r-DIMENSIONAL INTEGRATION IN n-SPACE

HASSLER WHITNEY

1. Introduction. There are various elementary and fundamental questions in integration theory as applied to geometry and physics that are not covered by the modern theory of the Lebesgue integral. In particular, basic problems concerning integrals over domains of dimension less than that of the containing space, as functions of the domain, are largely untouched. We shall present here a general approach to this type of problem.[1]

For an example from physics, consider the flux through a surface S in Euclidean 3-space E^3. Cut S into small pieces $\sigma_1, \sigma_2, \cdots$; we find the flux through each σ_i, and add. Take a typical small piece σ, in the form of a parallelogram with vectors v_1, v_2 along two of its sides, and containing the point p. The flux $X(\sigma)$ through σ depends on p, the area of σ, and the direction of σ. If we change the direction of σ, $X(\sigma)$ varies; $X(\sigma)$ is approximately proportional (for small σ) to the cosine of the angle between the normal $v_1 \times v_2$ to σ and the direction of the flux through p. In particular, for a rotation through π, $X(\sigma)$ is replaced by its negative. Thus σ must be taken as oriented, say by ordering the vectors (v_1, v_2).

We may represent the flux by a vector function $\omega(p)$; then for small σ as above, $X(\sigma)$ is approximately $\omega(p)\cdot(v_1 \times v_2)$, and thus is linear and skew-symmetric in the vectors v_1, v_2 determining σ. More generally, for surface integrals in E^n, an integral $\int_S$ may be determined by naming a function $\omega(p)$ which, for each p, is a linear skew-symmetric function of v_1, v_2; that is, ω is a differential form, or a "2-form" for short. An r-dimensional integral may be determined similarly by an r-form.

Given $\omega(p)$, say with $r = 2$, the integral $\int_S \omega$ may be defined as follows. Cut S into small triangles (approximately) $\sigma_1, \cdots, \sigma_m$. S being oriented, let v_{i1}, v_{i2} be the vectors of two of the sides of σ_i, so that (v_{i1}, v_{i2}) orient σ_i like S. Let p_i be a point of σ_i. Then

$$\int_S \omega = \lim \frac{1}{2} \sum_i \omega(p_i; v_{i1}, v_{i2}),$$

taking the limit as the mesh of the subdivision of S approaches 0. The factor $1/2$ is due to the use of triangles instead of parallelograms. The usual Riemann type theory may be set up in this fashion, essentially without use of coordinate systems.[2]

[1] Some of the theory given here has appeared in notes by the author, *Algebraic topology and integration theory*, Proc. Nat. Acad. Sci. vol. 33 (1947) pp. 1–6, and *La topologie algébrique et la théorie de l'intégration*, in the volume *Topologie algébrique*, Colloques Internationaux du Centre National de la Recherche Scientifique, Paris, 1949.

[2] This was carried out by P. Olum in a Senior Thesis at Harvard, 1940. See also H. Federer, *An introduction to differential geometry*, mimeographed notes, Stenographic Bureau, Brown University, Providence, R. I. (see Math. Reviews vol. 10 (1949) p. 264), and A. Lichnerowicz, *Algèbre et analyse linéaires*, Masson, Paris, 1947.

2. The general problem. We now ask what a general theory of integration should look like. For a given "integrand" X, the integral is a function of the domain A; we shall write it as $X \cdot A$. The basic property in the usual theory is additivity:

$$\int_{P \cup Q} = \int_P + \int_Q \quad \text{if } P \cap Q = 0.$$

In the general case, we may wish to integrate over domains with self-intersections. For instance, we may desire the line integral over a curve C consisting, in succession, of the oriented arcs C_1, C_2, C_3, C_2, C_4. We clearly should have

$$\int_C = \int_{C_1} + 2\int_{C_2} + \int_{C_3} + \int_{C_4}.$$

Thus a general r-dimensional domain is an "r-chain" $A = \sum a_i \sigma_i^r$, with oriented r-cells σ_i^r, each having a coefficient a_i (which we may take as any real number). Now $\int_A = \sum a_i \int_{\sigma_i^r}$, i.e.,

$$X \cdot \sum a_i \sigma_i^r = \sum a_i (X \cdot \sigma_i^r), \tag{2.1}$$

and the integral is a linear function of r-chains, generalizing the additivity above.

Recall the Theorem of Stokes: an r-form ω with differentiable coefficients has an exterior derivative $\delta\omega$; given an $(r + 1)$-chain A, with boundary ∂A,

$$\int_A \delta\omega = \int_{\partial A} \omega. \tag{2.2}$$

If X is an integral (abstractly defined) over r-chains, we may *define* an integral δX over $(r + 1)$-chains by the corresponding formula

$$\delta X \cdot A = X \cdot \partial A. \tag{2.3}$$

If we wish an integral $X \cdot A$ to be expressible in terms of A and local properties of X, such as is the case if X is defined by a differential form, we cannot let A be too general. For instance, for $r = 1$, there is a curve C and a differentiable function f in the plane, such that the partial derivatives of f vanish at all points of C, and yet f is not constant in C.[3] The natural assumption here is that C should be rectifiable. A similar assumption is in order for $r > 1$; see §8 below.

When we have chosen what domains should be allowable for setting up r-chains, we must decide how general the allowable linear functions X should be. Without some restrictions, little theory could be obtained. For instance, for $r = 0$, a typical domain is a point p; then $X \cdot p = \phi(p)$ would be an arbitrary function. In the usual Lebesgue theory, bounded measurable functions as integrands form a special class of summable functions; yet they illustrate much of the basic theory. We might invent some kind of corresponding bound, which we

[3] See H. Whitney, *A function not constant on a connected set of critical points*, Duke Math. J. vol. 1 (1935) pp. 514–517.

call the "mass" $|X|$, of X. It turns out that assuming both $|X|$ and $|\delta X|$ are bounded leads to a very satisfactory theory, to be described below; see (4.2). In particular, any X can be represented by a differential form (§6). An important problem is to obtain general results under weaker hypotheses than the boundedness of $|X|$ and $|\delta X|$.

3. Polyhedral chains. We introduce integration theory in a space R as follows. First choose a set of elements, "r-chains", forming a linear space; introduce a norm in this space; when completed, this gives a Banach space $\mathbf{C}^r$. Then the space of "r-cochains" X is the conjugate space $\bar{\mathbf{C}}^r$ of $\mathbf{C}^r$. The function $X(A) = X \cdot A$ is the "integral" of X over A. By making the norm in $\mathbf{C}^r$ small, we enlarge the set of elements which can be in $\mathbf{C}^r$ (i.e., for which the norm exists); the corresponding norm in $\bar{\mathbf{C}}^r$ is then large, which restricts the elements occurring in $\bar{\mathbf{C}}^r$.

We shall remain mostly in Euclidean space E^n; later we consider briefly the case of manifolds and more general spaces. The norms in $\mathbf{C}^r$ and $\bar{\mathbf{C}}^r$ will depend on the metric of E^n; but the sets of elements in these spaces are independent of the metric.

Among r-chains in E^n we must certainly include polyhedral r-chains $\sum a_i\sigma_i^r$, each σ_i^r and its boundary cells being flat. Define the *mass* by

$$|\sum a_i \sigma_i^r| = \sum |a_i||\sigma_i^r| \tag{3.1}$$

(the σ_i^r non-overlapping), $|\sigma_i^r|$ denoting the r-dimensional volume of σ_i^r.

The mass is too large a norm for our purposes. For instance, for $r = 0$, using $|p| = 1$ shows that any bounded function would define an element of $\bar{\mathbf{C}}^0$. Consider the case $r = n$ for a moment. If we take a domain P and translate it by a small vector v, giving $Q = T_v P$, then $\int_P$ and $\int_Q$ are nearly the same. For a general r, we might require $\int_{\sigma^r}$ and $\int_{T_v\sigma^r}$ to be nearly the same, even though v need not lie in the r-plane of σ^r. In terms of chains, we may require the norm $|A|^\#$ to satisfy

$$|T_v\sigma^r - \sigma^r|^\# \leqq |v||\sigma^r|/(r+1). \tag{3.2}$$

This does not relate r-cells in different r-planes. To take care of this, assume that for any $(r+1)$-cell σ^{r+1}, whose boundary is an r-chain $\partial\sigma^{r+1}$, we have $|\partial\sigma^{r+1}|^\# \leqq N|\sigma^{r+1}|$ for some fixed N. There is a uniquely defined largest norm satisfying these conditions, and also $|A|^\# \leqq |A|$. We call it the *tight norm*. It is independent of N for N large.

With this norm the conjugate space corresponds exactly to what we formerly called the set of "tensor cochains" in E^n. (See §5 below.) Though these cochains X are easy to work with, they are not general enough for some purposes. For instance, f^*X, defined in §9, is not tightly Lipschitz.

The most important norm for our purposes is the *Lipschitz norm* $|A|^*$, intermediate in size between $|A|$ and $|A|^\#$. It is the largest norm satisfying the conditions

$$|\sigma^r|^* \leqq |\sigma^r|, \qquad |\partial\sigma^{r+1}|^* \leqq |\sigma^{r+1}|. \tag{3.3}$$

It is easy to find an explicit expression for $|A|^*$, as follows. Considering all polyhedral $(r+1)$-chains D, $|A|^*$ is the greatest lower bound

$$|A|^* = \operatorname{GLB}_D (|A - \partial D| + |D|). \tag{3.4}$$

For example, let σ be a segment of length a, and let σ' be $T_v\sigma$, $|v| = b$, b small. Let D be the parallelogram formed by carrying σ into σ', oriented so that $\partial D = \sigma' - \sigma + C$, where C is composed of the remaining short sides of D. Then $|D| \leqq ab$, $|C| = 2b$, and hence

$$|\sigma' - \sigma|^* \leqq |C| + |D| \leqq (a+2)b.$$

Of course $|\sigma' - \sigma| = 2a$; also $|\sigma' - \sigma|^\# = ab/2$ for b small.

Given A, taking $D = 0$ in (3.4) shows that

$$|A|^* \leqq |A|. \tag{3.5}$$

If $A^r = \partial B^{r+1}$, choosing D_2^{r+2} so that

$$|B - \partial D_2| + |D_2| < |B|^* + \epsilon$$

and setting $D_1 = B - \partial D_2$ gives

$$|A - \partial D_1| + |D_1| = |D_1| < |B|^* + \epsilon;$$

this proves

$$|\partial B|^* \leqq |B|^* \leqq |B|. \tag{3.6}$$

In particular, (3.3) is proved.

For some special dimensions, considering $aq - bp$ as a 0-chain and $q - p$ as a vector, we have

$$r = 0\colon\ |1q - 1p|^* = |1q - 1p|^\# \leqq |q - p|,$$

with the sign $=$ if $|q - p| \leqq 2$, and

$$r = n\colon\ |A^n|^* = |A^n|,$$

since there are no nontrivial $(n+1)$-chains in E^n.

If $A \neq 0$, then $|A|$, $|A|^*$, and $|A|^\#$ are all >0.

For any σ^r, $|\sigma^r| = |\sigma^r|^* = |\sigma^r|^\#$. In fact, $|A| = |A|^* = |A|^\#$ if $A = \sum a_i\sigma_i^r$, the σ_i^r are parallel and similarly oriented, and the a_i are $\geqq 0$.

4. Lipschitz cochains. A Lipschitz r-cochain in E^n (or in a subset R of E^n) is an element of the conjugate space $\bar{\mathbf{C}}^r$ of $\mathbf{C}^r$ (or of $\mathbf{C}^r(R)$, using chains in R only). The Lipschitz norm $|X|^*$ is the norm of X in $\bar{\mathbf{C}}^r$; the mass $|X|$, corresponding to $|A|$, is defined. The definitions are

$$(4.1) \qquad |X|^* = \mathop{\mathrm{LUB}}_{|A|^*=1} |X \cdot A|, \qquad |X| = \mathop{\mathrm{LUB}}_{|A|=1} |X \cdot A|.$$

hough $\mathbf{C}^r$ is separable, $\bar{\mathbf{C}}^r$ is not; hence $\mathbf{C}^r$ is not reflexive.

Note that $\delta\delta X = 0$; for $\delta\delta X \cdot A^{r+2} = \delta X \cdot \partial A = X \cdot \partial\partial A = 0$.

The following relation shows that $X \in \bar{\mathbf{C}}^r$ if and only if $|X|$ and $|\delta X|$ are nite:

$$(4.2) \qquad |X|^* = \max(|X|, |\delta X|).$$

o prove this, note first that $|X| \leqq |X|^*$, because of (3.5) and (4.1). Next, 2.3), (4.1), and (3.6) give, using any B^{r+1},

$$|\delta X \cdot B| = |X \cdot \partial B| \leqq |X|^* |\partial B|^* \leqq |X|^* |B|^*;$$

ence δX is a Lipschitz $(r+1)$-cochain, and $|\delta X| \leqq |X|^*$. This proves $\geqq$ ı (4.2). To prove the reverse inequality, we need merely show that for any olyhedral r-chain A,

$$(4.3) \qquad |X \cdot A| \leqq \max(|X|, |\delta X|)\, |A|^*.$$

iven $\epsilon > 0$, choose $D = D^{r+1}$ so that

$$|A - \partial D| + |D| < |A|^* + \epsilon.$$

hen if $C = A - \partial D$,

$$\begin{aligned} |X \cdot A| \leqq |X \cdot C| + |X \cdot \partial D| &\leqq |X|\,|C| + |\delta X|\,|D| \\ &\leqq \max(|X|, |\delta X|)\,(|A|^* + \epsilon), \end{aligned}$$

hich gives (4.3).

Consider the case $r = 0$. For any Lipschitz 0-cochain X, $w(p) = X \cdot p$ is a al-valued function. The relations

$$\begin{aligned} |w(p)| &= |X \cdot p| \leqq |X|\,|p| = |X|, \\ |w(q) - w(p)| &= |X \cdot (1q - 1p)| = |\delta X \cdot (pq)| \leqq |\delta X|\,|q - p| \end{aligned}$$

ıow that w is bounded and satisfies a Lipschitz condition. Conversely, any ıch w defines an X.

Now take $r = n$. With E^n oriented, $X \cdot Q$ is defined for polyhedral regions Q iented like E^n. Since $|X \cdot Q| \leqq |X|\,|Q|$, X is extendable to be an additive t function over measurable sets, satisfying the same inequality. Define the ullness" of a set Q by

$$(4.4) \qquad \Theta_n(Q) = |Q| / [\operatorname{diam} Q]^n.$$

y standard Lebesgue theory, there is a measurable function $D_X(p)$, $|D_X(p)| \leqq |X|$, such that

$$(4.5) \qquad X \cdot Q = \int_Q D_X(p)\, dp \qquad (Q \text{ oriented like } E^n).$$

Moreover, using sequences of cells $\sigma_1, \sigma_2, \cdots$ containing p and with

$$\Theta_n(\sigma_i) \geqq \eta > 0$$

for all i,

$$D_X(p) = \lim_{i \to \infty} X \cdot \sigma_i / | \sigma_i | \qquad \text{a. e. in } E^n. \tag{4.6}$$

5. The λ-norms. Suppose, instead of (3.3), we require

$$| \sigma^r |_\lambda^* \leqq | \sigma^r |, \qquad | \partial \sigma^{r+1} |_\lambda^* \leqq | \sigma^{r+1} | / \lambda. \tag{5.1}$$

We then obtain the *Lipschitz λ-norm,* with the property

$$| A |_\lambda^* = \operatorname*{GLB}_D (| A - \partial D | + | D | / \lambda). \tag{5.2}$$

The corresponding λ-norm for cochains satisfies

$$| X |_\lambda^* = \max (| X |, \lambda | \delta X |). \tag{5.3}$$

We obtain $| A |_\lambda^\#$ similarly, using $| v | \, | \sigma^r | / (r + 1)\lambda$ in (3.2). Define the *Lipschitz constant* of X by

$$\mathfrak{L}(X) = \text{LUB} \frac{| X \cdot (T_v \sigma - \sigma) |}{| \sigma | \, | v |}; \tag{5.4}$$

call X *tightly Lipschitz* (a "tensor cochain" in[1]) if this is finite. If this holds for the Lipschitz r-cochain X, then

$$| \delta X | \leqq (r + 1) \mathfrak{L}(X), \tag{5.5}$$

$$| X |_\lambda^\# = \max [| X |, (r + 1) \lambda \mathfrak{L}(X)]. \tag{5.6}$$

The sets of elements in the spaces $\overline{\mathbf{C}}_\lambda^r$ for various λ are the same; only the norms differ; similarly for $\overline{\mathbf{C}}_\lambda^{r\#}$.

Given σ^r, it is not hard to construct a tightly Lipschitz X vanishing outside an arbitrary neighborhood of σ, such that $| X | = 1$, $X \cdot \sigma = | \sigma |$. Then for λ small enough, $| X |_\lambda^\# = 1$. Using this, we may prove, for polyhedral A,

$$\lim_{\lambda \to 0} | A |_\lambda^\# = \lim_{\lambda \to 0} | A |_\lambda^* = | A |. \tag{5.7}$$

From this we may prove that $| A |$ is a lower semicontinuous function of polyhedral chains A in $\mathbf{C}^r$ (in fact, in any $\mathbf{C}_\lambda^r$ or $\mathbf{C}_\lambda^{r\#}$). For a general A in $\mathbf{C}^r$, *define* $| A |$ as the lower bound of $\lim\inf | A_i |$ for sequences of polyhedral chains A_i with $| A_i - A |^* \to 0$; it may be infinite. $| A |$ is still lower semicontinuous.

For $r = 0$, $| A |_\lambda^\# = | A |_\lambda^*$.

6. Lipschitz cochains and differential forms. We present here the theorem of Wolfe,[4] that the Lipschitz r-cochains in an open subset R of E^n correspond to

[4] J. H. Wolfe, *Tensor fields associated with Lipschitz cochains,* Harvard Thesis, 1948.

he differential forms in R satisfying certain conditions. Suppose X is given. .t any point p we wish to find a corresponding linear skew-symmetric function $(p;\, v_1, \cdots, v_r)$, or equivalently, a linear function $\omega(p)\cdot\alpha$ of contravariant -vectors α. The natural definition is the following. The set of points

$$p + \sum_{i=1}^{r} t_i v_i \qquad (0 \leqq t_i \leqq t,\, i = 1, \cdots, r)$$

ırms a parallelopiped σ_t, oriented by the ordered set $(v_1, \cdots, v_r)$. Set

$$\omega(p; v_1, \cdots, v_r) = \lim_{t \to 0} X\cdot\sigma_t/t^r, \tag{3.1}$$

this exists. (Cf. (4.6).) Then for almost all p in R, this exists and defines a near function $\omega(p)$; for fixed α, $\omega(p)\cdot\alpha$ is measurable. Also $|\,\omega(p; v_1, \cdots, v_r)\,| \leqq$ $X \,|\, |\, v_1 \,| \cdots |\, v_r \,|$. The same is true for $\delta\omega$, defined from δX. Moreover, for ll σ,

$$X\cdot\sigma = \int_\sigma \omega(p; e_1, \cdots, e_r)\, dp, \tag{3.2}$$

$_1, \cdots, e_r)$ being an orthonormal set in the plane of σ, oriented like σ.

The theorem is proved by first smoothing X by an averaging process (giving tightly Lipschitz cochain), in which case the corresponding form is more easily ound, and then passing to the limit. The reason $\omega(p)$ turns out to be linear may e illustrated for the case $r = 1$ as follows. Given vectors u, v, set

$$q_t = p + tu, \qquad q_t' = q_t + tv = p + t(u + v);$$

:t $\sigma_t = pq_tq_t'$. For small t, $\omega(p)\cdot tu = \int_{pq_t}\omega$ approximately, etc.; thus

$$\begin{aligned} |\,\omega(p)\cdot tu + \omega(p)\cdot tv - \omega(p)\cdot t(u+v)\,| &=_{ap} |\, X\cdot\partial\sigma_t \,| \\ &= |\,\delta X\cdot\sigma_t\,| \leqq t^2 \,|\, \delta X \,|\, |\, \sigma_1 \,| ; \end{aligned}$$

ividing by t and letting $t \to 0$ gives the result.

Conversely, suppose $\omega(p)$ is defined and linear a. e. in R, with $\omega(p)\cdot\alpha$ measrable for each α; suppose $|\,\omega(p)\,|$ is bounded as above, and $|\int_{\partial\sigma}\omega\,| \leqq N\,|\,\sigma\,|$ ır each $(r + 1)$-cell σ for which the integral is defined. Such a form we call a *ipschitz r-form* in R. (The conditions can be weakened; for instance, we need ıerely continuity in α of $\omega(p)\cdot\alpha$ for simple unit r-vectors α, not linearity.) Then ıere is a corresponding Lipschitz r-cochain X in R, for which (6.2) holds whenver the integral is defined.

Say ω_1, ω_2 are *equivalent* if for each α, $\omega_1(p)\cdot\alpha = \omega_2(p)\cdot\alpha$ a. e. in R. Then *ıe Lipschitz r-cochains in R correspond exactly to the classes of equivalent Lip- :hitz r-forms in R.*

Note that for any Lipschitz r-form ω in R, we may find the corresponding ipschitz r-cochain X, take δX, and find the corresponding $\delta\omega$, even if the comonents of ω are not differentiable; whenever they are, this definition of $\delta\omega$ agrees ith the analytic one.

A "simple" contravariant r-vector is one expressible as a product of r vectors. Define the "simple norm" $|\xi|_s$ of a covariant r-vector ξ by

$$(6.3)\qquad |\xi|_s = \text{LUB}\,|\xi\cdot\alpha|,\qquad \alpha \text{ simple},\qquad |\alpha| = 1.$$

Then for any X and corresponding ω,

$$(6.4)\qquad |X| = \text{essential LUB}_p\,|\omega(p)|_s.$$

7. General r-chains. Recall that $\mathbf{C}^r$ was the completion of the space of polyhedral r-chains; the new elements of $\mathbf{C}^r$ we call "general Lipschitz r-chains".

Consider first the case $r = 0$. Let R be a locally compact separable metric space. An additive set function Φ in R assigns to each Borel set Q a number $\Phi(Q)$, such that $\Phi(\sum Q_i) = \sum\Phi(Q_i)$ if the Q_i are disjoint. Any such set function defines uniquely a general Lipschitz 0-chain A_Φ, as follows. Let $R = R_1 \cup \cdots \cup R_m$ be a partition of R into disjoint Borel sets; choose $p_i \in R_i$, and set

$$(7.1)\qquad B = \sum\Phi(R_i)p_i;$$

this is an approximation to A_Φ by a polyhedral 0-chain, if the R_i are "small" enough (i.e. cut up Φ finely enough).

The mass $|A_\Phi|$ is the total variation of Φ. The expression for $|A_\Phi|^*$ is more complicated. Not all 0-chains are expressible in this manner; but the 0-chains described are dense in $\mathbf{C}^0$. For any Lipschitz 0-cochain X, corresponding to the function $\omega(p)$,

$$(7.2)\qquad X\cdot A_\Phi = \int_R \omega(p)\,d\Phi(p),$$

using the Lebesgue Stieltjes integral. The integral is defined for more general functions ω; on the other hand, $X\cdot A$ is defined for more general A.

Of course $\Phi(Q)$ is defined without regard to orientation properties. This corresponds to the fact that a 0-cell, i.e., a point, has a natural orientation. We may consider the theory of additive set functions and the Lebesgue Stieltjes integral as 0-dimensional integration; r-dimensional integration for $r > 0$ requires orientation properties.

Consider next n-chains in E^n. Recall the L^1-norm for summable functions F:

$$(7.3)\qquad |F| = \int_{E^n} |F(p)|\,dp.$$

If F is cellwise constant for some subdivision of E^n, it defines an n-chain A with $|A|^* = |A| = |F|$; this shows that summable functions define elements of $\mathbf{C}^n$; in fact, they give the whole of $\mathbf{C}^n$.

Take the case $n = 1$. If F is not only summable but differentiable, it is easy to show that $-dF/dx$ defines a 0-chain which is exactly the boundary (see below) of A_F. On the other hand, suppose

$$F_a(x) = \log|x|, \quad x < 0; \qquad F_a(x) = \log x + a, \quad x > 0;$$

hen $-dF_a(x)/dx = -1/x$, all $x \neq 0$. For each a, F_a defines a 1-chain A_a, nd the same function $-1/x$ (which is not summable) "corresponds" to ∂A_a.

The boundary ∂A of a general chain A may always be defined. For if A is lefined by the sequence $A_1, A_2, \cdots$ of polyhedral chains A_i, then

$$|\partial A_j - \partial A_i|^* \leqq |A_j - A_i|^*$$

y (3.6), and hence $\partial A_1, \partial A_2, \cdots$ is a Cauchy sequence, and defines ∂A.

Now take $0 < r < n$, in E^n. Let $\alpha(p)$ be any field of contravariant r-vectors, he components being summable functions. It may be shown that α defines niquely a general Lipschitz r-chain A_α, with the property that for any X, orresponding to ω,

7.4) $$X \cdot A_\alpha = \int_{E^n} \omega(p) \cdot \alpha(p)\, dp.$$

uch A_α we call "spread out" r-chains. They are dense in $\mathbf{C}^r$; this holds even ' we require the components to be analytic functions. (This holds in open subets of E^n.)

If the components $\alpha^{\lambda_1 \cdots \lambda_r}$ of α are differentiable, and

7.5) $$\beta^{\lambda_1 \cdots \lambda_{r-1}} = \sum_k \partial\alpha^{\lambda_1 \cdots \lambda_{r-1} k}/\partial x_k,$$

hen

7.6) $$\partial A_\alpha = (-1)^r A_\beta.$$

Define the "mass" of a contravariant r-vector α by

7.7) $$|\alpha|_m = \mathrm{GLB} \sum |\alpha_i|,$$

r expressions $\alpha = \sum \alpha_i$ of α as a sum of simple r-vectors. (Then the norms $\alpha|_m$ and $|\xi|_s$ of (6.3) correspond, considering the spaces of covariant and conavariant r-vectors as conjugate spaces of each other.) Then

7.8) $$|A_\alpha| = \int |\alpha(p)|_m\, dp.$$

'here seems to be no simple expression for $|A_\alpha|^*$.

8. Lipschitz r-chains. In the applications, one integrates commonly over omains which are not polyhedral (curved surfaces, etc.). The domains are enerally expressible as images of flat domains, under mappings f which are ifferentiable or piecewise differentiable. We shall assume merely that f satisfies Lipschitz condition: If ρ denotes distance,

3.1) $$\mathfrak{L}_f = \mathrm{LUB}\, \rho(f(p), f(q))/\rho(p, q) \text{ is finite.}$$

If f is a Lipschitz mapping of an r-dimensional polyhedron P, in which an chain A is given, into a metric space R, we call (A, f), or fA for short, a *Lip-*

schitz r-chain in R. We may as well take the cells of P as disjoint cells in E^r. With E^r oriented, A may be replaced by a summable function φ (see §7); φ defines a general r-chain $\tilde{\varphi}$ in E^r; we call $(\tilde{\varphi}, f) = f\tilde{\varphi}$ a *Lipschitz-Lebesgue r-chain* in R.

If R is a metric space satisfying certain conditions, Lipschitz chains in R can be used to set up integration theory. The mass $|A|$ and norm $|A|^*$ can be defined; see below. Particularly important is the case that R is a smooth manifold M; if a metric is not given, one may be introduced, and the spaces of chains and cochains (though not the norms) are independent of the metric (at least in compact subsets of M).

A Lipschitz-Lebesgue r-chain $f\tilde{\varphi}$ in E^n is a general Lipschitz chain, given by approximating φ by a cellwise constant function φ' and f by a simplicial mapping f' (thus defining an approximating polyhedral chain). The following continuity theorem holds: Given $f\tilde{\varphi}$ and numbers L and $\epsilon > 0$, there is a $\zeta > 0$ with the following property. For any φ' with $|\tilde{\varphi} - \tilde{\varphi}'|^* \leqq \zeta$ and for any f' with $\mathfrak{L}_{f'} \leqq L$ and $|f'(p) - f(p)| \leqq \zeta$ (all p), we have $|f'\tilde{\varphi}' - f\tilde{\varphi}|^* < \epsilon$. For given polyhedral $A = \tilde{\varphi}$, and L, keeping $\varphi' = \varphi$, we may take $\zeta = c\epsilon$ for some c ($c = L^r|A| + L^{r-1}|\partial A|$); but this is not possible in the general case.

A Lipschitz chain A, being a general chain, has a boundary ∂A; a Lipschitz $(r-1)$-form ω defines X and hence δX. General theory now gives Stokes' Theorem:

$$\int_{\partial A} \omega = X \cdot \partial A = \delta X \cdot A = \int_A \delta\omega.$$

It is always possible to represent a Lipschitz-Lebesgue chain in E^n as $f\tilde{\varphi}$ with f one-one in the carrier $\mathrm{Car}(\varphi)$ of φ, i.e., the set of points p where $\varphi(p) \neq 0$. For such a representation, the mass is given by the formula

$$|f\tilde{\varphi}| = \int |\varphi(p)|\,|J_f(p)|\,dp, \tag{8.2}$$

$J_f(p)$ being the Jacobian, which exists a.e. since f is Lipschitz (Rademacher's Theorem). For any representation, (8.2) holds with $\leqq$. In the proof, we make use of the existence of a tightly Lipschitz cochain X such that

$$|X| = 1, \qquad |f\tilde{\varphi}| - \epsilon < X \cdot f\tilde{\varphi} \leqq |f\tilde{\varphi}|, \tag{8.3}$$

for arbitrary $\epsilon > 0$.

Let $S_1, S_2, \cdots$ be a sequence of subdivisions of E^r with mesh $\to 0$. Then given $f\tilde{\varphi}$, if $\tilde{\varphi}_{ki}$ is the part of $\tilde{\varphi}$ in the cell σ_{ki} of S_k, we have $\sum_i \tilde{\varphi}_{ki} = \tilde{\varphi}$, and

$$\lim_{k\to\infty} \sum_i |f\tilde{\varphi}_{ki}|^* = |f\tilde{\varphi}|. \tag{8.4}$$

Given $f\tilde{\varphi}$ in E^n, with f one-one in $\mathrm{Car}(\varphi)$, and given $\eta > 0$ and $\epsilon > 0$, there is a $\zeta > 0$ with the following property. Let S be any simplicial subdivision of P of mesh $< \zeta$, whose simplexes σ_i satisfy $\Theta_r(\sigma_i) \geqq \eta$. Let φ' in σ_i be the verage of φ in σ_i, and let f' be the simplicial mapping which coincides with f at he vertices of S. Then

8.5) $$\big|\,|f\tilde{\varphi}| - |f'\tilde{\varphi}'|\,\big| < \epsilon.$$

Given a Lipschitz mapping f of a subset Q of E^r into a metric space R, define he "reduced Lipschitz constant"

8.6) $$\bar{\mathfrak{L}}_{f,Q} = \underset{\varphi}{\mathrm{GLB}}\, \mathfrak{L}_{f\varphi^{-1}(Q)}\,,$$

vhere φ is an affine volume-preserving mapping of E^r into itself, using f in Q nd φ in $\varphi^{-1}(Q)$. Define the "local constant" $\mathfrak{L}_f^*(p)$ of f at p as follows. Let $\rho_r a^r$ e the volume of the set $x_1^2 + \cdots + x_r^2 \leqq a$. Let $U_\zeta(p)$ denote the ζ-neighborhood f p. Given ζ and η such that $\eta < \rho_r\zeta^r$, set

8.7) $$\mathfrak{L}_f^*(p, \zeta, \eta) = \underset{Q}{\mathrm{GLB}}\, \bar{\mathfrak{L}}_{f,Q}\,, \qquad Q \subset U_\zeta(p), \qquad |Q| > \eta,$$

8.8) $$\mathfrak{L}_f^*(p) = \lim_{\mu\to 0}\ \liminf_{\zeta\to 0}\ \mathfrak{L}_f^*[p, \zeta, (1-\mu)\rho_r\zeta^r].$$

If f is a Lipschitz mapping of a measurable subset Q of E^r into E^n, then

8.9) $$|J_f(p)| = [\mathfrak{L}_f^*(p)]^r \text{ a. e. in } Q.$$

Now the set of $(\tilde{\varphi}, f)$, with certain equivalence relations, forms the space of ipschitz r-chains A in R. For any such A, considering the various expressions $\tilde{\varphi}, f)$ of A, define

8.10) $$|A| = \mathrm{GLB} \int |\varphi(p)|\, [\mathfrak{L}_f^*(p)]^r\, dp.$$

ctually (with some restrictions on R) we may choose $(\tilde{\varphi}, f)$ so that f is one-one 1 $\mathrm{Car}(\varphi)$; then $|A|$ equals the integral above. In particular, taking $\varphi(p) = 1$ ives the "r-volume" of the "rectifiable" manifold $f(\mathrm{Car}(\varphi))$ in R. The definitions gree with the previous ones if R is an open subset of E^n.

The norm $|A|^*$ is definable from $|A|$; then cochains may be introduced as efore.

9. Properties of Lipschitz mappings. Let f be a Lipschitz mapping of E^n into E^m. Then each polyhedral r-chain A in E^n is carried into a Lipschitz r-chain fA in E^n. The definition may be extended to general Lipschitz r-chains in E^n. 'he usual properties hold. In particular,

9.1) $$|fA| \leqq \mathfrak{L}_f^r |A|, \quad |fA|^* \leqq \max(\mathfrak{L}_f^r, \mathfrak{L}_f^{r+1})\, |A|^*.$$

Given X in E^m, f^*X in E^n is defined by $f^*X \cdot A = X \cdot fA$. Then $|f^*X| \leqq \mathfrak{L}_f^r |X|$, tc.

If ω is a Lipschitz r-form in E^m, it defines a Lipschitz r-cochain X, this defines f^*X, and this gives ω^* in E^n; call this $f^*\omega$. Recall that X defines δX and hence $d\omega$; also $\delta f^*X = f^*\delta X$; hence $\delta f^*\omega = f^*\delta\omega$. The usual analytic theory requires the differentiability of both f and ω for any parts of this last formula even to be defined.

Nevertheless, with the "Lipschitz" hypotheses used here, an analytic treatment of $f^*\omega$ can be given. We first note a fundamental difficulty: ω may fail to be defined in a set of measure 0 in E^m, and f might even map the whole of E^n into this set. Thus the first necessity is to enlarge or improve the definition of ω. This is done by determining the corresponding X, and finding a new ω' from X; then $\omega' = \omega$ a.e. (see §6). Then the usual analytic formula defines $f^*\omega'$ a.e. in E^n.

The proof requires the following approximation theorem. Let X be a Lipschitz r-cochain in E^m, and let p be any point at which the corresponding $\omega(p)$ exists and is linear (see §6). Then for any $\eta > 0$ and $\epsilon > 0$ there is a $\zeta > 0$ with the following property. Let σ be any oriented r-simplex with

$$\sigma \subset U_\zeta(p), \qquad \Theta_r((p) \cup \sigma) \geqq \eta; \tag{9.2}$$

if $\{\sigma\}$ denotes the r-vector defined by σ, then

$$| X\cdot\sigma - \omega(p)\cdot\{\sigma\} | \leqq \epsilon\, | \sigma |. \tag{9.3}$$

As a particular case, suppose $r = 1$, and X is the coboundary δY of a Lipschitz 0-cochain Y, Y corresponding to the real-valued function $w(p)$. Then the theorem gives the Rademacher Theorem on the total differentiability a.e. of the Lipschitz function w.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS., U. S. A.