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Geometric Applications of Algebraic K-theory

Algebraic K-theory has been one of the most important mathematical developments of the last two decades. In the reports [81], [68], [6] of past International Congresses of Mathematicians, emphasis was given to the algebraic aspects of the theory. In this report, I shall concentrate on its geometric applications. After all, the theory was initiated by Reidemeister [70], Franz [38], de Rham [24], and J. H. C. Whitehead [89] (see also [90]) who introduced in the 30's some invariants for solving geometric problems. The revival of the interest in these invariants in the early 60's, which were the seeds of algebraic K-theory (Milnor [62], Smale [76] and Kervaire's exposition on the S-cobordism theorem of Barden–Mazur–Stallings [54]) also arose from geometric considerations. At the end of this note, I shall make a few conjectures.

Due to limitation of space, I skip the Hermitian K-theory and Novikov's conjecture on higher signatures of closed aspherical manifolds altogether.

Some of the geometric problems dealt with here have very interesting and equally important Hermitian analogues. The interested readers might consult [30], [34], [88].

I. $K_1(A)$, $Wh_1(\pi)$, simple homotopy type and h-cobordism

Let A be an associative ring with unit 1. The group of all non-singular $n\times n$ matrices over A will be denoted by $\mathrm{GL}(n, A)$. Identifying each $M \in \mathrm{GL}(n, A)$ with $\begin{bmatrix} M & 0 \\ 0 & 1\end{bmatrix} \in \mathrm{GL}(n+1, A)$, we obtain inclusions $\mathrm{GL}(1, A) \subset \ldots \subset \mathrm{GL}(n, A) \subset \ldots$ The union is called the infinite *general linear group* $\mathrm{GL}(A)$. A matrix is *elementary* if it coincides with the identity matrix except for one off-diagonal entry. It was observed by J. H. C.

Whitehead [4, p. 226], [63, p. 359] that the subgroup $E(A) \subset \mathrm{GL}(A)$ generated by all elementary matrices is a perfect group and is precisely equal to the commutator subgroup of $\mathrm{GL}(A)$. We define

$$K_1 A = \mathrm{GL}(A)/E(A) \tag{1}$$

which may be viewed as a generalization of the determinant function for matrices. Let π denote a multiplicative group and $Z[\pi]$ the corresponding integral group ring. We have natural inclusions $\pm\pi \subset \mathrm{GL}(1, Z[\pi]) \subset \mathrm{GL}(Z[\pi])$, where $\pm\pi$ denotes the subgroup of (1×1)-matrices $(\pm g)$, $g \in \pi$. The cokernel $K_1(Z[\pi])/\mathrm{image}(\pm\pi)$ is called the *Whitehead group* $Wh_1(\pi)$. Clearly, $K_1(A)$ and $Wh_1(\pi)$ are covariant functors of rings and groups to Abelian groups respectively.

Whitehead [90] introduced the notion of *simple homotopy* which is finer than *homotopy*. Let L_0 and L_1 be finite CW-complexes such that L_1 is obtained from L_0 by attaching a k-cell e_k to L_0 along a $(k-1)$-cell $e^{k-1} \subset \partial e^k$. Call this procedure *simple expansion* and the reverse procedure *simple collapsing*. Simple homotopy is the equivalence relation generated by simple expansion and simple collapsing. Let X and Y be the underlying topological space of the CW-complexes L and K, and let $f\colon X \to Y$ be a homotopy equivalence. Using the CW-complex structures of L and K, we may homotope f to a cellular map $g\colon L \to K$. By introducing the mapping cylinder

$$M_g = X \times [0, 1] \cup Y/\{(x, 1) = g(x) \mid x \in X\} \tag{2}$$

we obtain a CW-complex pair (M_g, L) such that L is a deformation retract of M_g. It is not difficult to see that the inclusion $K = K\times 0 \subset M_g$ is a simple homotopy equivalence, we shall say that f is simple if $L \subset M_g$ is simple. It was proved in [63, pp. 378–384] and [21] that this definition only depends on the underlying spaces X, Y and the map f, i.e., it is independent of the CW structures L and K of X and Y, and the map g.

Using simple expansions and simple collapsings repeatedly, we may replace (M_g, L) by a CW-complex pair (K_1, L_1), satisfying the following conditions:

(a) (M_g, L) are simply homotopic to (K_1, L_1) respectively and $L_1 \subset K_1$;

(b) K_1 arises from L_1 by attaching a finite number of k-dim cells $\{e_i^k\}$ and $(k+1)$-dim cells $\{e_j^{k+1}\}$ for $k \geqslant 2$. (3)

Consider the universal covering complexes $\hat{L}_1 \subset \hat{K}_1$ of $L_1 \subset K_1$. The fundamental group π will be identified with the group of covering transformations, so that each $\sigma \in \pi$ determines a mapping

$$\sigma\colon (\hat{K}_1, \hat{L}_1) \to (\hat{K}_1, \hat{L}_1), \tag{4}$$

which is *cellular*. If $C_*(\hat{K}_1, \hat{L}_1)$ denotes the cellular chain complex, then each $\sigma \in \pi$ determines a chain map

$$\sigma_{\#}\colon C_*(\hat{K}_1, \hat{L}_1) \to C_*(\hat{K}_1, \hat{L}_1) \tag{5}$$

and this action makes the chain group $C_p(\hat{K}_1, \hat{L}_1)$ into a free $Z[\pi]$-module with a basis obtained by making a choice of a lift to $\hat{K}_1$ of each p-cell of $K_1 - L_1$. Therefore, we have an isomorphism

$$0 \to C_{k+1}(\hat{K}_1, \hat{L}_1) \xrightarrow{\hat{d}_{k+1}} C_k(\hat{K}_1, \hat{L}_1) \to 0 \tag{6}$$

of free $Z[\pi]$-modules with the liftings $\{\hat{e}_j^{k+1}\}$ and $\{\hat{e}_i^k\}$ of $\{e_j^{k+1}\}$ and $\{e_i^k\}$ as bases. Using these bases, $\hat{d}_{k+1}$ determines an element in $\mathrm{GL}(Z[\pi])$ and thus an element $\tau(K_1, L_1)$ in $Wh_1(\pi)$. It was proved in [63] and [54] that the torsion $\tau(K_1, L_1)$ is independent of the choices[1] and it only depends on f. Denote it by $\tau(f) \in Wh_1(\pi)$. Let us summarize these fatcs in the following theorem.

THEOREM 1.1. *Let X, Y be the underlying topological spaces of the* CW-*complexes K and L, and let $f\colon X \to Y$ be a continuous map. Let $g\colon K \to L$ be a cellular map homotopic to f. Then f determines an element $\tau(f) \in Wh_1(\pi)$ depending only on $f\colon X \to Y$ such that g is simple if and only if $\tau(f) = 0$.*

Applying simple homotopy theory to manifolds, let us consider the following geometric problem. Let $(W^{n+1}; M_0^n, M_1^n)$ be a triad of compact manifolds such that $\partial W^{n+1} = M_0^n \cup M_1^n$. We say that W^{n+1} is an *h-cobordism* between M_0^n and M_1^n if M_i^n $(i = 0, 1)$ are deformation retracts of W^{n+1}. The simplest example of an h-cobordism is $(W^{n+1} = M^n \times [0, 1]$; $M_0^n = M^n \times 0$, $M_1^n = M^n \times 1)$. If $(W^{n+1}; M_0^n, M_1^n)$ is a smooth h-cobordism (i.e., W^{n+1} is a smooth manifold), $\pi_1(W^{n+1}) = 1$ and $n \geqslant 5$, then the remarkable theorem of Smale [76], [64] asserts that W^{n+1} is diffeomorphic to $M_0^n \times [0, 1]$ (and also to $M_1^n \times [0, 1]$). Our interest is focused on the case $\pi = \pi_1 W^{n+1} \neq 1$. If $(W^{n+1}; M_0^n, M_1^n)$ is a smooth h-cobordism,

[1] Since we may have different liftings of the cells, we pass from $K_1(Z[\pi])$ to $Wh_1(\pi)$ in order to make the invariant well-defined.

then a C^1-triangulation $t: (K; L_0, L_1) \to (W^{n+1}; M_0^n, M_1^n)$ gives rise to a combinatorial cobordism which has a handlebody structure from the triangulation [63]. Or, if $(W^{n+1}; M_0^n, M_1^n)$ is a topological h-cobordism and if $n \geqslant 5$, then (W^{n+1}, M_0^n) has a handlebody decomposition [56]. By a handlebody structure of W^{n+1} on M_0^n, we mean a filtration

$$Y^{(0)} = M_0^n \subset Y^{(1)} \subset \ldots \subset Y^{(l)} = W^{n+1} \tag{7}$$

such that

$$\begin{aligned}&\text{for each } i > 0, \text{ there is an embedding}\\ &f_i\colon S^{j_i-1} \times D^{n-j_i+1} \to Y^{(i-1)}, \text{ and a homeo-}\\ &\text{morphism } Y^{(i)} \to Y^{(i-1)} \cup_{f_i} D^{j_i} \times D^{n-j_i+1} \text{ rel } Y^{i-1}.\end{aligned} \tag{8}$$

The union of a k-handle and a $(k+1)$-handle $J = D^k \times D^{n-k+1} \cup D^{k+1} \times \times D^{n-k}$ along $D^k \times H_1 = H_2 \times D^{n-k}$, where $H_1 \subset \partial D^{n-k+1}$ and $H_2 \subset \partial D^{k+1}$ are codim 0 discs, will be called a *trivial pair of handles*. Introducing and cancelling a trivial pair of handles correspond to elementary expansion and elementary collapsing in simple homotopy theory. Following these procedures, if $n \geqslant 5$, we may produce a handlebody structure of (7) such that $j_1 = \ldots = j_k = k$ and $j_{k+1} = \ldots = j_{l-1} = k+1$ $(2 \leqslant k \leqslant n-3)$; i.e., W^{n+1} is obtained from M_0^n by attaching k-handles and $(k+1)$-handles. By making a choice of lifting of the cores[2] of the handles to $(\hat{W}^{n+1}, \hat{M}_0^n)$, the universal covers of the pair (W^{n+1}, M_0^n), we get a chain complex

$$0 \to C_{k+1}(\hat{W}^{n+1}, \hat{M}_0^n) \xrightarrow{\hat{d}_{k+1}} C_k(\hat{W}^{n+1}, \hat{M}_0^n) \to 0 \tag{9}$$

of based $Z[\pi]$-modules as in (6). The map $\hat{d}_{k+1}$ determines an element $\tau \in Wh_1(\pi(W^{n+1}))$ which is the torsion[3] of the inclusion of $M_0^n \subset W^{n+1}$. This element only depends on the underlying topological space and not on the handle structure of (W^{n+1}, M_0^n) and it is denoted by $\tau(W^{n+1}, M_0^n)$.

In correspondence to Theorem 1.1, we have the following S-cobordism theorem due to D. Barden, B. Mazur and J. Stallings [3], [54], [63].

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[2] I.e., $f(S^{j_i-1} \times 0)$ of (8) where 0 is the center of D^{n-j_i+1}.

[3] Note the asymmetry of M_0, M_1 in the definition. If we wish to consider (W, M_1), we have to consider the duality of [63, pp. 393–398].

THEOREM 1.2 (A) *Let* $(W^{n+1}; M_0^n, M_1^n)$ *be a smooth,* PL *or topological h-cobordism for* $n \geq 5$. *Then,* W^{n+1} *is diffeomorphic,* PL *homeomorphic or homeomorphic to* $M_0^n \times [0,1]$ *if and only if* $\tau(W^{n+1}, M_0^n) = 0$.

(B) *For a given manifold* M_0^n $(n \geq 5)$ *and an element* $\tau_0 \in Wh_1(\pi_1(M_0^n))$, *there exists an h-cobordism* $(W^{n+1}; M_0^n, M_1^n)$ *such that* $\tau(W^{n+1}, M_0^n) = \tau_0$ *and* W^{n+1} *is smooth or* PL *if* M_0^n *is so.*

II. Higher K-groups and pseudo-isotopy theory

For A a ring with unit 1, we observed in § I that $E(A) = [\mathrm{GL}(A), \mathrm{GL}(A)] \subset \mathrm{GL}(A)$ is perfect. Let $\mathrm{BGL}(A)$ be the classifying space of $\mathrm{GL}(A)$. Construct $\mathrm{BGL}^+(A)$, Quillen's "+" construction [67], from $\mathrm{BGL}(A)$ by attaching 2-cells and 3-cells such that there is a homology equivalence $\mathrm{BGL}(A) \to \mathrm{BGL}^+(A)$ over Z, and $\pi_1 \mathrm{BGL}^+(A) = \mathrm{GL}(A)/E(A) = K_1(A)$. In fact, $\mathrm{BGL}^+(A)$ is an infinite loop space. *$K_i(A)$ is defined to be $\pi_i \mathrm{BGL}^+(A)$* $(i \geq 1)$. Waldhausen [84] generalized this definition to construct higher algebraic K-groups of a pointed connected CW-complex X as follows. Let G be the loop group[4] of X and let $R = \Omega^\infty S^\infty[G_+]$ be the "group ring"[5] of G over $\Omega^\infty S^\infty$. Form the matrix ring $M_n(R)$. Consider the pull-back diagram

$$\begin{array}{ccc} \hat{\mathrm{GL}}_n(R) & \longrightarrow & M_n(R) \\ \downarrow & & \downarrow{\scriptstyle \pi_0} \\ \mathrm{GL}_n(\pi_0 R) & \longrightarrow & M_n(\pi_0 R) \end{array} \tag{10}$$

and let $\hat{\mathrm{GL}}(R) = \varinjlim_{n} \hat{\mathrm{GL}}_n(R)$. It turns out that $\mathrm{B}\hat{\mathrm{GL}}(R)$ exists, and using the fact that $[\pi_0 \hat{\mathrm{GL}}(R), \pi_0 \hat{\mathrm{GL}}(R)] = E(Z[\pi_0 G])$ we may perform the "+" construction for $\mathrm{B}\hat{\mathrm{GL}}(R)$ such that $\mathrm{B}\hat{\mathrm{GL}}^+(R)$ is an infinite loop space [84], [79]. Waldhausen defined

$$A(X) = \mathrm{B}\hat{\mathrm{GL}}^+(R) \tag{11}$$

and $K_i(X) = \pi_i A(X)$ for $i \geq 1$.

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[4] G is either a simplicial group model or a topological group model of the loop space ΩX of X.

[5] Since $\Omega^\infty S^\infty$ is not a topological ring, R is only a group ring in an appropriate sense. This causes most of the technical difficulties.

Based on this model, some computations for $\pi_i(A(X))\otimes Q$ were given in [49], [35], [15]. Invariant theory plays an important rôle in these results.

Due to the fact that $\Omega^\infty S^\infty$ is not an honest (topological) ring, Waldhausen's original way of introducing $A(X)$ is different from the above. See [84], [78], [79] for the details.

$A(X)$ is closely related to pseudo-isotopy theory. Let us recall some developments before the publication of [84]. If M is a compact differentiable manifold (generally with boundary), a pseudo-isotopy of M is a diffeomorphism $f\colon M\times I\to M\times I$ such that $f|M\times 0 = \mathrm{id}$. Let $P(M)$ denote the space of pseudo-isotopies endowed with the C^∞-topology. We are interested in computing $\pi_i(P(M))$. This problem was first studied by Cerf [20] and later by Hatcher and Wagoner [42]. Their idea roughly goes as follows. Connect a given pseudo-isotopy $f\colon M\times I\to M\times I$ to the identity pseudo-isotopy $f_0\colon M\times I\to M\times I$ by means of a generic map $F\colon M\times I\times I\to I\times I$ such that $F|M\times I\times 0 = pf_0$ and $F|M\times I\times 1 = pf$ where p denotes the projection to the second I factor. Choosing F carefully, the construction produces a one-parameter family of handlebodies as follows: Let t denote the coordinate of the second I factor. For $t=0$ (resp. $t=1$), $M\times I$ is the given product structure induced by a gradient-like vector field associated to the Morse function pf_0 (resp. pf). There is a finite number of birth points[6] for $0<t<\varepsilon_0$ (ε_0 a small positive number) such that at $t=\varepsilon_0$ the Morse function F_{ε_0} gives rise to "trivial pairs of handles". Similarly, there is a finite number of death points for $1-\varepsilon_1 < t < 1$ (ε_1 a small positive number) such that the handles are cancelled in "trivial pairs" at $t = 1-\varepsilon_1$. The one-parameter family of Morse functions F_t ($\varepsilon_0\leqslant t\leqslant 1-\varepsilon_1$) gives rise to a one-parameter family of handlebodies over the subinterval $[\varepsilon_0, 1-\varepsilon_1]\subset[0,1]$. Based on analysis of the parametrized handlebodies, Hatcher and Wagoner [42] relate $\pi_0 P(M)$ to $Wh_2(\pi_1 M)$, a quotient of $K_2(Z[\pi_1 M])$ for $\dim M\geqslant 6$.

In [40], Hatcher studied the space of PL pseudo-isotopy spaces $P_{\mathrm{PL}}(M)$ and the following stability question:

> Let $P_{\mathrm{PL}}(M)\subset P_{\mathrm{PL}}(M\times I)$ (resp. $P(M)\subset P(M\times I)$) be the natural inclusion essentially given by $f\times\mathrm{id}$. Is $\pi_i(P_{\mathrm{PL}}(M))\to\pi_i(P(M\times I))$ an isomorphism for $i\leqslant\dim M$?

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[6] See [20], [42] for the precise definitions.

He claimed the stability theorem for $P_{\mathrm{PL}}(M)$ and then Burghelea and Lashof extended it to $P(M)$ [16]. Unfortunately, there are some flaws in the proof of [40]. Based on his work on pseudo-isotopy by eliminating the higher order singularities [51] and a modification of Hatcher's argument, K. Igusa in a still unpublished paper has proved that the stability theorem is valid for $P(M)$ with a somewhat smaller range. Elaborating on the multi-disjunction lemma, in his thesis, T. Goodwillie has claimed that the stability ranges for $P(M)$ and $P_{\mathrm{PL}}(M)$ are the same (yet unpublished). The interest in the stability theorem stems from the observation that

$$\begin{gathered} \boldsymbol{P}(M) = \lim_i P(M \times I^i), \\ (\text{resp. } \boldsymbol{P}_{\mathrm{PL}}(M) = \lim_i P_{\mathrm{PL}}(M \times I^i)) \end{gathered} \tag{13}$$

becomes an infinite loop space and we can apply homotopy theory and the categorical machinery. This is the starting point for Waldhausen.

Let us study $Wh^{\mathrm{Diff}}(M)$, the double delooping of $\boldsymbol{P}(M)$. Motivated by consideration of the parametrized handlebodies for studying $\pi_0 P(M)$, consider a "rigid handlebody theory", a manifold model of Waldhausen's expansion space [84]. (We follow the exposition of [46].)

Let ∂_0 be an $(n+k-1)$-dim manifold and π_0: $\partial_0 \to \Delta^k$ a differentiable bundle map such that the fibers are $(n-1)$-dim manifolds (generally with boundary). Suppose that ∂_0 is a codim 0 submanifold of ∂Y for a manifold Y and π: $Y \to \Delta^k$ is a bundle projection extending π_0. We say that Y is a *k-parameter family of rigid handlebodies on* ∂_0 if there is a filtration:

$$Y^{(0)} = \partial_0 \subset Y^{(1)} \subset \ldots \subset Y^{(l)} = Y \tag{14}$$

satisfying the following conditions:

(a) For each $i > 0$, there is an embedding

$$f_i\colon S^{j_i-1} \times D^{n-j_i} \times \Delta \to Y^{(i-1)}$$

and a homeomorphism

$$Y^{(i)} \xrightarrow[\simeq]{h_i} Y^{(i-1)} \cup_{f_i} D^{j_i} \times D^{n-j_i} \times \Delta^k$$

rel $Y^{(i-1)}$ such that f_i and d_i preserve the projection onto Δ^k.

(b) $M^{(i)} = Y^{(i)} \cup_{\partial_0 = \partial_0 \times \{1\}} \partial_0 \times I$ is a manifold, even though $Y^{(i)}$ need not be.

(c) Let $\partial_+ Y^{(i)} = \mathrm{cl}\left(\partial M^{(i)} - (\partial_0 \times \{0\} \cup \partial M^{(i)} | \partial \Delta^k)\right)$ where $\partial M^{(i)} | \partial \Delta^k$ is the part lying over $\partial \Delta^k$. Then, $f_i(S^{j_i - 1} \times D^{n-j_i} \times \Delta^k) \subset \partial_+ Y^{(i-1)}$ and f_i is a differentiable embedding into $\partial_+ Y^{(i-1)}$ (after we smooth the corners). Note that h_i has an obvious extension to

$$M^{(i)} = M^{(i-1)} \cup D^{j_i} \times D^{n-j_i} \times \Delta^k,$$

and also assume that this is a diffeomorphism (again after smoothing the corners).

The attaching data, the f_i's and d_i's, are a part of the structure, but independent handles may be attached in any order. We may construct a *category E_k^n which has the k-parameter families of rigid handlebodies as objects and the compositions of isomorphisms and cancelling of trivial pairs of handles*[7] *as morphisms.*

The obvious definitions of face and degeneracy make $E_{\bullet}^n$ a simplicial category and appropriate inclusions and quotients make it into a cofibration category in the sense of [84], [46], [59].

Let X be a space and let ξ be a stable vector bundle over X. The categories $E_k(X, \xi)^n$, $k, n = 0, 1, \ldots,$ are defined as follows. The objects are diagrams

$$f\colon (Y, \partial_0) \to X \tag{16}$$

where (Y, ∂_0) is an object of E_k^n and $f\colon Y \to X$ is a continuous map, together with a stable bundle isomorphism $\psi\colon tY \to f^* \xi$ where tY is the stable tangent bundle of Y. The morphisms and cofibrations of E_k^n, appropriately modified with the data on the induced bundles from ξ, define morphisms and cofibrations of $E_k(X, \xi)^n$. Note that $E_k(X; \xi)^n$ has a composition law "+" — disjoint union of (Y, ∂_0)'s — and hence the classifying space has an infinite delooping in the sense of Γ-spaces [72]. On the other hand, we can make use of the cofibration structure such that the $S_{\bullet}$ construction of [84] and the Q construction of [66], [46] apply to this situation as explicit deloopings of $E_k(X; \xi)^n$.

Let $E_k^h(X; \xi)^n \subset E_k(X; \xi)^n$ be the full subcategory of objects such that $\partial_0 \subset Y$ is a homotopy equivalence. We can also deloop $E_{\bullet}^h(X; \xi)^n$ by means of $S_{\bullet}$ construction.

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[7] For technical reasons, we don't really cancel the trivial pairs geometrically. See [46] for details.

Multiplying by D^1, we define functors

$$\begin{gathered} S_.E_.(X;\,\xi)^n \xrightarrow{\Sigma} S_.E_.(X;\,\xi)^{n+1}, \\ S_.E^h_.(X;\,\xi)^n \xrightarrow{\Sigma} S_.E^h_.(X;\,\xi)^{n+1}. \end{gathered} \tag{17}$$

Set

$$\begin{aligned} BE(X;\,\xi) &= \lim_n |S_.E_.(X;\,\xi)^n|, \\ BE^h(X;\,\xi) &= \lim_n |S_.E^h_.(X;\,\xi)^n|, \end{aligned} \tag{18}$$

where the limit is taken with respect to Σ. It turns out that $BE(X;\,\xi)$ and $BE^h(X;\,\xi)$ are weakly homotopically equivalent to $\Omega^{\mathrm{fr}}(X)$ — the infinite loop space associated to the frame bordism and $Wh^{\mathrm{Comb}}(X)$ of [84], respectively, if X is a finite complex. (They are independent of ξ!) It has been shown that $Wh^{\mathrm{Comb}}(X)$ is rationally equivalent to $Wh^{\mathrm{Diff}}(M)$ if X is homotopy equivalent to M with the tangent bundle of M, $t(M)$ stably equivalent to ξ. (This is the reason why we keep ξ in the construction.) In fact, Waldhausen has recently claimed that $Wh^{\mathrm{Comb}}(X)$ is $Wh^{\mathrm{Diff}}(M)$.

Finally, let us state the remarkable result of Waldhausen [84]. It comes out of his "localization theorem" [84], [59], [83].

THEOREM 2.1 *There is a fibration up to homotopy*

$$\Omega BE(X;\,\xi)\to A(X;\,\xi)\to BE^h(X;\,\xi)$$

which is weakly homotopically equivalent to

$$\Omega^{\mathrm{fr}}(X)\to A(X)\to Wh^{\mathrm{Comb}}(X)$$

if X if a finite complex.

Since $Wh^{\mathrm{Comb}}(M)$ is (at least rationally) equivalent to $Wh^{\mathrm{Diff}}(M)$, one can easily see the importance of the functor $A(X)$ if one is interested in computing $\pi_i(P(M)) = \pi_{i+2}(Wh^{\mathrm{Diff}}(M))$.

III. $K_0(A)$, obstructions to being a finite CW-complex and to finding a boundary for an open manifold

Again let A be a ring with unit 1. Let K_0A be the additive group having one generator, $[P]$, for each finitely generated projective module P over A, and one relation, $[P]-[P_0]-[P_1]$, for a short exact sequence $0\to P_0 \to P\to P_1\to 0$. In other words, K_0A is the "Grothendieck group" associated to the category of finitely generated projectives over A. The class of free A-modules of rank 1 generates a cyclic subgroup of K_0A. The quotient

$$K_0A/(\text{subgroup generated by free modules})$$

is called the *projective class group*, $\tilde{K}_0(A)$. If $A = Z[\pi]$, then $\tilde{K}_0(Z[\pi])$ is sometimes written as $\tilde{K}_0(\pi)$.

Let X be a connected CW-complex and let Y be a connected *finite* CW-complex. We say that X is dominated by Y if there exist f: $X \to Y$, g: $Y \to X$ such that fg: $Y \to Y$ is homotopic to the identity. We would like to know whether X itself is of the homotopy type of a finite complex. It turns out that we may choose Y such that

$$H_i(\hat{M}_i, \hat{X}) = 0$$

for $i \neq k$ ($k \geqslant 2$) and $H_k(\hat{M}_f, \hat{X})$ is a finitely generated projective module over $Z[\pi_1 X]$ where M_f denotes the mapping cylinder of f and $(\hat{M}_f, \hat{X})$ is the universal covering of the pair (M_f, X). The class

$$\sigma(x) = (-1)^k [H_k(\hat{M}_f, \hat{X})] \in \tilde{K}_0(\pi_1 M) \tag{19}$$

turns out to be well-defined, independent of the choice of Y or of the integer k. In [86], the following fundamental theorem was proved:

THEOREM 3.1 (A) *For X a connected CW-complex dominated by a finite complex, X is of the homotopy type of a finite complex iff $\sigma(X) = 0$.*

(B) *Let $\sigma_0 \in \tilde{K}_0(\pi)$ be a given element with π a finitely presented group. There exists a connected CW-complex X dominated by a finite CW-complex with $\pi_1 X = \pi$ and $\sigma(x) = \sigma_0$.*

Let W^n ($n > 5$) be a smooth (or PL) open manifold. If there exists an arbitrarily large compact set with 1-connected complement and if $H_*(W)$ is finitely generated as an Abelian group, then, as was proved in [10], W is the interior of some smooth (or PL) compact manifold, $\overline{W}$. For the general case, L. Siebenmann developed the following theory [74]. Let W^n ($n > 5$) be a connected smooth open manifold and let ε be an end of W^n. We say that ε is *tame* if it satisfies the following conditions:

(A) There exists a sequence of neighborhoods of ε, $U_1 \supset U_2 \supset \ldots \supset U_i \supset \ldots$ such that $\bigcap U_i = \emptyset$ and $\pi_1(U_i) \leftarrow \pi_1(U_{i+1})$, $i = 1, \ldots$ are isomorphisms. We set $\pi = \pi_1(\varepsilon) = \varprojlim \pi_1(U_i)$ and call it π_1 of the end ε. (20)

(B) Each U_i is dominated by a finite CW-complex.

We may ask whether we can add a boundary to W^n at the tame end ε and reduce our problem to the case where W^n has only one end. In fact, we may choose each U_i of (19) to be a manifold with compact boundary ∂U_i such that $\pi_q(\partial U_i) \simeq \pi_1(U_i) \simeq \pi$ and

$$H_j(\hat{U}_i, \partial\hat{U}_i) = 0$$

for $i \neq k$ $(3 \leqslant k \leqslant n-3)$ where $(\hat{U}_i, \partial\hat{U}_i)$ is the universal covering of the pair $(U_i, \partial U_i)$ and $H_k(\hat{U}_i, \partial\hat{U}_i)$ is a finitely generated projective module over $Z[\pi]$. Then, $(-1)^k[H_k(\hat{U}_i, \partial\hat{U}_i)] = \sigma(U_i) \in \tilde{K}_0(\pi)$ is the obstruction defined in Theorem 3.1. Here is Siebenmann's theorem [74].

THEOREM 3.2. (A) $(-1)^k[H_k(\hat{U}_i, \partial\hat{U}_i)] \in \tilde{K}_0(\pi_1\varepsilon)$ *only depends on* ε, *and we denote it by* $\sigma(\varepsilon)$.

(B) *A boundary can be added to* W^n *at* ε *iff* $\sigma(\varepsilon) = 0$.

IV. Künneth formula for algebraic K-theory and its geometric application

Let T be an infinite cyclic group with a generator t, and let $A[T]$ be the *finite Laurent series ring* of A on t, which is just the group ring of T over A. If α is an automorphism of A, we also have the α-twisted finite Laurent series ring $A_\alpha[T]$. (See [32] for details.) Let $\underline{\underline{\mathrm{Nil}}}(A, \alpha)$ be the full subcategory of the category $\underline{\underline{\mathrm{P}}}(A)$ with objects (P, f) where P is a finitely generated projective module over A with f an α *semilinear nilpotent endomorphism.* Let $\mathrm{Nil}_0(A, \alpha)$ be the Grothendieck group of $\underline{\underline{\mathrm{Nil}}}(A, \alpha)$. The "forgetful functor" defined by "forgetting" the endomorphism f defines a homomorphism j: $\mathrm{Nil}_0(A, \alpha) \to K_0(A)$ and we let $\widetilde{\mathrm{Nil}}_0(A, \alpha)$ denote $\mathrm{Ker}\, j$. It is easy to see that we have a natural decomposition

$$\mathrm{Nil}_0(A, \alpha) = \widetilde{\mathrm{Nil}}_0(A, \alpha) \oplus K_0(A). \tag{21}$$

Let I denote the subgroup of $K_1 A$ generated by $x - \alpha_* x$ and let $(K_0 A)^{\alpha*}$ denote the subgroup of $x \in K_0(A)$ invariant under α_* (induced by α). Thinking of K_1 and K_0 as homology functors of rings, one might guess from the Künneth formula for the homology groups of a space E fibering over S^1, $F \to E \to S^1$, that there should be an exact sequence

$$0 \to K_1(A)/I \to X \to K_0(A)^{\alpha*} \to 0$$

such that $X \simeq K_1(A_\alpha[T])$. This is not true in general, unless A is (right or left) regular [7]. In fact, it was proved in [4, p. 628] that there is a canonical decomposition

$$K_1(A[T]) = K_1(A) \oplus \widetilde{\mathrm{Nil}}_0(A) \oplus \widetilde{\mathrm{Nil}}_0(A) \oplus K_0(A). \tag{22}$$

This was generalized in [32] to give:

$$K_1(A_\alpha[T]) = X \oplus \widetilde{\mathrm{Nil}}_0(A, \alpha) \oplus \widetilde{\mathrm{Nil}}_0(A, \alpha^{-1}), \tag{23}$$

where X fits into an exact sequence $0 \to K_1(A)/I \to X \to K_0(A)^{\alpha*} \to 0$, and we also have a natural projection:

$$p\colon K_1(A_\alpha[T]) \to K_0(A)^{\alpha*} \oplus \widetilde{\mathrm{Nil}}_0(A, \alpha). \tag{24}$$

For the group $\pi = G \times_\alpha T$, a semi-direct product, consider $A_\alpha[T] = Z[G \times_\alpha T]$ with $A = Z[G]$. By passing to $Wh_1(G \times_\alpha T)$, we have the formula

$$Wh_1(G \times_\alpha T) = \bar{X} \oplus \widetilde{\mathrm{Nil}}_0(A, \alpha) \oplus \widetilde{\mathrm{Nil}}_0(A, \alpha^{-1}), \tag{25}$$

where $0 \to Wh_1(G)/I \to \bar{X} \to \tilde{K}_0(A)^{\alpha*} \to 0$ $(I = \{y = x - \alpha_* x \mid x \in Wh_1(G)\})$. We also have a natural projection

$$\bar{p}\colon Wh_1(G \times_\alpha T) \to \tilde{K}_0(G)^{\alpha*} \oplus \widetilde{\mathrm{Nil}}_0(Z[G], \alpha). \tag{26}$$

Now consider the following geometric problem. Let M^n $(n \geqslant 6)$ be a closed smooth (PL or topological) manifold. Is M^n a fibration over S^1 with connected F^{n-1} as fiber? If so, we should have a projection $q\colon M^n \to S^1$ such that the fiber is homotopic to a connected finite complex X with $\pi = \pi_1 M^n = G \times_\alpha T$ being a semi-direct product of $G = \pi_1 X$ by $T = \pi_1 S^1$. For $\pi = Z$ (i.e., $G = 1$), it was proved in [12] that this condition is sufficient. For the general case, we need to find a codim 1 submanifold $F^{n-1} \subset M^n$ representing the homotopy fiber of q satisfying the following conditions:

$$\begin{array}{ll} \text{(A)} & \text{When we cut } M^n \text{ open along } F^{n-1}, \text{ we have} \\ & \text{an } h\text{-cobordism } (W^n; F_0^n, F_1^n) \text{ with } F_0^{n-1}, F_1^{n-1} \\ & \text{diffeomorphic to (PL or homeomorphic to) } F^{n-1}; \\ \text{(B)} & \tau(W^n, F_0) \in Wh_1(G) \text{ vanishes.} \end{array} \tag{27}$$

If only condition (A) is satisfied, we call it an *almost fibration.* It was proved in [28] that the obstruction to being an almost fibration is an element in $\tilde{K}_0(G)^{\alpha*} \oplus \mathrm{Nil}_0(Z[G], \alpha)$.

Generalizing the problem when a space fibers over S^1, let M_1^n be a closed smooth (PL or topological) manifold of $\dim n \geqslant 6$ with $\pi = \pi_1 M = G \times_\alpha T$ and let $F_1^{n-1} \subset M_1^n$ be a connected codim 1 submanifold with $G = \pi_1 F^{n-1}$ corresponding to the subgroup $G \subset \pi$. Let

$$f\colon M_2^n \to M_1^n \tag{28}$$

be a homotopy equivalence. We ask what is the obstruction $O(f)$ to finding an $(n-1)$-dim submanifold $F_2^{n-1} \subset M_2^n$ and a map

$$g\colon (M_2^n, F_2^{n-1}) \to (M_1^n, F_1^{n-1}) \tag{29}$$

such that

$$\begin{aligned} &(\mathrm{A})\ g \text{ is a homotopy equivalence of pairs,}\\ &(\mathrm{B})\ g^{-1}(F_1^{n-1}) = F_2^{n-1},\\ &(\mathrm{C})\ \text{the induced map } g\colon M_2^n \to M_1^n \text{ is homotopic to } f. \end{aligned} \tag{30}$$

$O(f)$ is called the *obstruction to splitting f with respect to F_1^{n-1}*.

THEOREM 4.1 [36]. *Assume that (M_1^n, F_1^{n-1}), $f\colon M_2^n \to M_1^n$ $(n \geqslant 6)$ are given as above. Then the obstruction $O(f)$ to splitting f with respect to F_1^{n-1} is equal to $\bar{p}\tau(f)$ where $\tau(f) \in Wh_1(\pi)$ is the torsion of f.*

One should read S. E. Cappell, A Splitting Theorem for Manifolds, *Invent. Math.* **33** (1976), pp. 66–170 for the generalization of the above theorem.

V. Negative K-groups $K_{-i}(A)$ and some of their geometric applications

In [4, pp. 657–674], H. Bass introduced the functor $K_{-i}(A)$ $(i \geqslant 0)$. He observed that the decomposition (22) is functorial and can be written as

$$0 \to K_1(A) \to K_1(A[t]) \oplus K_1(A[t^{-1}]) \to K_1(A[T]) \to K_0(A) \to 0. \tag{31}$$

It is natural to define K_{-i} $(i > 0)$ recursively using the formula

$$K_{-i}(A) = \mathrm{Coker}\{K_{-i+1}(A[t]) \oplus K_{-i+1}(A[t^{-1}]) \to K_{-i+1}A[T]\}. \tag{32}$$

Bass showed that (31) continued to hold with K_0, K_1 replaced by K_{-i} and K_{-i+1} $(i > 0)$ (and he also defined negative Nil groups). For T^n

$=T\times\ldots\times T$, we have the following decomposition formula [4], [1]:

$$\begin{aligned}Wh_1(\pi\times T^n) &= Wh_1(\pi\times T_1\times\ldots\times\hat{T}_i\times\ldots\times T_n)\oplus\\ &\quad\oplus\tilde{K}_0(\pi\times T_1\times\ldots\times\hat{T}_i\times\ldots\times T_n)\ (\text{mod nil terms})\\ &= Wh_1(\pi)\oplus n\tilde{K}_0(\pi)\oplus\\ &\quad\oplus\sum_{i=2}^{n-1}\binom{n}{i}K_{1-i}(Z[\pi])\oplus\\ &\quad\oplus K_{1-n}(Z[\pi])\ (\text{mod nil terms}).\end{aligned}\tag{33}$$

It was proved in [8], [19] that if π is a finite group, then

$$K_{-i}(Z[\pi]) = 0 \quad \text{for } i\geqslant 2.\tag{34}$$

Let us now turn to a geometric problem. A Top *stratification* of a space X is an increasing family of closed subsets of X, $\{X^n|n\geqslant 1\}$ such that $X^{(-1)}=$ fi, there is a positive integer N such that $X^{(N)}=X$, and for every n, each component of $X^{(n)}-X^{(n-1)}$ is a topological (Top) n-dim manifold without boundary (possibly empty). The stratification is *locally cone-like* if for every $x\in X^{(n)}-X^{(n-1)}$, there exists a compact Top stratified space L and a stratum-preserving open embedding $h\colon R^n\times cL\to X$ such that $h(0,v)=x$ where cL denotes the open cone over L and v is its vertex. The space L is called a *link of x* and h is called a *local chart*. We shall call a space X a Top CS space [2], [75], if it has a locally cone-line stratification. In [2], we define (combinatorial) PL structures on a Top CS space X compatible with the stratification (see [2] for the precise definition), and then study the existence and uniqueness of such structures on a given Top CS space. Of course, these questions are the refined forms of the problem of triangulating a topological space and the Hauptvermutung. The simplest example of a Top CS space is the suspension $\sum M$ of a closed manifold M. In [62], [77] counterexamples to Hauptvermutung were given with $X=\sum M^n$ $(n\geqslant 5)$ as the underlying topological space and with elements in $Wh_1(\pi_1 M)$ as the invariants to distinguish them. [1], [2] globalize the examples of Milnor and Stallings into an obstruction theory such that the obstruction invariants are generally in the subquotients of K_{-i} of the group ring of the links of various strata.

The obstruction theory of [1], [2] has been explicitly applied to the following problem. Let $R_1, R_2\in O(n)$, the group of orthogonal transformations of R^n. We say that R_1, R_2 are *topologically* (resp. *linearly*) *equiv-*

alent if there is a homeomorphism (resp. linear automorphism) $f: R^n \to R^n$ such that $f^{-1}R_1 f = R_2: R^n \to R^n$. The conjecture that the notions of topological and linear equivalence of rotations should be equivalent was stated by de Rham in 1935 [24] and he reduced it to the case where the rotations have finite order. Note that f induces a homeomorphism

$$h: X_1 \to X_2, \tag{34}$$

where $X_i = R^n/\langle R_i \rangle$, the quotient space of R^n by the finite subgroup of $O(n)$ generated by R_i, $i = 1, 2$. Given X_i, the preferred PL structures induced from the rotation R_i, we may then try to deform h to a PL homeomorphism. This is the problem studied in [2]. (See also [69].) Modifying the topologically equivalent R_1, R_2 to new ones, R_1', R_2', if necessary, we manage to kill most of the obstructions in the subquotients of K_{-i} and then apply a version of G-signature theorèm to obtain the following result [47].

Let $R_1, R_2 \in O(n)$ have order $k = l2^m$ where l is odd and $m \geqslant 2$. Suppose that (a) *R_1 and R_2 are topologically equivalent, and* (b) *the eigenvalues of R_1^l and R_2^l are either* 1 *or primitive 2^m-th roots of unity. Then R_1 and R_2 are linearly equivalent.*

If k is odd, then condition (b) is superfluous. In this case, it was proved independently, by Madsen and Rothenberg [60] using a different method from [47]. However, the K_{-i} groups of [1], [2] (see also [57]) still play an important rôle in their work.

The interest of de Rham's problem was revived [55] and there are remarkable counterexamples of this conjecture in [18] if $k = l2^m$, $m \geqslant 2$, $l \neq 1$, and the above condition (b) is not satisfied.

VI. Concluding remarks and some conjectures

One of the problems in algebraic K-theory is to compute $K_i(A)$ $(-\infty < i < \infty)$. Emphasizing the geometric applications, we are mostly interested in the case of $A = Z[G]$ for G a finitely presented group. Most algebraic calculations have been carried out for G finite. Let me pose some conjectures about the case when G is *not necessarily* finite or *torsion-free*. In fact, I believe that these problems are more geometrically interesting and they should serve as guide posts for future development.

CONJECTURE 1. *Let G be a finitely presented group. Then $K_{-i}(Z[G]) = 0$ for $i \geqslant 2$. At least, $K_{-i}(Z[G]) = 0$ for $i \gg 0$.*

Before I state the next conjecture, let me single out a class of infinite groups. We say that a closed manifold M^n is a $K(\Gamma, 1)$-manifold (an aspherical manifold) if $\pi_i(M^n) = 0$ for $i > 1$ and $\pi_1 M^n = \Gamma$. Note that Γ is *necessarily torsion-free.*

CONJECTURE 2. *Let Γ be the fundamental group of a closed $K(\Gamma, 1)$-manifold. Then* $Wh_1(\Gamma) = \tilde{K}_0(\Gamma) = K_{-i}(Z[\Gamma]) = 0$ $(i \geqslant 1)$. (See [31] for supporting evidence.)

It is clear that the following conjecture is much stronger than Conjecture 2.

CONJECTURE 3. *Let Γ be a torsion-free group such that $B\Gamma$ has the homotopy type of a finite* CW*-complex. Then* $Wh_1(\Gamma) = \tilde{K}_0(\Gamma) = K_{-i}(Z[\Gamma]) = 0$ $(i \geqslant 1)$.

For the higher K-groups, let us consider the map of [58]:

$$\lambda_*: \ h_*\big(BG; \underline{\underline{K}}(Z)\big) \to K_*(Z[G]), \tag{35}$$

where $h_*\big(BG; \underline{\underline{K}}(Z)\big)$ denotes a generalized homology theory with coefficients in the spectrum of the algebraic K-theory of Z.

CONJECTURE 4. *If Γ is a torsion-free group such that $B\Gamma$ is of the homotopy type of a finite* CW*-complex, then*

$$\lambda_* \otimes \mathrm{id}: \ h_*\big(B\Gamma; \underline{\underline{K}}(Z)\big) \otimes Q \to K_*(Z[\pi]) \otimes Q$$

is an isomorphism.

For $B\Gamma$ having the homotopy type of an aspherical manifold, Conjecture 4 was verified in some special cases [31]. As we pointed out in [35], Conjecture 2 is the algebraic K-theory analogue of Novikov's conjecture on higher signatures. (So are Conjectures 2, 3!) Interested readers should consult [30], [34], [88] for further details about this conjecture.

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(A suggested list)

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