Proceedings of the International Congress of Mathematicians
Helsinki, 1978

Control under Incomplete Information and Differential Games

N. N. Krasovskii

Certain problems of control theory under incomplete information may be formalized within the framework of differential games. This report will be devoted to one such formalization developed by the author and his students. The size of this report leaves no opportunity for the discussion of many valuable contributions due to other authors in this field. I would only like to mention that our investigations are related to those of Bellman, Bensoussan, Boltyansky, Breakwell, Chernousko, Elliot, Fleming, Friedman, Gamkrelidze, Ho, Isaacs, Kalton, Lions, Markus, Mischenko, Nikolskii, Pontriagin, Pschenichnyi, Roxin, Varaiya, Young and certain other investigators in adjacent fields.

1. Let us first give an informal description of our problems. We will consider systems formed of the controlled plant, of the controller and of the environment. The current state of the plant is determined by its state variable $x[t]$. The evolution of $x[t]$ is described by a differential equation. The action of the controller on the plant will be named as the control and denoted by a minor u. The action of the environment will be called as the disturbance and denoted by a minor v. The accessible information on the current state of the system will be given by a certain informational variable $y[t]$ that is related in a certain way to $x[t]$ and $v[t]$. In particular it may be $y[t]=x[t]$.

In our case of uncertain information the values of the disturbance $v[\tau]$ are unknown in advance. At time t we are informed only of the domains $Q(\tau)$, $\tau \geqslant t$, that will contain the future values $v[\tau]$. These domains $\{Q(\tau), \tau \geqslant t\}$ may be included in the informational variable $y[t]$.

2. We will consider problems of closed-loop control when the desired law of control U should assign the current action $u[t]$ on the basis of the accessible realization $y[t]$.

Assume a certain functional

$$\gamma = \gamma(x[\cdot]),\ x[\cdot] = \{x[t],\ t_0 \leqslant t \leqslant \vartheta\}, \tag{2.1}$$

for the process to be given. We will say that an optimal law is a law of control U^0 that gives a minimax

$$U^0 : \gamma^0 = \min_{U} \sup_{x[\cdot]} \gamma \tag{2.2}$$

where the minimum is taken over all the admissible laws of control, and the maximum over $x[\cdot]$ is determined by all the possible realizations of the uncertain factors, namely, of the disturbance $v[t]$. It is then convenient to treat the situation as a two-person game. In this game we select the law of control—our strategy U. The realization of the uncertain factors is determined by the second player which is in general a fictitious one.

3. Consider a model problem. Assume the plant to be a heat conductor in the form of a rod $0 \leqslant \xi \leqslant 1$ on the axis ξ. Assume the informational variable $y[t]$ to coincide with the state variable $x[t]$. This is the temperature distribution ζ:

$$y[t] = x[t] = \zeta(t, \cdot) = \{\zeta(t, \xi),\ 0 \leqslant \xi \leqslant 1\}. \tag{3.1}$$

The control u may be an action of heating concentrated at point $\xi = v[t]$, so that we have the standard heat equation

$$\frac{\partial \zeta}{\partial t} = a^2 \frac{\partial^2 \zeta}{\partial \xi^2} + u\delta(\xi - v[t]) \tag{3.2}$$

under certain boundary conditions. Certain restrictions on the control and on the disturbance are also given:

$$|u| \leqslant \lambda,\ v_* \leqslant v \leqslant v^*. \tag{3.3}$$

The following problem is possible. Suppose the initial condition $\{t_0, x_0\}$ is given. Among the admissible functions $U(t, y)$ that generate the control

$$u[t] = U(t, y[t]) \tag{3.4}$$

one is to specify a law of control U^0, that ensures a minmax

$$U^0\colon \min_{U} \sup_{v[\cdot]} \max_{0 \leqq \xi \leqq 1} |\zeta(\vartheta, \xi)| \tag{3.5}$$

where ϑ is a given instant of time. (A precise setting of the problem must of course include the description of the admissible functions U, v, ζ, etc.)

4. The problem may be made somewhat more complex. For example assume the coefficient a in the heat equation (3.1) and the current distribution $\zeta(t, \xi)$ to be unknown precisely. However at each time t the value $v[t]$ turns out to be known. Then the informational variable $y[t]$ may be the informational domain

$$y[t] = \{\zeta_*(t, \xi) \leqslant \zeta(t, \xi) \leqslant \zeta^*(t, \xi),\ a_*(t) \leqslant a \leqslant a^*(t),\ v[t]\} \tag{4.1}$$

where ζ^*, ζ_* and a^*, a_* are the results of an observation of the system until time t.

5. Let us return to the general case. The given concept of differential game has been developed for both ordinary and partial differential equations. It is natural that in the finite-dimensional case the results are more accomplished. We will therefore discuss the latter case in more detail. At the end of the report I will demonstrate how these results may be propagated to the case of infinite dimensions.

Assume the plant to be described by ordinary differential equation with given restrictions

$$\dot{x} = f(t, x, u, v),\ u \in \mathscr{P},\ v \in Q. \tag{5.1}$$

Here x, u, v are finite-dimensional vectors, the function f is continuous, while $\mathscr{P}$ and Q are compact. Assume the process to start at time t_0 from the state x_0.

The informational variable $y[t]$ may be the history

$$y[t] = x[\cdot]_t = \{x[\tau],\ t_0 \leqslant \tau \leqslant t\} \tag{5.2}$$

of the motion until time t, or $y[t]$ may be the pair $y[t] = \{x[\cdot]_t, v[t]\}$. Very often $y[t]$ may be the state $x[t]$ itself or $y[t]$ may be the pair $\{x[t], v[t]\}$.

6. The motion

$$x[\cdot] = \{x[t],\ t_0 \leqslant t \leqslant \vartheta\} \tag{6.1}$$

will be evaluated by the given functional

$$\gamma = \gamma(x[\cdot]),\ x[\cdot] \in C[t_0, \vartheta] \tag{6.2}$$

Further on, if there is no additional reservation, the functional γ will be assumed to be continuous in the space C of continuous functions. In many practical problems one may encounter for example the functional

$$\gamma = \min_t \left[\sigma(t, x[t]),\ t \in \theta \subset [t_0, \vartheta]\right]. \tag{6.3}$$

Here $\sigma(t, x)$ is a continuous function and θ is compact.

7. It is well known that one is incapable of presenting a good formalization of minmax problems for γ if one identifies strategies with functions $u(t, x)$ and $v(t, x)$ while treating $x[\cdot]$ as classical solutions of the equation

$$\dot{x} = f\big(t, x, u(t, x), v(t, x)\big). \tag{7.1}$$

Indeed in many cases the optimal strategies $u^0(t, x), v^0(t, x)$ could not be found among the functions that are suitable for a direct integration of equation (7.1).

8. Therefore we will assume the following generalized formalization. Let us name the pair $\{t, x\}$ as the position. Assume that we have selected a certain variety $\mathscr{L}$ of conditional probabilistic measures

$$\mu = \mu_v(B),\ B \subset \mathscr{P},\ v \in Q,\ \mu_v(\mathscr{P}) = 1. \tag{8.1}$$

A positional closed-loop $\mathscr{L}$-strategy U is a function

$$\mu = \mu_v(du|t, x) \tag{8.2}$$

that transforms the positions into measures from $\mathscr{L}$.

9. The strategy U generates Euler splines. These are the continuous solutions of the step-by-step equation

$$x_\Delta[t] = \int\limits_{[\tau_i, t]\times\mathscr{P}\times Q} f(\tau, x_\Delta[\tau], u, v)\mu_v(du|\tau_i, x_\Delta[\tau_i])\nu(d\tau, dv) + x_\Delta[\tau_i],\ \tau_i \leq t \leq \tau_{i+1}, \tag{9.1}$$

along a certain subdivision Δ with increment α:

$$\Delta = \{\tau_i\},\quad \tau_0 = t_0,\quad \tau_m = \vartheta,\quad \alpha = \max_i(\tau_{i+1} - \tau_i). \tag{9.2}$$

Here $\nu(d\tau, dv)$ is any measure

$$\nu = \nu(d\tau, dv) \quad \text{on} \quad [t_0, \vartheta]\times Q,\ \nu([\tau_*, \tau^*)\times Q) = \tau^* - \tau_*. \tag{9.3}$$

Here and further on all the measures are assumed to be Borel measures.

Our main assumption on the function f is that for any initial state $x_\Delta[\tau_i]$ in (9.1) and for any certain control $\eta = \mu_v \times \nu$ the solution of the equation (9.1) on the interval $\tau_i \leq t \leq \tau_{i+1}$ is unique. And all such program motions are assumed to be equibounded for each given position $\{\tau_i, x_\Delta[\tau_i]\}$.

The positional motion $x[t]$ is the limit

$$x[\cdot] = \lim x^{(k)}_{\Delta(k)}[\cdot],\ k \to \infty,\ \alpha^{(k)} \to 0, \tag{9.4}$$

of a certain sequence of Euler splines, that converges in the space C with $\alpha^{(k)}$ tending to zero.

10. In particular, a pure strategy is the measure μ (8.2) concentrated at point $u = u(t, x)$:

$$\mu(du|t, x) = \delta\big(u - u(t, x)\big)\, du.$$

Therefore it may be identified with a function $u(t, x)$. A counterstrategy is identified with a function $u(t, x, v)$: $\mu_v(du|t, x) = \delta\big(u - u(t, x, v)\big)\, du$. A mixed strategy is a function $\mu = \mu(du|t, x)$ weakly Borel in x.

11. Similarly but with substitution of u for v, μ for ν (and vice versa) and with a substitution of set $\mathscr{L}$ for a certain set K of conditional measures

$$\nu = \nu_u(B),\ B \subset Q,\quad u \in \mathscr{P}, \tag{11.1}$$

we may determine the strategies V for the second player.

12. Together with such positional closed-loop strategies we will consider the more general historical closed-loop strategies

$$\mu_v(du|x[\cdot]_t),\quad \nu_u(dv|x[\cdot]_t). \tag{12.1}$$

For a transition to these it is necessary in the former constructions to substitute position $\{t, x\}$ for the history $x[\cdot]_t$ of the motion until time t. If for any value c in which we are interested the inequality $\gamma(x[\cdot]) \leqslant c$ is equivalent to the condition $\{\tau, x[\tau]\} \in \mathcal{M}_c$, $\{t, x[t]\} \in \mathcal{N}_c$, $t_0 \leqslant t \leqslant \tau(x[\cdot]) \leqslant \vartheta$, where $\mathcal{M}_c$ and $\mathcal{N}_c$ are closed sets in the space $\{t, x\}$ we will make use of the positional strategies. In other cases we will make use of the historical strategies with no additional explanation whatever.

13. The starting position $\{t_0, x_0\}$ and the strategy U or the strategy V determine certain boundles $X(t_0, x_0, U)$, $X(t_0, x_0, V)$ of motions $x[\cdot] = \{x[t], t_0 \leqslant t \leqslant \vartheta\}$.

Assume the starting position to be given and the classes of strategies to be selected. We will formulate the following two problems that form our differential game.

The first problem is to select an optimal strategy U^0 that gives a minmax

$$U^0:\ \min_U \max_{x[\cdot]} \gamma(x[\cdot]) = c^0(t_0, x_0). \tag{13.1}$$

The second problem is to select an optimal strategy V^0 that gives a maxmin

$$V^0:\ \min_V \max_{x[\cdot]} \gamma(x[\cdot]) = c_0(t_0, x_0). \tag{13.2}$$

14. We will say that the class of $\mathcal{L}$-strategies U and the class of K-strategies V are coordinated (with respect to the function f) if for any possible position $\{t, x\}$ and for any vector s the following equality is true

$$\begin{aligned} &\min_{\mu\in\mathcal{L}} \max_{\nu} \int \langle s\cdot f(t, x, u, v)\rangle\, \mu_v(du)\times\nu(dv) \\ &= \max_{\nu\in K} \min_{\mu} \int \langle s\cdot f(t, x, u, v)\rangle\, \nu_u(dv)\times\mu(du). \end{aligned} \tag{14.1}$$

Here the symbol $\langle s\cdot f\rangle$ denotes a scalar product. In particular the classes {pure strategies—counterstrategies} and {mixed strategies—mixed strategies} are always coordinated.

15. One of the principal results is as follows.

THEOREM. *Assume that the classes of strategies $\{U\}$ and $\{V\}$ are coordinated. Then the differential game has a value*

$$\gamma^0(t_0, x_0) = c^0(t_0, x_0) = c_0(t_0, x_0) \tag{15.1}$$

and it has a saddle point—a pair of optimal strategies

$$\{\mu_v^0(du|t, x),\ \nu_u^0(dv|t, x)\}. \tag{15.2}$$

In particular, for any function f the game always has a saddle point in the classes of mixed strategies

$$\{\mu^0(du|t, x),\ \nu^0(dv|t, x)\}. \tag{15.3}$$

16. Assume the functional γ to be only lower semicontinuous. Then there exists an optimal strategy μ^0 and in general, only an optimalizing sequence ν_k^0, $k=1, 2, \ldots,$ that approximates the value γ^0. Such an example is given by certain problems with a functional

$$\gamma(x[\cdot]) = \tau(x[\cdot]), \tau(x[\cdot]) = \min (t: \{t, x[t]\} \in \mathscr{M}) \tag{16.1}$$

where $\mathscr{M}$ is a certain given closed set $\mathscr{M} = \{x \in \mathscr{M}(t),\ t_0 \leq t \leq \vartheta\}$. Naturally in the general case of functional γ the game may have no value γ^0. An example is given already by certain problems with functonal $\gamma(x[\cdot]) = \sigma(\tau(x[\cdot]), x[\tau(x[\cdot])])$. Here $\sigma(t, x)$ is a continuous function and the value $\tau(x[\cdot])$ is determined by the equality (16.1). In these cases one may indicate some additional sufficient conditions for the existence of a value of the game and of a saddle point.

17. The essence of the given formal theorem on the saddle point may be clarified with the aid of approximations. Suppose for example that the game is formalized in the pair of classes of positional strategies; {pure strategies—counterstrategies}. Then the pure optimal strategy $u^0(t, x)$ ensures an inequality

$$\gamma(x_\Delta[\cdot]) < \gamma^0(t_0, x_0) + \varepsilon \tag{17.1}$$

for any $\varepsilon > 0$ selected in advance. This is true for any Euler spline described by the equation

$$\begin{gathered} \dot{x}_\Delta = f(t, x_\Delta[t], u^0(\tau_i, x_\Delta[\tau_i]), v[t]), \\ \tau_i \leq t \leq \tau_{i+1}, \ v[t] \in Q, \end{gathered} \tag{17.2}$$

provided the increment α of subdivision Δ is sufficiently small: $\tau_{i+1} - \tau_i \leqq \alpha(\varepsilon)$, $\alpha(\varepsilon) > 0$. Here the measurable realization of the variable $v[t]$ is generated by the environment on the basis of the one or the other of its laws. In particular if the disturbance $v[t]$ will be formed on the basis of its optimal counterstrategy $v^0(t, x, u)$ with its own subdivision $\Delta^* = \{\tau_i^*\}$ that means that the motion $x_\Delta[t]$ will also satisfy the equation

$$\begin{gathered} \dot{x}_\Delta = f(t, x_\Delta[t], u[t], v^0(\tau_i^*, x_\Delta[\tau_i^*], u[t])), \\ \tau_i^* \leq t \leq \tau_{i+1}^*, \end{gathered} \tag{17.3}$$

where $u[t]$ is any measurable realization of the control, then the following inequality will be fulfilled

$$\gamma(x_\Delta[\cdot]) > \gamma^0(t_0, x_0) - \varepsilon \tag{17.4}$$

provided increment α^* of the subdivision Δ^* is sufficiently small: $\tau_{i+1}^* - \tau_i^* \leq \alpha^*(\varepsilon)$.

18. Unfortunately these approximations are unstable with respect to minor informational errors $\Delta x[t] = x^*[t] - x[t]$. Indeed, the inequalities (17.1), (17.4) determined above may be destroyed if the actual realizations $x_\Delta[t]$ are determined by the equations

$$\dot{x}_\Delta = f(t, x_\Delta[t], u^0(\tau_i, x_\Delta^*[\tau_i]), v[t]) \tag{18.1}$$

or

$$\dot{x}_{\Delta^*} = f(t, x_\Delta[t], u[t], v^0(\tau_i^*, x_{\Delta^*}^*[\tau_i^*], u[t]) \tag{18.2}$$

even in the case of arbitrary small errors Δx. And this occurs not only for our concept. An instability with respect to minor informational errors is typical for many well-known solutions of differential games.

19. However we may offer the following improvement of the approximations. To the primary plant $\mathscr{F}$ we add a certain model $\mathscr{H}$ whose current state may be characterized by a suitable variable $w[t]$. This model may be materialized in the actual circuit of control on some computer. The variable $w[t]$ is a guide that directs the motion $x[t]$ to the desired target. The relation between $x[t]$ and $w[t]$ is constructed on the basis of the stability theory. The motion $x[t]$ is governed by an appropriate substrategy $\mu_v(du|t, x, w)$. The model $\mathscr{H}$ is constructed on the basis of one of the formal models of the game of which we will speak in the sequel. Therefore, if we follow the terminology of a chess game, we will have a game "on two boards". In the plant $\mathscr{F}$ we are playing with nature while in the model $\mathscr{H}$ we are playing with ourselves. When this procedure is implemented we always achieve a stable procedure of control that ensures a suboptimal result (17.1) or (17.4) for the player that uses its guide $w[t]$.

20. Let us now discuss an approximation for the case of mixed strategies. A mixed positional strategy $\mu(du|t, x)$ in approximation generates already a random motion $x_\Delta[t]$ that satisfies a step-by-step equation

$$\dot{x}_\Delta = f(t, x_\Delta[t], u[\tau_i], v[t]), \quad \tau_i \leqslant t \leqslant \tau_{i+1}. \tag{20.1}$$

Here $u[\tau_i]$ is the result of a random test with probability distribution $\mu(du) = \mu(du|\tau_i, x_\Delta[\tau_i])$ on $\mathscr{P}$. Such a procedure of control based on anoptimal strategy μ^0, ensures the inequality

$$P(\gamma(x_\Delta[\cdot]) < \gamma^0(t_0, x_0) + \varepsilon) > \beta \tag{20.2}$$

for any $\varepsilon > 0$ and $\beta < 1$ selected in advance provided increment α of subdivision Δ is sufficiently small: $\tau_{i+1} - \tau_i \leqslant \alpha(\varepsilon, \beta)$, $\alpha(\varepsilon, \beta) > 0$. Here the symbol $P(\ldots)$ denotes the probability of the respective event.

The disturbance $v[t]$ may be formed in an arbitrary manner that may also allow an appropriate statistical interpretation.

If the disturbance $v[t]$ is formed on the basis of its optimal strategy $v^0(dv|t, x)$, with its own subdivision then the following inequality will be ensured

$$P(\gamma(x_\Delta[\cdot]) > \gamma^0(t_0, x_0) - \varepsilon) > \beta \tag{20.3}$$

provided the increment α^* of the subdivision Δ^* will be sufficiently small: $\tau^*_{i+1} - \tau^*_i \leqslant \alpha^*(\varepsilon, \beta)$, $\alpha^*(\varepsilon, \beta) > 0$.

It is important to note that each of the propositions (20.2) and (20.3) is true under the condition that the actions $u[t]$ and $v[t]$ are stochastically independent within minor intervals of time or at least that they are sufficiently weakly correlated. If we have a game with nature, then the a priori given assumption on the disturbance $v[t]$ in equation (20.2) seems to be a completely tolerable independent postulate.

However if we speak of a game between two intelligent players, each of which may select its mixed strategy with its own subdivision then this assumption cannot be taken as an independent postulate. It should then be founded. Indeed it may be well founded if we take that the current realizations $x_\Delta[t]$ are available for each player with sufficiently small errors Δx. Then the optimal strategies in appropriate schemes of control with a guide will ensure the inequalities (20.2) and (20.3) under the condition that the informational errors are sufficiently small and that the increments α and α^* are also sufficiently small. I would like to emphasize that all the approximations described here had been formulated and proved in precise terms. Here however due to a lack of space I was capable of giving only a partial and rather loose presentation of these topics.

21. The proof for the existence of saddle points for our game and the construction of control algorithms for the actual approximations are based on various formalized models of the game. One such model based on the limit motions $x[t]$ has been already described above. Let us now describe some other model. For determinicity we will further restrict ourselves to the case of mixed strategies. Let us consider a formalization based on quasistrategies. A quasistrategy $\mathscr{U}$ for the interval $[\tau, \vartheta]$ is a transformation

$$\{\mu_t(du), \tau \leqslant t \leqslant \vartheta\} = \mathscr{U}\{\nu_t(dv), t_* \leqslant t \leqslant \vartheta\} \tag{21.1}$$

that transforms conditional stochastic measures $\nu_t(dv)$ onto conditional stochastic measures $\mu_t(du)$. The transformation $\mathscr{U}$ must satisfy a condition of physical realizability. That is for any τ_* from the given interval $[\tau_*, \vartheta]$ the histories of the images μ_t, $t<\tau$, coincide provided the histories of the arguments ν_t, $t<\tau_*$, have already coincided.

The starting position $\{\tau_*, w_*\}$, the quasistrategy and the conditional measure ν_t, $\tau_*<t$, generate a quasimotion $w[t]$ that is a solution of the equation

$$\dot{w} = \int_{\mathscr{P}\times Q} f(t, w, u, v)\mu_t(du)\times\nu_t(dv), \quad w[\tau_*] = w_*, \tag{21.2}$$

where $\mu_t(du)$ is defined by condition (21.1).

The quasistrategy $\mathscr{V}$ is defined similarly with appropriate substitutions.

22. The formal model considered here is formed of two problems. Assume a certain initial history $w[\cdot]_{\tau_*}=\{w[\tau], t_0\leqslant\tau\leqslant\tau_*\}$ is given.

The First Problem is to select an optimal quasistrategy $\mathscr{U}^0$ that gives a minmax

$$\mathscr{U}^0\colon \min_{\mathscr{U}} \max_{\nu} \gamma(x[\cdot]) = c_*^0(w[\cdot]_{\tau_*}). \tag{22.1}$$

The Second Problem is to select an optimal quasistrategy $\mathscr{V}^0$ that gives a maxmin

$$\mathscr{V}^0\colon \max_{\mathscr{V}} \min_{\mu} \gamma(x[\cdot]) = c_0^*(w[\cdot]_{\tau_*}). \tag{22.2}$$

It has been proved that this formal game has a value γ^0_* and a pair of optimal strategies

$$\gamma^0_*(w[\cdot]_{\tau*}) = c^*_0 = c^0_*, \ \{\mathscr{U}^0, \mathscr{V}^0\}. \tag{22.3}$$

The main result here is that the value $\gamma^0(t_0, x_0)$ of the initial closed-loop game considered above is equal to the value $\gamma^*_0(w[\cdot]_{\tau_0})$ of this formal game for the same starting position $w[\cdot]_{t_0} = w[t_0] = x_0$.

23. If the optimal quasistrategy $\mathscr{U}^0$ and $\mathscr{V}^0$ has been determined then the closed loop strategy μ^0 and ν^0 is constructed in principle without a great difficulty. Moreover this formal model in terms of quasistrategies is very suitable for the construction of an actual model $\mathscr{H}$ in the control scheme with a guide $w[t]$. However the search for the optimal quasistrategies is complexified by conditions of physical realizability. Let us omit this condition say for the case of quasistrategies for the first player. We will obtain the operators Π:

$$\{\mu_t(du), \tau_* \leqslant t \leqslant \vartheta\} = \Pi\{\nu_t(dv), \tau_* \leqslant t \leqslant \vartheta\} \tag{23.1}$$

that will be named as the programs. Let us formulate the first open-loop problem. This problem is to select an optimal program that gives a minmax

$$\Pi^0\colon \min_{\Pi} \max_{\nu} \gamma(w(\cdot)) = c_*(w[\cdot]_{\tau*}). \tag{23.2}$$

Here the program motions $w(t)$ are the solutions of the equation (21.2) where $\mu_t(du)$ is determined by condition (22.1).

In general this open-loop problem is not equivalent with respect to the value c to a similar problem for quasistrategies. However one may indicate certain regularity conditions when the equality $c^0_*(w[\cdot]_{\tau_*}) = c_*(w[\cdot]_{\tau_*})$ is true. In these cases one may construct a closed-loop strategy $\mu^0(du|t, x)$ on the basis of solving some auxiliary open loop problems (23.2).

24. The solution

$$\{\mu^0_t(du), \tau_* \leqslant t \leqslant \vartheta\} = \Pi^0\{\nu^0_t(dv), \tau_* \leqslant t \leqslant \vartheta\} \tag{24.1}$$

of the open-loop problem (23.2) is determined under certain assumptions by a minmax condition

$$\begin{aligned}&\langle s(t)\cdot \int f(t, w^0(t), u, v)\mu^0_t(du)\times \nu^0_t(dv)\rangle \\ &= \min_{\mu} \max_{\nu} \langle s(t)\cdot \int f(t, w^0(t), u, v)\mu(du)\times \nu(dv)\rangle\end{aligned} \tag{24.2}$$

that corresponds here to the well-known maximum principle of Pontriagin. Here $w^0(t)$ and $s(t)$ are the solutions of certain ordinary differential equations, that very often turn out to be of the Hamiltonian type.

The main point in the regularity conditions is as follows. Let $S(w[\cdot]_{\tau_*})$ denote the set of all vectors $s(\tau_*)$, that correspond to all of the possible optimal solutions.

of the mentioned equation. Then the condition will consist in the feature that for any selection of the vector l the intersection

$$W_l^+ \cap \left(\bigcap_s W_s^-\right) \neq \emptyset, \quad s \in S(w[\cdot]_{\tau*}), \tag{24.3}$$

would be nonvoid. Here the symbols W_s^+ and W_s^- denote the semispaces

$$W_s^+ = \{w: \langle s \cdot w \rangle \geqslant \varkappa\},\ W_s^- = \{w: \langle s \cdot w \rangle \leqslant \varkappa\}$$
$$\varkappa = \min_{\mu} \max_{\nu} \int \langle s \cdot f \rangle \mu(du) \times \nu(dv). \tag{24.4}$$

Beyond such regular cases we have a great deal of very peculiar cases of the games. One may give some classification of these cases, and obtain the regular case of the first rank, of the second rank, etc., until infinity.

25. Thus in the regular cases the problem of synthesizing an optimal closed-loop strategy may be reduced to the solution of auxiliary open loop control problems on the basis of ordinary Hamiltonian equations. We have therefore arrived at a typical procedure of analytical mechanics. Another scheme of solving extremal problems that is also standard for analytical mechanics is related to the Hamilton–Jacobi partial equation. In our case this scheme leads to the partial equations of dynamic programming. Unfortunately the value γ^0 that must be a solution of this equation often turns out to be a nondifferentiable function of the position $\{t, x\}$ or of the history $x[\cdot]_t$ of the motion. It is known however that as a rule a transition to related stochastic games for systems with a minor Wiener noise yield a regularization of the value of the game. Within the given concept this appears in the following way. Again for the sake of determinicity we consider only the case of mixed strategies and for example only for the functional γ of type $\gamma = \sigma(x[\vartheta])$ where time ϑ is given, $\sigma(x)$ is a continuous function.

Consider a plant $\mathscr{H}_\lambda$ described by the Ito equation

$$dw = \int_{\mathscr{P} \times Q} h_\lambda(t, w, u, v)\mu(du \mid t, w) \times \nu(dv \mid t, w)\, dt + \lambda\, dz[t] \tag{25.1}$$

and a functional

$$\gamma_\lambda(w[\cdot]) = M\{\sigma_\lambda(w[\vartheta]) \mid t_0, w[t_0] = x_0\} \tag{25.2}$$

where $M\{\ldots\}$ stands for the mean value.

Here $z[t]$ is a nongenerate Wiener process, $\lambda > 0$ is a small parameter, the function $\sigma_\lambda(x)$ is bounded and sufficiently smooth. Moreover in a sufficiently large domain G uniform limit relations $\lim h_\lambda = f$, $\lim \sigma_\lambda = \sigma$, $\lambda \to 0$ are valid.

The mixed strategies are identified with conditional probability measures $\mu(du|t, w)$, $\nu(dv|t, w)$ that are weakly Borel in $\{t, w\}$. The random motions are the weak solutions of equation (25.1).

26. It is known that this game on the minmax-maxmin of the functional γ_λ (25.2) has a value $\gamma^0(t_0, x_0)$ and a saddle point $\{\mu_\lambda^0(du|t, w), \nu_\lambda^0(dv|t, w)\}$.

The value $\gamma_\lambda^0(t, x)$ is a smooth function and satisfies the well-known parabolical partial equation

$$\frac{\partial \gamma_\lambda^0}{\partial t} + \frac{\lambda^2}{2} \Delta \gamma_\lambda^0 + \min_\mu \max_\nu \int_{\mathscr{P} \times Q} \left\langle \frac{\partial \gamma_\lambda^0}{\partial x} \cdot h_\lambda \right\rangle \mu(du) \times \nu(dv) = 0 \tag{26.1}$$

with boundary condition $\gamma_\lambda^0(\vartheta, x) = \sigma_\lambda(x)$.

27. The principal result consists here in the limit relation

$$\lim_{\lambda \to 0} \gamma_\lambda^0(t_0, x_0) = \gamma^0(t_0, x_0). \tag{27.1}$$

Unfortunately, in general there are no analogous limit relations for the strategies. However if we know the optimal strategies μ_λ^0 and ν_λ^0 for the stochastic game with minor $\lambda > 0$, it is always possible to construct a control for the given plant with a stochastic guide $\mathscr{H}_\lambda$ so that for any $\varepsilon > 0$ and $\beta < 1$ selected in advance the inequality

$$P\big(\sigma(x[\vartheta]) < \gamma^0(t_0, x_0) + \varepsilon\big) > \beta \tag{27.2}$$

would be fulfilled for the first player or the inequality

$$P\big(\sigma(x[\vartheta]) > \gamma^0(t_0, x_0) - \varepsilon\big) > \beta \tag{27.3}$$

for the second player, provided the parameter λ, the informational errors Δx and the increment α are sufficiently small.

28. We will now pass to the discussion of systems with infinite dimensions. First of all, note that with no great difficulty the previous results may be propagated to systems with time lag

$$\dot{x} = f(t, x_{t-\varrho}[\cdot]_t, u, v) \tag{28.1}$$

where f is a functional of the history of the motion $x_{t-\varrho}[\cdot]_t = \{x[\tau], t-\varrho \leqslant \tau \leqslant t\}$. As a specific fact we note that under rather general assumptions the differential games for a functional differential system (28.1) are well approximated by appropriate games for finite-dimensional systems described by ordinary differential equations.

29. Further on the results are propagated to certain parabolical and hyperbolical systems under standard initial and boundary conditions. Here the assumptions on the admissible classes of parameters and spacial and boundary control actions are related to the conventional assumptions of the general theory of parabolical or hyperbolical systems and in particular, to the theory of optimal open-loop control for such systems. Here first of all I have in mind the investigations of the group led by Lions.

Some theorems were proved within the framework of our concept. These theorems concern the existence of saddle points for the respective closed-loop differential games. Respective formal models as well as algorithms for the construction of optimal closed-loop strategies have been developed. In particular these algorithms include those that are based on solving auxiliary open-loop problems. In the actual approximations the motions or Euler splines are understood to be the splines formed by generalized solutions of the corresponding equations for the intervals $\tau_i \leq t \leq \tau_{i+1}$. It is natural that the functional nature of the problem yields a lot of additional difficulties. However under certain assumptions we may overcome them. Here in each case the result depends greatly on the selection of an appropriate functional space. In many cases of parabolical systems for example the approximate motions are considered in the space $\mathscr{L}^{(2)}$ but the respective formal constructions were developed through special transformation in other appropriate spaces, say in the state space $\mathscr{H}^{(-1)}$.

For the parabolical and hyperbolical systems considered here it is also possible to achieve a good approximation by related game theoretic problems for finite-dimensional systems. Such problems of approximation sufficient tolerably may be solved as on the bases of the method of Galerkin as well as within difference schemes.

30. Let us return for example to the model of the heat conducting rod at the beginning of our report. The respective differential game has a value $\gamma^0(t_0, \zeta_0)$ and a saddle point in mixed strategies

$$\{\mu^0(du|t, \zeta(t, \cdot)), \nu^0(dv|t, \zeta(t, \cdot))\}. \tag{30.1}$$

This denotes that in the actual approximation schemes that are based on the equations

$$\frac{\partial \zeta_\Delta}{\partial t} = a^2 \frac{\partial^2 \zeta}{\partial \xi^2} + u[\tau_i]\delta(\xi - v[t]), \ \tau_i \leq t < \tau_{i+1}, \tag{30.2}$$

or

$$\frac{\partial \zeta_\Delta}{\partial t} = a^2 \frac{\partial^2 \zeta}{\partial \xi^2} + u[t]\delta(\xi - v[\tau_i^*]), \ \tau_i^* \leq t < \tau_{i+1}^*, \tag{30.3}$$

for the first or second player respectively we have that for any $\varepsilon > 0$ and $\beta < 1$ selected in advance the inequalities

$$P\left(\max_{0 \leq \xi \leq 1} |\zeta_\Delta(\vartheta, \xi)| < \gamma^0(t_0, \zeta_0(t_0, \cdot)) + \varepsilon\right) > \beta \tag{30.4}$$

and

$$P\left(\max_{0 \leq \xi \leq 1} |\zeta_\Delta(\vartheta, \xi)| < \gamma^0(t_0, \zeta_0(t_0, \cdot)) - \varepsilon\right) > \beta \tag{30.5}$$

would be fulfilled provided the increments α and α^* of the subdivisions Δ and Δ^* are sufficiently small. Here $u[\tau_i]$ and $v[\tau_i^*]$ are respectively the results of stochastic tests with distributions

$$\mu(du) = \mu(du|\tau_i, \zeta(\tau_i, \cdot)) \quad \text{on } \mathscr{P} \tag{30.6}$$

and

$$\nu(dv) = \nu(dv|\tau_i^*, \zeta(\tau_i^*, \cdot)) \quad \text{on } Q. \tag{30.7}$$

Here the results and the considerations of above that concern the similar finite-dimensional case still remain true.

31. Let us now discuss the case when the informational variable $y[t]$ is the informational domain $y[t] = G[t]$, $x[t] \in G[t]$ in the space $\{x\}$ that includes the actual state $x[t]$. The problem is now transferred to the one of controlling the evolution of these domains. For the construction of the laws for this evolution a respective theory of differential games of observation has been developed. This theory may be considered as a certain minmax analogy for the statistical filtering theory. The combination of dual closed-loop differential games of control and observation includes sufficiently general theorems for the existence of saddle points as well as certain methods of constructing of optimal strategies of observation and control. However a practical realization of the solutions is achieved here for more or less simple model problems. The investigations are more effective for the case when the domains $G[t]$ are convex. Then the problem may be reduced to differential games in one or another functional space that includes the support function $g[t, l]$ of these domains. In this form the problems may be placed within a sufficiently general framework of differential games for differential evolutionary systems.

References

1. N. N. Krasovskii and A. I. Subbotin, *Positional differential games,* Nauka, Moscow, 1974.
2. N. N. Krasovskii, *Mixed strategies in differential games,* Soviet Math. Dokl. **235** (1977). 519—522.
3. A. B. Kurzhanskii, *Control and observation under conditions of uncertainty,* Nauka, Moscow, 1977.
4. Yu. S. Osipov, *Closed-loop control in parabolic systems,* Prikl. Mat. Meh. **41** (1977), 195—201.
5. V. D. Batuhtin, *A programmed construction for the positional control,* Optim. Techn. IFIP Techn. Conf., Springer-Verlag, Berlin, 1975, 435—439.
6. A. G. Chentsov, *On a problem of tracking,* Soviet Math. Dokl. **226** (1976), 73—76.
7. A. V. Krjažimskii, *On the theory of positional differential games of convergence-evasion,* Soviet. Math. Dokl. **239** (1978), 779—782.

СССР, 620066, г. Свердловск, К—66,
ул. С. Ковалевской, 16
Институт математики и механики
УНЦ АН СССР