i}. The theorem is proved in [29] (statement 1.4).

The cohomology with coef cients in the spaces of smooth sections of smooth vector bundles. Suppose A is a finite dimensional $GL(n, R)$ module and suppose M is a smooth connected manifold (we do not assume M either orientable, or compact, or without boundary). Denote by α the vector bundle over M with fiber isomorphic to A, induced by the tangent bundle and by means of the representation of the group $CL(n, R)$ in A. By $\mathscr{A}$ denote the space of smooth sections of the fiber bundle α. The space $\mathscr{A}$ has an obvious $\mathfrak{A}(M)$ module structure. Our goal is the study of the cohomology of the algebra $\mathfrak{A}(M)$ with coefficients in the $\mathfrak{A}(M)$ module $\mathscr{A}$.

In the complex $C(M; A) = \{ C^a(\mathfrak{A}(M); \mathscr{A}); d^a \}$ we will introduce a filtration similar to the one considered above for $C(M)$. We shall say that the cocycle $L \in C^q(\mathfrak{A}(M); \mathscr{A})$ has filtration no greater than k if the section $L(\xi_1,\dots, \xi_q)$ of the bundle α is equal to zero for any point $x \in M$ with the following property: for any points $x_1,\dots, x_k \in M$ one of the vector fields $\xi_1,\dots, \xi_q$ equals zero in the neighbourhood of each of the points $x_1,\dots, x_k, x$.

The space of q-dimensional cocycles which have filtration no greater than k is denoted by $C^q_k(\mathfrak{A}(M); \mathscr{A})$. It is clear that $C_k(M; \mathscr{A}) = \{ C^q_k(\mathfrak{A}(M); \mathscr{A}) \}$ is a subcomplex of the complex $C(M; \mathscr{A})$.

The subcomplex $C_0(M; \mathscr{A})$ is called " diagonal ". We denote it by $C_\Delta(M; \mathscr{A})$.

THEOREM 3.5. — We have the following spectral sequence $\{ E^{p,q}_r, d^{p,q}_r \}$ which converges to $\mathfrak{H}^*_\Delta(M; \mathscr{A})$ and is such that $E^{p,q}_r = H^p(M; R) \otimes H^q(L_0; A)$. In the multiplicative case the spectral sequence is a multiplicative one and the isomorphism considered above is an isomorphism of rings.

CONJECTURE. — $H^*_\Delta(M, \mathscr{A}) = H(T_M, R) \otimes \mathrm{Hom}_{CL(n)}(A, H^*(L, R))$ where T is the principal $U(n)$ bundle over M induced by the complexification of the tangent bundle.

This conjecture has been proved in the case when $A = \Lambda^q$ is the exterior power of the standard representation. The case $q = 0$ was independently studied by Locik [33].

In the end of this part of the report I would like to introduce a general concept of formal differential geometry. It arises when one formalises and generalises the methods of construction of Pontryagin and Chern classes (by means of metrics and connections); also in the expression of the index of a differential operator in terms of the symbol and the metric of the manifold.

Suppose we have an algebra W_n of formal vector fields. Consider the jet space and, in it, a invariant algebraic submanifold X. Examples of such manifolds are the space of all symmetric tensors of rang 2, the set of all affine connections.

Let us define the complex $\Omega(X)$. Any rational map of X into the complex of formal differential forms will be called a chain of $\Omega(X)$, the differential will be obtained by differentiation in the image. Set $\Omega(X) = \mathrm{Hom}(X, \Omega)$, where Ω is the complex of formal differential forms, and call the maps of the rational cohomology of $\Omega(X)$-generalised Chern classes. It can be shown, in the case when X is the manifold of symmetric tensors of rang 2, that they coincide with Pontryagin classes ($q < n$).

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