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Non-Standard Characteristics in Asymptotical Problems

Introduction: Examples of non-standard characteristics. General problems

1. An example of the Klein–Gordon equation. If we asked a physicist what Hamilton–Jacobi equation naturally corresponds to the Klein–Gordon equation

$$h^2\frac{\partial^2 u}{\partial t^2} - h^2\frac{\partial^2 u}{\partial x^2} + m^2c^4u = 0 \tag{0.1}$$

(m, c, h are physical constants), he would write of course the Hamilton–Jacobi equation

$$\left(\frac{\partial S}{\partial t}\right)^2 - \left(\frac{\partial S}{\partial x}\right)^2 - m^2c^4 = 0 \tag{0.2}$$

describing the free motion of a relativistic particle. But the mathematician specializing in hyperbolic equations would write another Hamilton–Jacobi equation,

$$\left(\frac{\partial \Phi}{\partial t}\right)^2 - \left(\frac{\partial \Phi}{\partial x}\right)^2 = 0. \tag{0.3}$$

This equation is a characteristic equation in a standard mathematical sense. Who is right?

The answer is: both the physicist and the mathematician are wrong or, to say it more politely, both of them are right. The nature of the disagreement is easy to see. The physicist is looking for the semi-classical asymptotics of the solution of the Klein–Gordon equation with respect to the parameter h. He sets this parameter to be small.

The mathematician is looking for the asymptotics with respect to "smoothness". Namely, he looks for example for the solution of a problem with singular initial data modulo differentiable functions.

To obtain the characteristic equation (0.3) the mathematician substitutes in the Klein–Gordon equation a solution of the form

$$\theta(\Phi)\varphi_0+\Phi\theta(\Phi)\varphi_1+\ldots, \qquad (0.3.1)$$

where θ is the Heaviside function, $\Phi\in C^\infty$, $\varphi_i\in C_0^\infty$. Then he puts the coefficient by the main singularity equal to zero.

The physicist obtains the Hamilton–Jacobi equation corresponding to the free motion of a relativistic particle by substituting in the Klein–Gordon equation the function $\varphi\exp(iS/h)$ ($S\in C^\infty$, $\varphi\in C_0^\infty$). Then he puts the coefficient by the main term in the resulting polynomial in h equal to zero.

The problem of constructing the semi-classical asymptotics can easily be reduced to the problem of constructing the asymptotics with respect to smoothness. Namely, consider the solution of the Klein–Gordon equation as a function of one more variable. This variable is our small parameter h. Further, it is convenient to introduce $\lambda = 1/h$. Evidently, the asymptotical expansion of the solution $u(x,t,\lambda)$ of the Klein–Gordon equation is equivalent to the expansion with respect to smoothness of the function $v(x,t,\xi)$, which is the Fourier transform with respect to λ of the function $u(x,t,\lambda)$. And we see, the function $v(x,t,\xi)$ satisfies the hyperbolic equation

$$\frac{\partial^2 v}{\partial t^2}-\frac{\partial^2 v}{\partial x^2}-m^2c^4\frac{\partial^2 v}{\partial \xi^2}=0. \qquad (0.4)$$

Indeed, by setting

$$v(x,t,\xi)=(2\pi)^{-1/2}\int e^{-i\lambda\xi}u(x,t,\lambda)\,d\lambda$$

and passing to the Fourier transform in (0.1), we obtain equation (0.4). Note that c in (0.1) and (0.4) may be a smooth function in x.

We search for the parametrix (0.4), i.e., the asymptotic expansion with respect to the smoothness of the solution of equation (0.4) with initial data

$$v|_{t=0}=0, \qquad \left.\frac{\partial x}{\partial t}\right|_{t=0}=\delta(x-y)\,\delta(\xi-\eta),$$

in the form

$$\int e^{i|p|\Phi(x,\xi,\nu,\omega,t)}\big(\varphi_0(x,\xi,\nu,\omega,t)+|p|^{-1}\varphi_1(x,\xi,\nu,\omega,t)+\ldots\big)\,dp,$$

where $p=(p_1,p_2)$ are dual to the variables x, ξ, $|p|=\sqrt{p_1^2+p_2^2}$, $\nu=p_1/|p|$, $\omega=p_2/|p|$, $\Phi,\varphi_0,\varphi_1,\ldots\in C^\infty$.

But the Fourier transform property $|p|$ is considered as a large parameter.

Analogically to the previous procedure, as $|p|\to\infty$ for $\Phi(x, \xi, \nu, \omega, t)$ we obtain the equation

$$\left(\frac{\partial\Phi}{\partial t}\right)^2-\left(\frac{\partial\Phi}{\partial x}\right)^2-m^2c^4\left(\frac{\partial\Phi}{\partial\xi}\right)^2=0. \tag{0.5}$$

By using the expansion of δ-function

$$\delta(x-y)\,\delta(\xi-\eta)=\frac{1}{2\pi}\int\exp\Big(i|p|\big(\nu(x-y)+\omega(\xi-\eta)\big)\Big)dp$$

we obtain the condition on Φ:

$$\Phi|_{t=0}=\nu(x-y)+\omega(\xi-\eta).$$

Set

$$\Phi(x, \xi, \nu, \omega, t)=S(x, \xi, \nu, t)+\omega(\xi-\eta),$$

where S is independent of ξ, η (such substitution is possible since equation (0.5) is independent of ξ). Then the equation for S has the form

$$\left(\frac{\partial S}{\partial t}\right)^2-\left(\frac{\partial S}{\partial x}\right)^2-m^2c^4\omega^2=0. \tag{0.6}$$

It is easy to see that for $\omega^2=1$, i.e., when the asymptotics is constructed with respect to the variable ξ only, this equation coincides with (0.2).

Despite such evident correspondence between the semi-classical asymptotics and the asymptotics with respect to smoothness, they were studied independently for a long time. In the middle of the sixties the method of characteristics describing the asymptotics of solutions both with respect to smoothness and to the parameter was expanded upon a wide class of pseudodifferential and h-pseudodifferential operators (i.e., operators with a small parameter h by the derivatives).

Thus there exist two types of characteristics for equations with a small parameter: for the so-called h-differential equations and for more general h-pseudodifferential equations.

The characteristics of the first type describe asymptotical expansions in terms of powers of this parameter. The characteristics of the second type describe asymptotical expansions in terms of smoothness. It is natural to pose for h-pseudodifferential equations the problem of constructing of such compound asymptotical expansions as would include asymptotical expansions both in terms of the parameter and in terms of smoothness. Namely, the problem is to find solutions modulo functions simultaneously smooth and small. For the Klein–Gordon equation the characteristics corresponding to our problem are defined by a one-parametr family of the Hamilton–Jacobi equations (0.6) depending on the parameter $\omega^2 \in [0, 1]$ (such an interval of variation of the parameter ω^2 is the consequence of the equality $\nu^2+\omega^2 = 1$). For $\omega = 0$ these characteristics coincide with the characteristics of the problem of construction asymptotics with respect to smoothness, and for $\omega = 1$ with that of the problem of rapidly oscillating asymptotics.

The compound asymptotics and the family of characteristics corresponding to them play an essential role. For example, the compound asymptotics describe mathematically effects of the Cherenkov type, namely, the phenomenon in with there is a domain of rapid oscillations (light beaming) in the tail of a particle. The particle is described by the δ-function. This domain can be described exactly by a corresponding family of characteristics.

2. Systems of equations of crystal lattice oscillations and difference schemes. Analogous compound asymptotics and corresponding characteristics can be constructed for difference schemes and for systems of a large number of ordinary differential equations.

Consider a simple example of such a system, namely, the atom oscillations in a one-dimensional crystal lattice with the step h on the circle of length $2\pi r_N = Nh$ (N is the number of atoms):

$$\frac{d^2u_n}{dt^2} = \frac{c^2}{h^2}(u_{n+1}-2u_n+u_{n-1}), \qquad n = 1, \ldots, N. \tag{0.7}$$

Here u_n denotes the deviation of the n-th atom from the equilibrium state, $u_0 = u_N, u_1 = u_{N+1}$.

Assume that the circle radius r_N remains finite where $N \sim 1/h$ is a large number, and let us search for the asymptotical solution under these assumptions.

Consider a smooth $(2\pi r_N)$-periodic function $u(x, t)$ taking the values u_j at the lattice points. Equation (0.7) can be rewritten as follows:

$$\frac{\partial^2 u}{\partial t^2} = \frac{c^2}{h^2}[u(x+h, t)-2u(x, t)+u(x-h, t)].$$

The solution of this equation at the lattice points coincides with the solution of (0.7) and is independent of the initial data outside the lattice points.

Using the identity

$$\exp\left[i\left(-ih\frac{\partial}{\partial x}\right)\right]u(x,t) = u(x+h,t),$$

which can be verified by means of the Taylor expansion or by means of the Fourier transform, we rewrite the latter equation in the form

$$h^2\frac{\partial^2 u}{\partial t^2} + 4c^2\sin^2(-ih\tfrac{1}{2}D)u = 0, \quad D \equiv \frac{\partial}{\partial x}. \tag{0.8}$$

Even in this simplest case the family of characteristic equations appears not to be standard (Maslov, 1965, [103]). It has the form

$$\left(\frac{\partial S}{\partial t}\right)^2 - 4\frac{c^2}{\omega^2}\sin\left(\frac{\omega}{2}\frac{\partial S}{\partial x}\right) = 0, \tag{0.9}$$

where the parameter ω varies in the interval $[0, 2\pi]$. This fact concerns equation (0.7) but not equation (0.8), and it follows from the fact that the characters of the discrete group, corresponding to the lattice $2\pi k/N$, vary just in this interval.

In the three-dimensional case for the lattice corresponding to an Abelian discrete group the parameters in the characteristic equation vary in the Brillouin zone (well known in the crystal theory [3], p. 100). If the lattice corresponds to a non-Abelian group, then the parameters in the characteristic equation may vary in some extraordinary domains.

Consider the simplest example of a finite-difference equation, namely, the difference scheme

$$h^{-2}(u_n^{m+1} - 2u_n^m + u_n^{m-1}) = c^2h^{-2}(u_{n+1}^m - 2u_n^m + u_{n-1}^m), \tag{0.10}$$

which approximates the wave equation. The family of characteristic equations for (0.10) has the form:

$$\frac{\partial S}{\partial t} \pm 2\omega^{-1}\arcsin\left[c\sin\left(\frac{\omega}{2}\frac{\partial S}{\partial x}\right)\right] = 0,$$

where the parameter ω varies from 0 to 2π. These characteristics define the spread zone of the oscillations for the so-called unity error, i.e., of the oscillation of the solution, which at the initial moment is equal to unity at a point of the net and is equal to zero at other points. First such characteristics for difference schemes were introduced in [106].

For the equations with constant coefficients the spread zone can be calculated directly from the exact solution [146], [39].

3. General setting of the problem of asymptotics compound with respect to smoothness and to the parameter. The problem of constructing asymptotics with respect to smoothness or to the parameter can be regarded as follows: for a given pseudodifferential operator $L\colon H_s \to H_{s-m}$ in the scale of the Sobolev spaces H_s an almost inverse operator R_N such that

$$LR_N = 1 + Q_N \tag{0.11}$$

should be constructed. Here $Q_N\colon H_s \to H_{s+N}$ is a smoothing operator in the problem of asymptotics with respect to smoothness, or a "small" operator, namely $\|Q_N\|_{H_s \to H_s} = o(h^N)$, in the problems on asymptotics with respect to the parameter h. For the compound asymptotics the operator Q_N is simultaneously a "small" and a smoothing one:

$$\|Q_N\|_{H_s \to H_{s+N}} = O(h^N).$$

When the asymptotics is constructed, the initial equation is reduced to an integral equation of the second kind with a kernel which is not only smooth but also small with respect to the parameter. This fact enables us to prove the existence theorem and to construct estimates of the solution uniform with respect to the parameter. The compound asymptotics can naturally be interpreted as the construction of an almost inverse operator in the scale generated by a pair of commuting operators $A_1 = 1/h$ (the multiplication by the inverse of the parameter) and $A_2 = -i\,\partial/\partial x = -iD$. The Fourier transform with respect to $\lambda = 1/h$ transfers this scale into the usual Sobolev scale.

4. Compound asymptotics with respect to smoothness and the decrease at infinity. More complicated situation arises when we construct asymptotics with respect to an n-tuple of non-commuting operators. For example, for differential equations with growing coefficients, i.e., equations containing both the powers of the differentiation operator $A_2 = iD$ and the powers of the operator $A_1 = x$ it is natural to pose the following problem: an almost inverse operator R_N should be constructed such that the remainder Q_N in (0.11) not only is a smoothing operator but also transforms any function from L_2 into a function rapidly decreasing as x tends to infinity. In this problem the characteristics are defined by an n-tuple of non-commuting operators A_1 and A_2.

Consider the example

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - x^2\big(1 + b(x)\big)u, \qquad b \in C_0^\infty(R),$$

The characteristics are defined by means of the following family of non-standard Hamilton–Jacobi equations (Maslov, 1973, [109]):

$$\left(\frac{\partial S}{\partial t}\right)^2 = \left(\frac{\partial S}{\partial x}\right)^2 + \omega\left(1+b(x)\right),$$

where ω is a parameter varying from zero to one.

For general equations with growing coefficients analogous characteristics are constructed and the global asymptotic of the solution with respect to smootheness and to the growth at infinity is obtained, i.e., an almost inverse operator in the scale induced by the pair A_1, A_2 is constructed (Maslov, 1973, [109]; Maslov and Nazaikinskij, 1979, [117]).

5. The case of degenerate characteristics. In the same way, for equations with singularities or with singular standard characteristics it is sometimes possible to find appropriate self-adjoint operators $A_1, \ldots, A_n$ with respect to which the non-standard characteristics of the equation are non-singular. Then it is possible to construct an almost inverse operator with respect to the scale generated by A_j. The scale is given by the sequence of norms

$$\|u\|_s = \|(1+A_1^2+ \ldots +A_n^2)^{s/2}u\|.$$

The operator R_N almost inverse to the operator L, satisfies equation (0.1), in which the remainder Q_N is a smoothing operator with respect to the scale:

$$\|Q_N\|_{s\to s+N} < \infty.$$

The operator R_N can be explicitly calculated in terms of functions of non-commuting model operators $A_1, \ldots, A_n$ in the case where they define a representation of a Lie algebra or a nilpotent algebra with nonlinear commutation relations ([112], [64], [66]).

Even in the case of geometrically simplest characteristics, namely, for some degenerating elliptic equations, this idea enables us to construct almost inverse operators in the scale generated by the n-tuple of vector fields $A_1, \ldots, A_n$ which induce a nilpotent Lie algebra (Stein and Folland, 1974, [41]; and others [141], [140], [142], [43], [54]).

6. Example of an asymptotics in the case of characteristics with singularities. We show by means of the simplest example how the ideas discussed above enable us to solve the problem of oscillating solutions for a hyperbolic equation with a singularity in the characteristics.

Consider the problem for the following wave equation:

$$\frac{\partial^2 u}{\partial t^2} - c^2(x)\Delta u = f,$$
$$u|_{t=0} = 0, \quad u'_t|_{t=0} = 0, \tag{0.12}$$
$$f = \exp\left(\frac{i}{h} S_0(x)\right)\varphi_0(x), \quad S_0(x) = |x|^2,$$

where $c(x) \in C^\infty(R^n)$, $c(x) \geqslant \delta > 0$, $\varphi_0(x) \in C_0^\infty(R^n)$, $\varphi_0(0) \neq 0$. By the Duhamel principle the solution of this problem is expressed in terms of the solution of the Cauchy problem

$$\frac{\partial^2 v}{\partial t^2} - c^2(x)\Delta v = 0,$$
$$v|_{t=0} = f, \quad v'_t|_{t=0} = 0.$$

The standard scheme of constructing the asymptotic solution of the above problem is the following: the solution is presented in the form $\exp(iS/h)\varphi$. Evidently the wave operator acts on the exponent as follows:

$$e^{-iS/h}\left[-\left(-ih\frac{\partial}{\partial t}\right)^2 + c^2(-ihD)^2\right]e^{iS/h}\varphi$$
$$= \left[-\left(\frac{\partial S}{\partial t}\right)^2 + c^2\left(\frac{\partial S}{\partial x}\right)^2\right]\varphi +$$
$$+(-ih)\left[-2\frac{\partial S}{\partial t}\frac{\partial \varphi}{\partial t} + 2c^2\frac{\partial S}{\partial x}\frac{\partial \varphi}{\partial x} - \varphi\frac{\partial^2 S}{\partial t^2} + c^2\varphi\Delta S\right] +$$
$$+h^2\left[\frac{\partial^2 \varphi}{\partial t^2} - c^2\Delta\varphi\right].$$

Hence $\partial S_0/\partial x = 0$ for $x = 0$, and the solution of the characteristics equation

$$-\left(\frac{\partial S}{\partial t}\right)^2 + c^2\left(\frac{\partial S}{\partial x}\right)^2 = 0, \quad S|_{t=0} = S_0,$$

is a non-smooth function at this point. Thus the standard scheme of constructing the asymptotics with respect to the parameter cannot be applied to this case.

It is evident that the characteristics of the wave equation corresponding to the asymptotics with respect to smoothness (parametrix) (0.11) have no singularities. Indeed, the equation is homogeneous in operators iD; hence the bicharacteristics start from the sphere $|p| = 1$ and the singular point $p = 0$ drops out in this case. The asymptotics with respect to smoothness enables us to represent the solution (0.12) in the form $u = (R_N + r_N)f$ where $R_N\colon H_s \to H_{s+1}$ is an explicitly calculated operator, and $r_N\colon H_s \to H_{s+N}$ is a smoothing operator. The operator r_N in this example transfers the right side to a function whose norm in the space H_{s+N} is of order $h^{n/2}$, since this function is a convolution of the exponent $\exp(iS/h)$ with the (smooth) kernel of the operator r_N. The other terms of the asymptotics of function $r_N f$ with respect to h are obtained by means of the solution of the wave equation with zero Cauchy data and smooth right sides. This problem differs from the initial one in that the wave operator should be inverted on a function uniformly smooth (non-oscillating) with respect to the small parameter. Such a solution can easily be obtained by means of computer.

The accuracy of the approximation of the solution by the leading term of the asymptotics depends on the type of the initial conditions. For example, if $S_0(x) = |x|^4$ then the leading term $R_N f$ approximates the solution modulo $O(h^{n/4})$. Thus the estimate of the leading asymptotical term is connected with the individual initial condition. The leading term $R_N f$ can asymptotically be represented in the form of an integral whose integrand oscillates in the parameter h and has a singularity. For example in case $c = 1$, $n = 3$ we have

$$R_N f = \frac{1}{4\pi} \int\limits_{|x-\xi|\leqslant t} |x-\xi|^{-1} \exp(i|\xi|^2/h)\varphi_0(\xi)\, d\xi .$$

The leading term of the asymptotics in the case of variable coefficients and an arbitrary dimension has the same properties.

This scheme is applicable to the case of a general operator equation with characteristics non-singular with respect to an n-tuple of operators and singular with respect to another n-tuple of operators.

Apparently, this scheme covers the results of Guillemin, Sternberg, Uhlmann and others [37], [43], [44], [49], [50], [127], [5].

7. General setting of the problem of characteristics. We have considered a number of simple examples which show that it is necessary to give a general definition of characteristics. The main part of this general notion of characteristics is the possibility of connecting different areas in physics and mathematics. Then the methods of one area can be applied in another. For example, it turns out that the following problems, apparently unconnected, are of the same mathematical nature: the behaviour of the solutions of the equations of the dynamics of a viscous liquid far ahead of the shock wave and large deviations in the theory of probabilities; the Cherenkov effect and the diffusion of unity error in a difference scheme; effects of dissociation of molecules and conical refraction in acoustics and so on. These effects often involve difficulties, which arise in proving the existence theorems for pseudodifferential equations.

First of all, in order to define characteristics, we need an n-tuple of operators, namely, model operators with respect to which the asymptotics is constructed. Further, the original equation should be expressed in terms of those model operators. This procedure itself demands preliminary investigation, namely, the construction of the calculus of non-commuting operators.

In this paper I have no opportunity to give a general definition, since it demands preliminary considerations. I prefer to show, by means of simple but specific examples, the basic ideas which enable us to construct the characteristics in a more general situation.

Part I. The non-standard characteristics of linear equations and equation with non-local nonlinearity

1. Pseudodifferential operators with symbols and characteristics on arbitrary symplectic manifolds. Quantization conditions for coordinate-momenta. Consider the first generalization of the notion of characteristics. We shall describe below the difficulty which arises in asymptotical problems for asymptotics in terms of the parameter. It also arises for compound asymptotics in terms of an n-tuple of operators. The phase space for these problems is not necessarily a cotangent bundle of manifolds.

Even in the simplest example of the equations of oscillations of atoms which we mentioned above the phase space is not a cotangent bundle. In fact, the configuration space in this problem is a two-parameter family of circles. This family depends on the continuous parameter h tending to 0 and on the discrete parameter N, which tends to infinity. Note that the length of the circle is equal to Nh. The character arguments of the discrete group of shifts of the lattice vary from 0 to 2π, and the space of momenta is the unit radius circle. Thus, in this case the phase space is a family of two-dimensional tori. The surface of the torus is obviously equal to $2\pi Nh$; hence the following equality holds:

$$\frac{1}{2\pi h}\int dp\wedge dxN. = \tag{1.1}$$

Consider now a one-parameter family of tori. If the left-side expression in the equality (1.1) is an integer, then it is possible to construct a calculus of pseudodifferential operators; namely, with any symbol f which is a smooth function on the torus one can associate an operator $\hat{f}$ which is defined on functions on the circle $S^1 = 0 \leqslant x \leqslant 2\pi r_N$, so that the following formulas hold: the commutation formula

$$[\hat{f}, \hat{g}] = -ih\widehat{\{f, g\}} + O(h^2) \tag{1.2}$$

and the formula of the operator action on the exponent

$$\begin{aligned} e^{-iS/h}\hat{f}(e^{iS/h}\varphi) &= f(x, dS(x))\varphi(x) + \\ &+(-ih)\left\{\frac{\partial f}{\partial p}(x, dS)\frac{\partial\varphi}{\partial x} + \frac{\varphi}{2}\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial p}(x, dS)\right) - \right.\\ &\left. -\frac{\varphi}{2}\operatorname{tr}\frac{\partial^2 f}{\partial x\,\partial p}(x, dS)\right\} + O(h^2). \end{aligned} \tag{1.3}$$

Here curly brackets denote the Poisson brackets, φ is an arbitrary smooth function on the circle, S is a real function on the circle which defines a smooth curve $p = dS(x)$ on the torus, and the estimate $O(h^2)$ in (1.3) is in the norm L_2.

As in the case of the Euclidean space the operators $\hat{f}$ will be called h-pseudodifferential operators (with symbols on the torus). We shall use the notation

$$\hat{f} = f\left(x, \frac{h}{i}D\right).$$

Now to a symbol on the torus $f = c^2\sin^2(p/2)$ there corresponds an h-pseudodifferential operator of the form

$$\hat{f} = c^2\sin^2\left(\frac{h}{i}D\right).$$

Consider an equation with this operator

$$h^2\frac{\partial^2 u}{\partial t^2} + 4c^2\sin^2\left(\frac{h}{i}D\right)u = 0, \qquad u|_{x=2\pi r_N} = u|_{x=0}. \tag{1.4}$$

Denote by u_k the value of the function u at the point $x = kh$. Then we can write

$$u_{k+1} - 2u_k + u_{k-1} = (e^{h\partial/\partial x} - 2 + e^{-h\partial/\partial x})\, u|_{x=kh}$$
$$= -4\sin^2\left(-\frac{ih}{2}\frac{\partial}{\partial x}\right) u\Big|_{x=kh}.$$

Thus equation (0.7), describing the oscillations of the atom lattice on a circle, is equivalent to the h-pseudodifferential equation (1.4), whose phase space is a torus satisfying condition (1.1).

Now consider the general case: the family of symplectic manifolds depending on a parameter μ, varying in a compact. It is convenient to consider that the manifold is fixed, and that the symplectic structure on it depends on the parameter μ (i.e., it is a closed non-degenerate 2-form $\Omega^{(\mu)}$ or the Poisson bracket $\{\ldots,\ldots\}^{(\mu)}$). We assume that for $\mu = \mu(h)$ the following condition holds: over any two-dimensional cycle the integral of the symplectic form $\Omega^{(\mu(h))}$ divided by $2\pi h$ coincides (modulo integer numbers) with half of the value of the second Stiefel–Whitney class, i.e.,

$$\frac{1}{\pi h}\,[\Omega^{(\mu(h))}] = W_2(\mathrm{mod} 2Z). \tag{1.5}$$

Then with each smooth function f on the phase manifold one can associate an operator $\hat{f}$, so that the commutation formula (1.2) (with the Poisson bracket $\{\ldots,\ldots\}^{(\mu(h))}$) holds and formula (1.3) holds locally (Karasev and Maslov, 1981, [66], details in [67], [68], reproduced in [120], [124]).

Call equation (1.5) the quantization condition for coordinate-momenta. The value $\frac{1}{2}W_2$ in (1.2) will be called the vacuum correction. It is remarkable that condition (1.5) (with the zero vacuum correction $W_2 = 0$) arose already in the construction of a rather narrow class of pseudodifferential operators with locally linear in ϱ symbols, namely, the differential operators of the first order (Kostant, 1970, [77]; Souriau, 1966–1970, [149–151]). For the half-integer vacuum correction (there are important examples, see e.g. [42]) within the framework of the first order operators, condition (1.5) was obtained for the Kaehler manifolds (Czyż, 1979, [23]) and in the case of general real manifolds (Hess, 1981, [55]).

By constructing the calculus of pseudodifferential operators on the orbits of a compact Lie group condition (1.5) numerates the irreducible representations of the group. Thus, the quantization of coordinate-momenta on the orbits coincides with the Weyl rule of integer major weights of irreducible representations (Borel and Hirzebruch, 1959, [16]; Kirillov, 1968, [74]). The vacuum correction is then equal to zero.

It should be noted that in general phase manifolds the quantization condition for coordinate-momenta is a sufficient condition for the existence of the canonical operator on Lagrangian submanifolds [68]. This fact enables us to apply to general phase manifolds the theory of global asymptotical solutions of h-pseudodifferential equations, which is constructed in detail in R^{2n} [103]. The bicharacteristics of such h-pseudodifferential operators belong to those phase manifolds on which the operator symbols are given. The global calculus of h-pseudodifferential operators can also be defined on symplectic V-manifolds [67].

Note that in constructing the calculus of ordinary pseudodifferential operators on an important class of homogeneous symplectic manifolds (Boutet de Monvel and Guillemin, 1981, [18]) the quantization conditions do not arise, see [68].

2. Electron terms. Now we consider the second generalization of the notion of characteristics. Recall that the characteristics equations for a hyperbolic system of equations of the first order are obtained from the characteristic matrix by equating its determinant to zero. In the same way we obtain the characteristics for the systems of h-pseudodifferential equations

$$ih\frac{\partial u}{\partial t} + L\left(x, \frac{h}{i} D\right) u = 0, \qquad x \in R^n. \tag{1.5.1}$$

Here $u = (u_1, \ldots, u_m)$ and the symbol $L(x, p)$ denotes a smooth matrix-valued function, diagonalized by a smooth transformation. The equations of characteristics for system (1.5.1) are defined by the eigenvalues $\lambda_j(x, p)$, $j = 1, \ldots, l$ $(l \leqslant m)$ of the matrix $L(x, p)$ and have the form

$$\frac{\partial S}{\partial t} + \lambda_j\left(x, \frac{\partial S}{\partial x}\right) = 0, \quad j = 1, \ldots, l. \tag{1.5.2}$$

It is natural to pass from the finite-dimensional space, where the symbol $L(x, p)$ acts to an infinite-dimensional space, which results in equations with the operator-valued symbol $\hat{L}(x, p)$. If the spectrum of the operator $\hat{L}(x, p)$ is discrete and its eigenvalues have constant multiplicity, the asymptotical solution is constructed as in the finite-dimensional case (Maslov, 1965, [103], [104]). In this case the equations of characteristics are defined in the same way as the equations of characteristics for systems. They have the form of equations (1.5.2), but the functions $\lambda_j(x, p)$ are now the eigenvalues of the operator $\hat{L}(x, p)$. *If the spectrum of the operator $\hat{L}(x, p)$ is continuous, then the equations of characteristics are defined by the poles of the analytical continuation of its resolvent which are closest to the real axis.* This fact is used in the collision theory and in the theory of the decay of nuclei (the Gamov theory). Since the poles are complex, the characteristics are also complex in this case.

The eigenvalues $\lambda_j(x, p)$ of the symbol of the h-pseudodifferential operator are called terms or effective Hamiltonians in physical literature.

As an example consider the problem of interaction of heavy and light particles (the nuclei and the electrons). This problem is studied in the quantum theory of molecules and in the theory of collisions. In this case the small parameter appears in the Schrödinger equation only at the derivative corresponding to heavy particles. In the simplest one-dimensional model the equation is

$$ih\frac{\partial\psi}{\partial t} = -h^2\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2} + V(x, y)\psi.$$

The operator-valued symbol has the form

$$\hat{L}(x, p) = p^2 - \frac{\partial^2}{\partial y^2} + V(x, y).$$

Its eigenvalues (the electron terms) define the characteristics which describe the motion of heavy particles along the classical paths in the field generated by the light quantum particles.

The concept of equations with operator-valued symbols has a general character and can be used, in particular, to obtain the characteristics and the corresponding asymptotical expansions in terms of "smoothness". Examples of such characteristics are given by Grushin, 1972, [46], [47], and, in fact, by Boutet de Monvel and Guillemin, 1981, [18], and by Guillemin and Sternberg, 1979, [49].

It should be noted, that this general concept allows us to connect problems which seem, on the surface, absolutely different. For example, physicists' papers on the problem of predissociation of molecules with the intersection of electron terms helped Kucherenko, 1974, [84] in constructing the parametrix for non-strictly hyperbolic equations.

3. Pseudodifferential operators with complex characteristics. Global asymptotics. Now consider the third generalization of the notion of characteristics. Difference schemes and Markovian chains can now be represented as h-pseudodifferential equations. Then it is possible to obtain their characteristics. Note that the equations of characteristics are complex for Markovian chains, non-symmetric difference schemes, equations of the principal type, and also for the problems of the decay of nuclei mentioned above. For example, the simplest difference scheme $u_i^{n+1}-u_i^n = \frac{\tau}{h}(u_{i+1}^n - u_i^n)$ on the circle (τ is the step in t, and h is the step on the circle), represented in the form of an h-pseudodifferential operator

$$[(e^{ah\partial/\partial t}-1)-a(e^{h\partial/\partial x}-1)]u = 0, \qquad a = \frac{\tau}{h},$$

leads directly to the complex characteristics equation

$$\frac{\partial S}{\partial t} + H\left(\omega, \omega\frac{\partial S}{\partial x}\right) = 0,$$

where $\omega \in [0, 2\pi]$ is the parameter (see equation (0.9)), and

$$H(\omega, p) = \frac{1}{i\omega a}\ln\big(1+a[\exp(ip)-1]\big)$$

is a complex-valued Hamilton function *with the non-positive imaginary part.* The equations describing processes with absorption lead to complex characteristic equations with analogical properties.

Complex solutions of real analytical equations of characteristics were considered in the famous paper by Leray, Gårding and Kotake, 1964, [96], in connection with the study of the singularities of non-analytical solutions of partial differential equations. In physical literature complex solutions of real analytical Hamilton–Jacobi equations and of the Hamilton system were used long ago (Keller, 1956, in the problem of reflection [70]; Maslov, 1963, in the problem of scattering [102]; Kravzov, 1967, in the analogous problem of refraction [81], and others). This approach encountered essential difficulties which arise in obtaining analytical solutions of the Hamilton–Jacobi equation and choosing the right branch of the multi-valued solution. These difficulties were avoided when the problem was solved modulo $O(h^N)$ (or modulo n times differentiable functions) with the help of constructions based on the following simple idea. Let the asymptotical solution have the form $\exp\left(\frac{i}{h}S(x)\right)\varphi(x)$, where $\operatorname{Im} S \geqslant 0$ (which is necessary for the boundedness of the solution as $h \to +0$). It is clear that the values of the functions S and φ in the domain $\operatorname{Im} S \geqslant \delta > 0$ are not essential, since the solution in this domain vanishes with the accuracy considered. Then the imaginary part of the function S acts as an additional small parameter which follows from the estimate:

$$(\operatorname{Im} S)^\nu \exp(ih^{-1}S) = O(h^\nu).$$

Thus one can construct asymptotical analogues of the analytic Hamiltonian formalism, in which *analyticity is reduced by almost analyticity*, namely, it is required that *the Cauchy–Riemann conditions should be satisfied* mod $((\operatorname{Im} S)^N)$.

This idea was used in [108] heuristically in the theory of the complex germ. Almost analytical formalism is also based directly on this idea. It was also used by Treves in [154] to construct the parametrix of the equation of the principal type.

Since 1965, [103], the n-dimensional submanifolds of the phase space $R_x^n \times R_p^n$, which annihilate the symplectic form, have been the basic geometrical construction in the theory of characteristics. The author called them Lagrangian manifolds.

The geometrical basis of almost analytical formalism is the notion of the analytical Lagrangian manifold. It is locally a $2n$-dimensional real submanifold in the complex phase space $C^{2n} = R^{4n}$. This approach is close to the analytical theory and has theoretical harmony.

The main geometrical construction in another approach is the "Lagrangian manifold with the complex germ". The structure of the complex germ is given on an n-dimensional real manifold Λ which is "almost Lagrangian" by means of imbedding into the complex phase space $C_Q^n \times C_p^n$. This means that on Λ a non-negative function D (called the dissipation) and the function W satisfying the condition $dW = PdQ + O(D)$ are given.

The dissipation condition is assumed to be satisfied. This means that the planes tangent to Λ are C-Lagrangian on the zero set of dissipation; namely, they are real-similar, they annihilate the form $dP \wedge dQ$ and satisfy the positivity condition $\operatorname{Im}(P_\alpha g, Q_\alpha g) \geqslant 0$, $\forall g \in C^n$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ are the coordinates on the plane.

The advantage of the theory of the complex germ over the almost analytical theory is the following. Asymptotical formulas obtained by means of the complex germ are more constructive and simpler. For example, for the construction of the global asymptotics of an h-pseudodifferential equation of the first order it is sufficient to solve only the Hamilton system and the system in variations.

Namely, let $H(x, p)$ be a smooth complex-valued Hamilton function corresponding to the asymptotical problem, $\mathscr{H} = \operatorname{Re} H$, $\tilde{H} = \operatorname{Im} H \leqslant 0$.

To construct the asymptotics in the theory of the complex germ the (real) Hamilton system and the system of equations in variations should be solved:

$$\dot{q} = \mathscr{H}_p, \quad \dot{p} = -\mathscr{H}_q,$$

$$\dot{z} = \mathscr{H}_{qq}z + \mathscr{H}_{qp}w + i\tilde{H}_q,$$

$$\dot{w} = -\mathscr{H}_{pq}z - \mathscr{H}_{pp}w - i\tilde{H}_p.$$

Almost analytical formalism and the theory of the complex germ enable us to construct global asymptotical solutions of the problems mentioned above and to obtain simpler formulas by using additionally only the solution of system in variations. The last fact is essential for constructing the asymptotics of solutions of stationary problems (Maslov, 1977, [113]). Thus the asymptotics of some concrete quantum-mechanical spectrum were obtained [113], [32].

Analogous spectral series were obtained by the method of model problems for the Laplace equation on a Riemann manifold (see Babich and Buldirev, 1972, [6]; Lazutkin, 1969, [92]).

The theory of the complex germ permitted the construction of a complete system of semi-classical coherent states for the Schrödinger operator, the Klein–Gordon operator, and the Dirac operator (Bagrov, Belov, and Ternov, 1982, [8], [7]).

I call asymptotic formulae obtained by means of the theory of the complex germ additive asymptotical formulae. They have the form $\psi = \psi_0 + O(h^N)$, where ψ is the exact solution and ψ_0 is the asymptotical one. Such asymptotical formulae in the diffraction theory describe only the domain of partial shadow. In the theory of probabilities they describe only normal deviations.

Distinct from additive asymptotical formulae, multiplicative formulae have the form $\psi = \psi_0\big(1 + O(h^N)\big)$ and describe in the diffraction theory the whole domain of shadow. In the theory of probabilities they describe large deviations.

The difference between these two situations can be shown by means of the following trivial example: the multiplicative asymptotical formula of the function $\exp\{(-x^2 - x^4)/h\}$ is the function itself but the additive asymptotical formula for this function has the form $\exp\{-x^2/h\} + O(h)$ as the following condition holds:

$$\operatorname*{Max}_x (x^4 \exp\{-x^2/h\}) = (2h/e)^2, \quad \text{and so on.}$$

4. Problems of the logarithmic asymptotics of the solution. The class of equations of the tunnel type. Instanton as the logarithmic limit.

4.1. *Equations of the tunnel type.* Now we consider the fourth generalization of the notion of characteristics. Equations whose solutions do not oscillate at all, but only damp, are of great importance. We call such equations tunnel equations. The asymptotical formulas for the solutions of such equations are also constructed with the help of characteristics, but the latter are obtained in another way, which is the result of the general definition of characteristics.

Compare the characteristics of equations with rapidly oscillating solutions and those of equations with rapidly damping solutions. For this purpose we consider the Schrödinger equation and a parabolic equation. They are very similar. On the left we write the Schrödinger equation with a small parameter h and its solution, on the right we write the parabolic equation, which also has a small parameter h.

Schrödinger equation	Parabolic equation
$ih\dfrac{\partial\psi}{\partial t} = -h^2\dfrac{\partial^2\psi}{\partial x^2} + V(x)\psi.$	$h\dfrac{\partial u}{\partial t} = h^2\dfrac{\partial^2 u}{\partial x^2} - V(x)u.$
The Green function for $V(x)=0$ has the form	The Green function for $V(x)=0$ has the form
$\psi_0 = \dfrac{1}{\sqrt{4\pi i t h}}\exp\left[\dfrac{i(x-\xi)^2}{4th}\right].$	$u_0 = \dfrac{1}{\sqrt{4\pi t h}}\exp\left[-\dfrac{(x-\xi)^2}{4th}\right].$

As $h\to 0$, the function ψ_0 rapidly oscillates and the function u_0 rapidly damps without oscillations. For $V(x)\in C_0^\infty$ and for small t the asymptotics of the Green function for the Schrödinger equation has the form:

$$\psi = \frac{1}{\sqrt{2\pi i h}}\sqrt{\left|\frac{\partial^2 S(x,\xi,t)}{\partial x\,\partial\xi}\right|}\exp\left[-\frac{i}{h}S(x,\xi,t)\right] + O(h),$$

where the function $S(x,\xi,t)$ satisfies the h-characteristic equation

$$\frac{\partial S}{\partial t} - \left(\frac{\partial S}{\partial x}\right)^2 - V(x) = 0.$$

The asymptotics of the Green function for the parabolic equation as $h \to 0$ has the form

$$u = \frac{1}{\sqrt{2\pi h}} \sqrt{\left| \frac{\partial^2 S_1(x, \xi, t)}{\partial x \partial \xi} \right|} \exp\left[-\frac{S_1(x, \xi, t)}{h} \right] \big(1 + O(h)\big),$$

where the function $S_1(x, \xi, t)$ satisfies the Hamilton–Jacobi equation

$$\frac{\partial S_1}{\partial t} + \left(\frac{\partial S_1}{\partial x} \right)^2 - V(x) = 0.$$

It is natural to consider the last equation as the equation of characteristics for the asymptotical problems corresponding to the parabolic equation.

Note the following. The Hamilton–Jacobi equation corresponding to the Schrödinger equation has a physical sense. Moreover, for the problem which is described by the Schrödinger equation with a small parameter h this equation was considered earlier, since it describes the classical motion of a particle. The physical sense of the Hamilton–Jacobi equation is not clear, but the solution u of the parabolic equation has a remarkable property: *there exists a limit*

$$\lim_{h \to 0} h \ln u = -S_1(x, \xi, t).$$

The global asymptotics of the parabolic equation in the multidimensional case was constructed by Maslov in 1965, [103]. For an important class of equations which are connected with the probability problems the logarithmic limits of the solutions were first obtained in the outstanding papers of Varadhan, 1966, [157]; 1967, [158]; and Borovkov, 1967, [17].

The global asymptotics for the heat conduction equation with varying coefficients in degenerated case was constructed by Molchanov, 1975, [130], Kifer, 1976, [73], Maslov and Chebotarev [20].

General systems of equations with this property were published by the present author as late as 1981, [121], though the term "tunnel equations" had been introduced in 1965 in "The perturbations theory and asymptotical methods". In the preface to that book the author promised to devote his next work to these equations. However, he has not done so. He has abandoned this theme, considering the class of tunnel equations to be too narrow. It is difficult to invent examples of systems of equations which would belong to this class, besides the parabolic equations, which are the only equations in the class of differential equations containing the derivative of the first order with respect to time. Then it turned out that the narrowness is not a defect of this class but on the contrary, its advantage, since it absorbs, from among all the possible

Hamiltonians, the right physical models. Probably, the only h-pseudo-differential (not differential) equation of the first order with respect to time which belongs to this class is the general integro-differential Kolmogorov equation in the theory of probabilities.

We give the tunnel conditions in the case of systems of h-pseudo-differential equations of the form

$$h\frac{\partial u}{\partial t} = A\left(-h\frac{\partial}{\partial x}, x, t\right)u. \tag{1.6}$$

Here $A(p, x, t)$ is a $(2n+1)$-parametric $m\times m$ matrix-valued function whose elements are entire functions of the argument p and smooth functions in x and t. Denote by $H_a(p, x, t)$, $a \leqslant m$ the eigenvalues (Hamiltonians) of the matrix $A(p, x, t)$.

DEFINITION. A system is called a *system of the tunnel type* if all its Hamiltonians H_a have constant multiplicity as $p \in R^n\setminus\{0\}$, $t \in [0, T]$ and satisfy the following conditions:

(1) for $t\in[0, T]$, $x \in R^n_x$, $p\in C^n\setminus\{\bigcap_{i=1}^{n}\operatorname{Re}p_i = 0\}$ the function $H(p, x, t)$ depends regularly on its argument p,

(2) $\max_\eta \operatorname{Re} H(p+i\eta, x, t) = H(p, x, t)$, $|p| \neq 0$,

(3) the Lagrangian L is equal to $\langle p, H_p(p, x, t)\rangle - H(p, x, t) \geqslant 0$,

(4) $\det\|H_{pp}(p, x, t)\| \neq 0$ for $|p| < \infty$.

These conditions are valid for the linearized Navier–Stokes system with small viscosity, for systems of equations arising in the theory of plasma, in magnetic hydrodynamics (e.g., see [87]), in the theory of elasticity, for the Boltzmann equation, for systems of the Semenov equations (e.g., see [145]), of the chain reaction, for the equation of drop coagulation (e.g., see [71]).

As an example we consider the equation which describes the Foycht model in the theory of elasticity:

$$\frac{\partial^2 u}{\partial t^2} = E\frac{\partial^2 u}{\partial x^2} + h\frac{\partial^3 u}{\partial x^2\,\partial t},$$

where $E = E(x) \geqslant 0$ is a smooth function, and $h > 0$ is a small parameter. To this system correspond the following Hamiltonians:

$$H_{1,2} = \frac{p^2}{2} \pm p\left(\frac{p^2}{4} + E\right)^{1/2}.$$

The verification of the tunnel conditions for these Hamiltonians requires hard calculations.

4.2. *Construction of the asymptotics of the solution at focal points.* The construction of the asymptotics of the Green function near the focal points for the equation of the tunnel type (1.6) is analogous to the corresponding construction for the oscillating solutions of h-pseudodifferential equations.

Consider this analogy in the case of the scalar equation $ih\,\partial/\partial t = H(-ih\,\partial/\partial x, x, t)\psi$. For simplicity, let $\mathrm{tr} H_{px} = 0$. In this case the asymptotical Green function is constructed with the help of the canonical operator[1] on the Lagrangian graph of the canonical transformation g_H^t, which corresponds to the shift along the trajectories of the Hamiltonian H through time t. The canonical operator defines the projective representation K of the group of canonical transformations of the phase space R^{2n} in the space of functions asymptotical $\mathrm{mod}\, O(h^2)$ on $L_2(R^n)$. This representation is characterized by the following properties:

(1) If $g(\xi, p_0)\to(x, p)$ has no focal points, i.e., its Lagrangian graph is diffeomorphically projected on the plane (ξ, x), then $K(g)$ is an integral operator with the kernel

$$(-2\pi ih)^{-n/2}\left|\frac{\partial^2 S}{\partial\xi\,\partial x}\right|^{1/2} e^{-iS(x,\xi)/h},$$

where $S(x, \xi)$ is the producing function,

(2) $K(g_1\otimes g_2) = K(g_1)\otimes K(g_2)$,

(3) $f_2\left(x, \frac{h}{i}D\right)K(g)f_1\left(x, \frac{h}{i}D\right) = 0$, if $g(\mathrm{supp} f_1)\cap(\mathrm{supp} f_2) = \emptyset$.

Denote $H_I(p) = \sum_{i\in I}\frac{p_i^2}{2}$, $I\subset\{1, \ldots, n\}$. If g is such a canonical transformation that for some $t>0$ and $I\subset\{1, \ldots, n\}$ the mapping $g_{H_I}^{-\tau}g$ has no focal points for $\tau\leqslant t$, then properties (1), (2) imply that $K(g)$ has the kernel

$$\mathscr{K}^g(x, \xi) = \tau^{-k/2}(-2\pi ih)^{-(n+k)/2}\int_{R^k}\left|\frac{\partial^2 S(\xi, y)}{\partial\xi\,\partial y}\right|^{1/2}\times$$
$$\times\exp\left\{-\frac{i}{h}\left(S(\xi, y)+\frac{1}{2\tau}\sum_{j\in I}(x_j-\eta_j)^2\right\}d\eta, \qquad (1.7)$$

[1] In the same way the canonical operator is introduced in the general problem of asymptotics with respect to smoothness (Maslov, 1965, [103]) and for the parametrix (Maslov, 1967, [105]; Duistermaat and Hörmander, 1972, [38]; Hörmander, 1971, [56]). In the last case the canonical operator is often called the integral Fourier operator. Moreover, the canonical operator is defined on a Lagrangian manifold [103], which enables us to solve stationary problems.

where $y_j = \eta_j$ for $j \in I$, and $y_j = x_j$ for $j \notin I$, and $S(\xi, y)$ is the producing function of the mapping $g_{H_I}^{-\tau} g$. Property (3) implies, that $K(g)$ can be represented as a zero-dimensional operator cocycle on the Lagrangian graph Λ_g of the mapping g: to any open set $\Omega \subset \Lambda_g$ corresponds a local kernel $\mathscr{K}_\Omega(x, \xi)$, $\mathscr{K}_{\Lambda g} = \mathscr{K}^g(x, \xi)$. This fact and the statement given below enable us to give the local version of formula (1.7).

LEMMA. *Any point $a \in \Lambda_g$ is non-singular with respect to the projection on a coordinate plane $\{p_0 = 0; x_i = 0, i \in I; p_i = 0, i \notin I\}$. If the point $a \in \Lambda_g$ is non-singular with respect to π_I, then for sufficiently small $\tau > 0$ the point $g_{H_I}^{-\tau} a \in \Lambda_{g_{H_I}^{-\tau} g}$ is non-singular with respect to the projection π on the plane $\{p_0 = 0, p = 0\}$.*

Let the point $a \in \Lambda_g$ be non-singular with respect to π_I. Then for a sufficiently small neighbourhood Ω of the point a the formula is valid, which is analogous to (1.7):

$$\mathscr{K}_\Omega(x, \xi) = \tau^{-k/2}(-2\pi i h)^{-(n+k)/2} e^{i\gamma} \int_{R^k} \left| \frac{\partial^2 S(\xi, y)}{\partial \xi \, \partial y} \right|^{1/2} \times$$

$$\times \exp\left\{-\frac{i}{h}\left(S(\xi, y) + \frac{1}{2\tau} \sum_{j \in I} (x_j - \eta_j)^2\right)\right\} \varphi(y)\, d\eta \qquad (1.8)$$

for (x, ξ) close to $\pi(a)$. Here $y_j = \eta_j$ for $j \in I$ and $y_j = x_j$ for $j \notin I$; φ is a "cutting" function equal to unity near $\pi\big(g_{H_I}^{-\tau}(a)\big)$; $S(\xi, y)$ is a local producing function of the mapping $g_{H_I}^{-\tau} g$; $\tau > 0$ is sufficiently small and the real number γ is defined by the Maslov index on the manifold Λ_g.

In the construction of the canonical operator one can use, instead of $g_{H_I}^{-\tau}$, other canonical transformations. In the original definition (Maslov, 1965, [103]) the rotation through an angle of $\pi/2$ with respect to some of the coordinates were used, i.e., $g_{\tilde{H}_I}^{\pi/2}$, where $\tilde{H}_I(x, p) = \frac{1}{2} \sum_{i \in I} (x_i^2 + p_i^2)$. More general canonical transformations were used in [56].

Formula (1.8) presents the method of calculating the asymptotical Green function

$$G(x, \xi, t) \approx \mathscr{K}_{g_H^t}(x, \xi)$$

for equation (1.6). In the neighbourhood of the point a we have

$$G(x, \xi, t) \approx \sum_{a \in \pi^{-1}(x_0, \xi_0)} \mathscr{K}_{\Omega(a)}(x, \xi), \qquad (1.9)$$

where $\Omega(a)$ is a small neighbourhood of the point a on $\Lambda_{g_H^t}$.

The asymptotics of the Green function for the tunnel equation (1.6) is given by the same formula as (1.9), where in the expression (1.8) for $\mathscr{K}_\Omega$ the parameter h should be changed into ih and the phase factor $e^{i\gamma}$ should be omitted, the parameter τ should be chosen so small as to keep the function $S(\xi, y)$ positive. Then the sum (1.9) is taken only at the points $\alpha = \{\xi_0, L_y[y(0), \dot{y}(0), 0], x_0, L_y[y(t), \dot{y}(t_0), t]\} \in R^{4n}_{\xi,p_0,x,p}$, where L is the Lagrangian corresponding to the Hamiltonian H, and $y(\cdot)$ runs over the set of solutions of the variational problem

$$\int_0^t L(y, \dot{y}, \tau)\, d\tau \to \inf, \qquad y(0) = \xi,\ y(t) = x.$$

The construction described here gives the asymptotics of the Green function uniform in t in the domain $0 < \delta \leqslant t \leqslant T$. The asymptotical solution which is uniform in t on $[0, T]$ has an essentially more complicated form. The proposed method permits minimizing the number of integrals at the focal point.

In the problem of the asymptotics of the Green function the focal points arise only in the case of equations with varying coefficients. As an example consider the Markovian chain

$$u_x^{t+h} = P_{x-h}^{+} u_{x-h}^{t} + P_{x+h}^{-} u_{x+h}^{t}, \qquad h > 0,$$
$$\{(k, j) \in Z_+ \times Z,\ t = kh,\ x = jh\},$$

where u_x^t is the probability of the chain being at time t in the state x,

$$P_x^{\pm} = \varphi^{\pm}|_{x=jh}, \qquad \varphi^{+}(x) = \frac{1}{4} + \frac{\cos^2 x}{2}, \qquad \varphi^{-}(x) = \frac{1}{4} + \frac{\sin^2 x}{2}.$$

The following h-pseudodifferential equation corresponds to this Markovian chain:

$$e^{h\partial/\partial t} u = \{e^{-h\partial/\partial x}\varphi^{+}(x) + e^{h\partial/\partial x}\varphi^{-}(x)\} u,$$

its Hamiltonian having the form

$$H(x, p) = \ln\{\varphi^{+}(x)e^{p} + \varphi^{-}(x)e^{-p}\}.$$

This Hamiltonian is of the tunnel type. The focal point for g_H^t arises at time $t = \pi\sqrt{3}/2$.

4.3. *Asymptotics of solutions of stationary problems.* An altered construction can be used to obtain the asymptotics of the solution of stationary problems, e.g., of the stationary Schrödinger equation

$$\left(-\frac{h^2}{2}\Delta+V(x)\right)\psi_k=\lambda_k\psi_k, \tag{1.10}$$

where $\psi_k=\psi_k(x)\in L_2(R^n)$ is the eigenfunction, λ_k is the eigenvalue corresponding to ψ_k, and $V(x)$ is a positive smooth function such that $|V(x)|\to\infty$ for $|x|\to\infty$. This function is called potential.

In such problems characteristics of a different type may arise. For example, consider the Schrödinger equation for the quantum oscillator

$$\left(-\frac{h^2}{2}\frac{d^2}{dx^2}+\omega^2\frac{x^2}{2}\right)\psi_k=\lambda_k\psi_k, \qquad \omega=\text{const},\ \omega>0,$$

where k is the number of eigenfunction. To such numbers $k\to\infty$ that $kh\to\mathscr{E}/\omega=\text{const}$ as $h\to 0$ correspond the classical characteristics Λ_k: $\{p^2+\omega^2x^2=2\lambda_k\}$, i.e., the closed curves in the phase space. The values of λ_k close to $\mathscr{E}$ are determined by the quantum condition $(2\pi h)^{-1}\oint p\,dq=\lambda_k/(\omega h)=k+1/2$ and the eigenfunctions ψ_k are asymptotically equal to the canonical operator constructed on the curve Λ_k and applied to the function equal to 1.

On the other hand, to the vacuum state (the state for $k=0$ and $\lambda_k=\omega h/2$) corresponds the eigenfunction $\psi_0=\exp(-\omega x^2/2h)$, whose logarithmic limit is equal to $S(x)=-\lim_{h\to 0}h\ln\psi_0=\omega x^2/2$. The function $S(x)$ is the characteristic corresponding to the Hamiltonian $H=p^2/2-\omega^2x^2/2$ with the potential $V(x)=-\omega^2x^2/2$, which is the "inverted" with respect to the classical potential $V(x)=\omega^2x^2/2$. Thus the function ψ_0 can be written in the form

$$\psi_0=\exp\left(-\frac{1}{h}\int_0^x p\,dx\right),$$

where (p,x) is the solution of the characteristics equation $p^2-\omega^2x^2=0$ distinguished by the condition $\int_0^x p\,dx\geqslant 0$.

In the general case the exponential asymptotics of the eigenfunctions corresponding to lower energetic states of the Schrödinger equation (1.10) is defined by the characteristic equation with the Hamiltonian $H(x,p)=p^2/2-V(x)$. This Hamiltonian differs from the known quantum Hamiltonian (1.10) in "turning over" the potential. If the potential $V(x)$ has some points $\xi_0,\xi_1,\ldots,\xi_l$ of the global minimum, then the logarithmic limit of the eigenfunction corresponding to the vacuum state is defined by the solution of the following variational problem:

$$\int (\dot{q}^2/2 - V(q)\, d\tau \to \min_k \inf_t \inf_l ,$$

$$l = \{q = q(\tau),\ \tau \in [0, t] |\ q(0) = \xi_k,\ q(t) = x\}.$$

Let the potential $V(x)$ have two points ξ_0, ξ_1 of the global minimum and be a function symmetrical with respect to, for example, the plane $\{x_1 = 0, x_2 = 0, \ldots, x_{n-1} = 0\}$. Then the two lower (vacuum) eigenvalues λ_0 and λ_1 differ by a value exponentially small with respect to the parameter h (whereas the distance between the eigenvalues is a value of the order h). Note that for $\Delta\lambda = \lambda_1 - \lambda_0$ there exists a logarithmic limit

$$-\lim_{h\to 0} h \ln \Delta\lambda = \min_l \int_l p\, dq, \tag{1.11}$$

where $l = \{p = p(t), q = q(t)\}$ is the solution of the Hamilton system

$$\dot{q} = p, \quad \dot{p} = V_x(q)$$

satisfying the conditions

$$q|_{t=-\infty} = \xi_0, \quad q|_{t=\infty} = \xi_1.$$

The trajectories l minimizing the integral in (1.11) correspond to the so-called instantons. The role played by instantons in the splitting of energetic levels of vacuum states is intensively studied in physical literature at present ([171], [164], [11]).

Note in conclusion that the tunnel canonical operator can be applied as well in nonlinear problems to the Boltzmann equation in the case of weak collisions, to large deviations, to the system of equations of gas dynamics with small viscosity, to the asymptotics of the shock wave running far ahead, and to some other problems. However, the formation of this asymptotics involves great mathematical difficulties.

5. Characteristics and global asymptotics as $h\to 0$ for equations with nonlocal nonlinearity. Equations of the Vlasov type for the propagation of wave fronts of oscillations. Bicharacteristics defining canonical transformations. Now we consider the fifth generalization of the notion of characteristics. We consider the characteristics of nonlinear h-pseudodifferential equations. Unfortunately, the equations which describe them are usually more complicated than the Hamilton–Jacobi equations. On the other hand, they usually correspond to well-known physical problems, though physically these problems are sometimes not connected with the original equations at all.

We first consider an important class of nonlinear equations of the following form. Assume that the symbol of a linear h-pseudodifferential operator depends on the solution of the given equation in some special way. Namely, it depends only on the square of its module, to which some linear smoothing operator has already been applied.

In this case the original symbol for a wide class of solutions has a limit as h tends to zero.

Owing to this fact it is possible to solve the preliminary problem of the propagation of the wave front of oscillations. This problem in the linear situation is similar to the problem of the Hörmander wave front for the singular solution. Then the global asymptotical solution can be constructed with rather general initial data. As an example we consider an analogue of the bicharacteristics system for the nonlinear Schrödinger equation with non-local interaction:

$$ih\frac{\partial\psi}{\partial t} = -\frac{h^2}{2}\Delta\psi - \psi\int K(x-\xi)|\psi(\xi,t)|^2 d\xi, \tag{1.12}$$

where $x = (x_1, \ldots, x_n)$, $\xi = (\xi_1, \ldots, \xi_n)$, $h \to 0$. Here the bicharacteristics are defined by the systems of partial integro-differential equations

$$\frac{\partial Q(x,p,t)}{\partial t} = \Xi(x,p,t), \quad Q(x,p,0) = x,$$

$$\begin{gathered}\frac{\partial \Xi(x,p,t)}{\partial t} = \iint K'\big(Q(x,p,t) - Q(\xi,\eta,t)\big) f_0(\xi,\eta)\, d\xi\, d\eta, \\ \Xi(x,p,0) = p, \quad p = (p_1, \ldots, p_n), \quad Q = (Q_1, \ldots, Q_n), \\ \Xi = (\Xi_1, \ldots, \Xi_n), \quad K'(z) = \left(\frac{\partial K}{\partial z_1}, \ldots, \frac{\partial K}{\partial z_n}\right),\end{gathered} \tag{1.13}$$

where f_0 is equal to the weak limit, as $h \to 0$, of the expression

$$\left(\frac{1}{2\pi h}\right)^n \int e^{ip(x-\xi)/h} \psi(\xi, 0)\bar{\psi}(x, 0)\, d\xi,$$

i.e., of the density of oscillation of the function $\psi|_{t=0}$.

It turns out that *the solution of problem* (1.13) *defines the canonical transformation* $g(t)$: $(x, p) \to (Q, \Xi)$. In this case the function $f_0\big(g(t)^{-1}(x, p)\big)$ is a solution of the classical Vlasov equation for a self-consistent field ([111]). The solution asymptotics for the initial equation (1.12) can be calculated by the formula

$$\psi = K\big(g(t)\big)\psi|_{t=0},$$

where K is the canonical operator defined on the Lagrangian graph of the canonical transformation $g(t)$ (Maslov, 1976, [111], [115]). The asymptotics of the solutions of spectral problems for equations with integral nonlinearities are also calculated by means of the canonical operator, in particular the asymptotics of soliton-like solutions for the Ukawa and Choqward equations (Karasev and Maslov, 1979, [64], [65]; Chernykh, 1982, [21]). In the case of the Ukawa equation the calculation of the characteristics is reduced to the classical Blodchett–Langmur equation for the spherical diode ([89]). In the case of the one-dimensional Choquard equation the characteristics are defined by the well-known Dowson ([25]) solution of the equation for the self-consistent field.

Such equations with non-local nonlinearity show in the most complicated physical situation the advantages of the general operator approach to the characteristics. The second part of the book "Algebras with general commutation relations and their applications" by Karasev, Maslov, and Nazailinskii, published in 1979, is entirely devoted to the operator approach to equations of this type.

We have followed all the basic paths of generalization of the notion of characteristics. The general definition contains complex characteristics, which in the case of pure oscillations or in the case of pure damping turn to be real. In general case one can consider an operator-valued n-dimensional symbol and the n-tuple of non-commuting operators, which should be substituted into this symbol. Besides, the symbol may depend, generally speaking, on the solution itself, which results in a nonlinear equation.

Note that the characteristics contain more information than the asymptotical behaviour of the solution of this problem. Just as the classical characteristics, they help to classify the problems and to specify the problem itself, which is absolutely necessary when we consider a new area in mathematical physics. For example, they help a great deal in problems of microelectronics, optical electronics and other problems arising in the electronic computer construction.

Part II. Characteristics of nonlinear equations

1. Linear equations with rapidly oscillating coefficients. Nonlinear wave equations whose characteristics are equations of the dynamics of gases. Characteristics equations for nonlinear wave superposition are equations of the dynamics of a gas mixture. It is well known that the consideration of linear problems with unsmooth coefficients may sometimes give a start to proving the existence theorems for nonlinear equations. For example, penetrating investigations made by Bony, 1981, [15] into problems of the parametrix construction for pseudodifferential equations with an unsmooth symbol led him to the existence theorems for nonlinear equations. The parametrix for the problem with unsmooth coefficients is closely connected with the problem of asymptotics for pseudodifferential equations with rapidly oscillating symbols. The solution of the last problem is evidently an important phase in the investigation of oscillating solutions of the semi-linear equations. There is no doubt that delicate methods used by Bakhvalov, 1974, [9], De Giorgi and Spagnolo, 1973, [26], Lions, 1980, [98], Oleinik and others ([15], [84], [104], [79]) in problems of equations with rapidly oscillating coefficients will be used in problems of semi-linear equations with a parameter. Here we consider in the most general way the analogy between these two groups of problems.

First we deal with the solution asymptotics of the equations with rapidly oscillating coefficients. For example, consider the Schrödinger equation with the rapidly oscillating potential

$$ih\frac{\partial\psi}{\partial t} = -\frac{h^2}{2}\frac{\partial^2\psi}{\partial x^2} + V\left(\frac{\Phi_1(x)}{h}, \ldots, \frac{\Phi_l(x)}{h}, x\right)\psi, \tag{2.1}$$

where $\Phi_j \in C^\infty$, $V(\tau_1, \ldots, \tau_l, x) \in C^\infty$ and V is 2π-periodic with respect to the variables $\tau = (\tau_1, \ldots, \tau_l)$. The solution of (2.1) can be represented in the form

$$\psi = W\left(\frac{\Phi_1(x)}{h}, \ldots, \frac{\Phi_l(x)}{h}, x, t, h\right), \tag{2.2}$$

where $W(\tau_i, \ldots, \tau_l, x, t, h)$ is a 2π-periodical function with respect to its first arguments which, generaly speaking, depends irregularly on h and satisfies the equation:

$$ih\frac{\partial\psi}{\partial t} = \left(\frac{h}{i}D_x - i\sum_{j=1}^{l}\frac{\partial\Phi_j(x)}{\partial x}\frac{\partial}{\partial\tau_j}\right)^2 W + V(\tau_1, \ldots, \tau_l, x)W. \tag{2.3}$$

It is an h-differential equation with the operator-valued symbol, since the small parameter appears by the derivatives with respect to part of the arguments (such equations were considered in Part II.2). The symbol of this equation is the operator

$$L(x, p) = \left(p - i\sum_{j=1}^{l} \frac{\partial\Phi_j(x)}{\partial x}\frac{\partial}{\partial\tau_j}\right)^2 + V(\tau_i, \ldots, \tau_l, x),$$

which is defined on the torus T^l for fixed x, p.

For example, let $H(x, p)$ be the one-tuple eigenvalue of this operator and let $\chi(x, p, \tau_1, \ldots, \tau_l)$ be the eigenfunction corresponding to it. Then the expression

$$\left[K(g_H^t)\circ\chi\left(x, \frac{h}{i}D, \tau_1, \ldots, \tau_l\right)\psi_0(x)\right]\Bigg|_{\tau_j=\Phi_j(x)/h},$$

where $\psi_0 \in C_0^\infty$ is the solution asymptotics for the initial equation (2.1).

The problem of eigenvalues and eigenfunctions of the operator $L(x, p)$ was essentially simplified by Novikov, 1974, [133], and others ([59], [36]) in the case where the gradients of the phase $\Phi_1, \Phi_2, \ldots, \Phi_l$ are colinear and the function $\omega = V\left(\eta\left|\frac{\partial\Phi_1}{\partial x}\right|, \eta\left|\frac{\partial\Phi_2}{\partial x}\right|, \ldots, \eta\left|\frac{\partial\Phi_l}{\partial x}\right|, x\right)$ is the finite gap potential of the operator $-d^2/d\eta^2+\omega$ for every x. Then the eigenfunctions and the eigenvalues of the operator $L(x, p)$ can be expressed in terms of the Riemann θ-function, and the solution asymptotics for the initial equation (2.1) has a beautiful geometrical interpretation.

Nonlinear equations with small dispersion also have rapidly oscillating solutions analogous to (2.2) (Whitham, 1965, [165], [166]). Such equations include, in particular, the nonlinear wave equation $h^2(\partial^2u/\partial t^2 - \Delta u)+g(u, x) = 0$, the Korteweg–de Vries equation and others. The difference from the linear case is that the functions Φ_j (phases) in (2.2) are not given in advance, as can be shown on the example of the following equation:

$$h^2\left(\frac{\partial^2u}{\partial t^2} - \Delta u\right) + a(x)\,\mathrm{sh}\,u = 0, \quad x \in R^3, \quad a \in C^\infty. \tag{2.4}$$

The function

$$u = f\left(\frac{S(x, t)}{h}, x, t, h\right) \tag{2.5}$$

($S \in C^\infty, f(\tau, x, t, h) \in C^\infty$ is a 2π-periodic function with respect to the argument τ) is the solution of (2.4) if

$$\left\{\left[\frac{\partial S}{\partial t}\frac{\partial}{\partial \tau} + h\frac{\partial}{\partial t}\right]^2 - \left[\frac{\partial S}{\partial x}\frac{\partial}{\partial \tau} + h\frac{\partial}{\partial x}\right]^2\right\} f + a(x)\operatorname{sh} f = 0. \qquad (2.6)$$

Differing from the linear case, the asymptotical solution f of equation (2.6) is regular with respect to the parameter h and hence, the leading term $f(\tau, x, t, 0)$ satisfies equation (2.6) (ordinary with respect to the argument τ) for $h = 0$. The function $S(x, t)$ and two "constants" of integration (which are functions in variables x, t), on which the function $f(\tau, x, t, 0)$ depends, can be obtained from the 2π-periodicity conditions with respect to the argument τ for the functions $f(\tau, x, t, 0)$ and $\frac{\partial f}{\partial h}(\tau, x, t, 0)$. These conditions lead to the "clutched" system of equations, which is equivalent to a system of relativistic hydromechanics equations without whirls [107]. The asymptotical rapidly oscillating solutions and the corresponding equations of characteristics were studied and applied in Luke, 1966, [99], Zabusky, 1967, [168], Berezin and Karpman, 1966, [14], Miura and Kruskal, 1974, [129], Povsner, 1974, [138], Scott 1970, [144], Ostrovskii, 1966, [135], Pelinovskii, 1982, [136], Gurevich and Pitaevskii, 1973, [52] and others.

In the wave equations connected with the interference and interaction of waves which arise, in particular, by reflection from the boundary there exist multi-phase solutions which are nonlinear superpositions of solutions (2.5). They have the form

$$u = \mathscr{F}\left(\frac{S_1(x, t)}{h}, \ldots, \frac{S_l(x, t)}{h}, x, t, h\right),$$

where $\mathscr{F}(\tau, x, t, h)$ is a smooth function with respect to the arguments $\tau = (\tau_1, \ldots, \tau_l), x, t,$ being 2π-periodic with respect to each τ_j and, generally speaking, irregularly depending on h. In this case the leading term $\mathscr{F}(\tau, x, t, 0)$ satisfies the equation with partial derivatives with respect to the variables $\tau_1, \ldots, \tau_l$ [1]. An effective solution of this equation became possible after the papers by Novikov, 1974, [133] and other authors [133], [59], [36], [78], [82], [90], [102], [125], [155], [131], which were devoted to the finite gap almost periodic solutions. The connection between the finite gap almost periodic solutions and the function $\mathscr{F}(\tau, x, t, 0)$ was investigated by means of the problem of the reflection from the boundary for the nonlinear equation (2.4) (Dobrokhotov and Maslov, 1979, [30], [31]). It turned out that in this problem the function $\mathscr{F}(\tau, x, t, 0)$ is expressed in terms of the almost periodic finite gap solution of the sine-Gordon equation $\partial^2 v/\partial\eta^2 - \partial^2 v/\partial\xi^2 + \sin v = 0$.

Thus the finite gap solutions define the "superpositions" of the asymptotical solutions of the multidimensional equations (2.4) with variable coefficients and a small parameter. The same correspondence between the finite gap almost periodic solutions and multi-phase asymptotics was used for constructing the solutions of the Korteweg-de Vries equation (Dobrokhotov and Maslov, 1980, [32], [34], Flaschka, Forest and McLaughlin, 1980, [40]).

Flaschka, Forest and McLaughlin have reduced the corresponding system of characteristics to an elegant geometrically invariant form, and Novikov and Dubrovin have recently shown that it is equivalent to the equation of the dynamics of a gas mixture, and have constructed for this system a beautiful Hamiltonian formalism.

The corrections of the leading term of the asymptotical solution of the corresponding nonlinear equation (the higher terms of the asymptotical expansion) are defined from the linear equation (with different right sides). Its coefficients are functions in the leading term of the asymptotical solution and oscillate rapidly. By applying the procedure given at the beginning of this section, this equation is reduced to an equation with an operator-valued symbol. Its eigenvalues for $l \geqslant 2$, as a rule, change multiplicity, and for the construction of the solutions [34] the methods developed by Kucherenko, 1974, [84], in problems of non-strictly hyperbolic systems of equations are applied.

2. Nonlinear equations with complex characteristics. Wave superposition law and its analogy with the soliton addition law. In the case where the characteristics of nonlinear equations are complex, the nonlinear WKB method (the Whitham method) together with the theorem on complex germ leads to essentially simpler answers compared with the case of real characteristics. The complex characteristics arise, in particular, in the problem with the impedance boundary condition for the class of elliptic nonlinear equations

$$h^2 \sum_{j=1}^{n} \frac{\partial}{\partial x_j} a_{ij} \frac{\partial}{\partial x_i} u = g(u, x), \qquad x \in \Omega \subset R^n, \tag{2.7}$$

where $a_{ij}(x) \in C^\infty$, $\|a_{ij}\| > 0$, $g(u, x)$ is a smooth function in the arguments x and entire in the argument u, $g(0, x) = 0$, $\partial g/\partial u(0, x) > 0$ (such equations are known, in particular, in the theory of semiconductors [72], [3]). In this case the characteristics coincide with the characteristics of the corresponding linear equation. For example, in problem (2.7) the characteristics are the same for all the functions having the same part $g'_u(0, x)u$ linear in u. Highly interesting and general is the "law of nonlinear superposition" of the solution asymptotics, which can be formulated by using the results [93], [94] of the Dirichlet series theory. For example, for equation (2.5) this law does not depend on the number of variables x or on the nonlinear component with respect to the argument u of the function $g(u, x)$ (Dobrokhotov and Maslov, 1980, [33]). The application of this law enables us, by using the "one-phase" always complex solutions $f(S(x)/h, x)$, $\operatorname{Im} S > 0$, defined by one-dimensional Dirichlet series for the function $f(\tau, x)$, to reconstruct the "multiphase" already both complex and real-valued solutions $F\big(S_1(x)/h, S_2(x)/h, \ldots, S_l(x)/h\big)$. They are defined by means of the multidimensional Dirichlet series for the function $F(\tau_1, \ldots, \tau_l, x)$. In the case of $g = \operatorname{sh} u$ the Dirichlet series are summed up, the function $F(\tau_1, \ldots, \tau_l, x)$ can be expressed in terms of the multi-soliton solutions of the sine-Gordon equation, and the "superposition" formulas for the Dirichlet series turn out to be equivalent to the "superposition" formulas for solitons (Dobrokhotov and Maslov, 1977, [113], [29]).

Though the complex characteristics of equation (2.5) coincide with the characteristics of the corresponding linear equations, the nonlinear effects in the solution asymptotics of these equations are essential. For the equation of a semiconductor, for example, this results in a considerable increase of conductivity in the neighbourhood of some curve on its surface [28].

In conclusion, we give the solution asymptotics for a bundary problem with the impedance condition for the equation of an electrically neutral semiconductor $h^2 \Delta u = \operatorname{sh} u$ in a half-infinite straight elliptic cylinder

$$\Omega = \left\{ x \in R^3 \,\middle|\, \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} \leqslant 1,\ x_3 \geqslant 0 \right\}, \qquad a < b\sqrt{2}.$$

They have the form [28]

$$u_n = 2 \ln \frac{1 + \varphi_1 e^{is_l^k/h} + \varphi_2 e^{is_2^k/h} + \frac{1}{4}\varphi_1\varphi_2(\lambda_k^2 - 1)\lambda_k^{-2} e^{i(s_1^k - s_2^k)/h}}{1 - \varphi_1 e^{is_1^k/h} + \varphi_2 e^{is_2^k/h} + \frac{1}{4}\varphi_1\varphi_2(\lambda_k^2 - 1)\lambda_k^{-2} e^{i(s_1^k - s_2^k)/h}},$$

where the characteristics have the form

$$s_1^k = -\bar{s}_2^k = \sqrt{\lambda_k^2-1}\left(x_1+a+\frac{\beta x_1^2}{2\big((1+\beta(x_1+1)\big)}\right)+i\lambda_k x_3, \qquad k \sim \frac{1}{h},\ k \in Z_+,$$

$$\varphi_1 = \bar{\varphi}_2 = \sqrt{1+\beta(x_1+a)}, \qquad \beta = (-a+i\sqrt{b^2-a^2})/b^2$$

and

$$\lambda_k = \frac{1}{h}\sqrt{1+\frac{h^2}{4a^2}\left(\pi k+\tfrac{1}{2}\arccos\left(1-\frac{2a^2}{b^2}\right)\right)}$$

is the impedance.

If in the formula for u_k we set

$$s_1^k = -i(\sqrt{1+V^2}\,\xi+V\eta), \qquad s_2^k = -i(\sqrt{1+V^2}\,\xi-V\eta),$$

$$\varphi_1 = \varphi_2 = \frac{iV}{(1+V^2)^{1/4}}, \qquad \lambda_k = iV,\ V = \text{const},$$

then the function u_k modulo the factor i coincides with the soliton–anti-soliton solution of the sine-Gordon equation.

3. Singularity propagation in nonlinear equations. Conditions of the Hugoniot type are the analogue of characteristics equations for this problem. Singularities of branching type. Solutions with finite outliers at single points.

3.1. *Singularities in nonlinear equations.* In linear hyperbolic equations the characteristics define the dynamics of the discontinuous part of the solution. In modern educational literature this fact is described by means of the theory of distributions and the so-called ray expansions. The solution substituted in the equation is of the form:

$$\varphi_0+\theta(s)\varphi_1+s\theta(s)\varphi_2+\ldots,$$

where $\theta(\tau)$ is the Heaviside function, $s = s(x,t) \in C^\infty$, $\nabla_x s|_{s=0} \neq 0$, $\varphi_i \in C_0^\infty$, the surface of zeros of the function defines the discontinuity surface. The equations for the functions s and for the coefficients $\varphi_i = \varphi_i(x,t)$ of this series are obtained by equating successively to zero the coefficients at the distributions $\delta(\tau)$, $\theta(\tau)$, $\tau\theta(\tau)$, etc. This procedure is transferred to the semi-linear hyperbolic equations, written in the so-called divergence form. The possibility of such generalization is based on the fact that such a series forms a commutative algebra with respect to usual multiplication. Consequently, the powers of this series are again a series of the same type, and the derivative of this series will again be a series of the same type (with the addition of δ-function only).

In the nonlinear case, however, the equations for the functions s and φ_i do not split, but are pairwise recurrently clutched (with the same "clutching coefficient" [116]). The first two of these equations are called conditions of the Hugoniot type and are the generalization of the notion of characteristics equation for the semi-linear hyperbolic equation.

However, such a generalization for the nonlinear case does not suit arbitrary types of solution singularities, since the following two conditions should be satisfied for the corresponding series. Firstly, these series should form a commutative algebra with respect to multiplication, secondly, the smoothness of the subsequent terms should increase by unity (if the initial equation is a differential one).

It turns out that only two algebras satisfy these conditions in the situation of "general position". If we admit the Ivanov concept, [60], [61], about the multiplication of distributions, then only three algebras are possible [122].

The first algebra corresponds to the solution describing shock waves, the second corresponds to detonation waves, and the third describes infinitely narrow solitons (which will be discussed below).

Nevertheless, when the nonlinearity in the equation has the character of a power, then for special coefficients and solutions vanishing on one side of the discontinuity surface ($\varphi_0 \equiv 0$) algebras of the type $\theta(s)s^a\sum_{k\geqslant 0} s^k\varphi_k$, where a depends on the nonlinearity degree, are possible.

The propagation of singularities of this type is possible not only for hyperbolic equations. In this case the leading terms of the series after the substitution in the equation are defined not only by the derivatives but also by the powers of the unknown function. Then the equation of the characteristics is of the Hugoniot type, and in this case it is nonstandard. Such propagation of singularities for nonlinear parabolic equations has a physical sense and was considered in [169], [10].

Note in conclusion that the same semi-linear differential equation can be reduced to different divergence forms by multiplication by different powers of the dependent variable, which leads to different conditions of the Hugoniot type for the same semi-linear equation. However, it seems that there is a contradiction. Singular solutions arise as limits for more exact solutions with a small parameter ε by the linear differential operator [57], [22], [58], [114]. Therefore to whatever distribution the solution of this more exact equation converges as $\varepsilon\to 0$, the summands containing ε tend in the equation to zero as distributions (in the weak sense). This property does not allow us to multiply the equation by the powers of the dependent variable and to pass to another divergence form.

Thus, from the point of view of singular solutions it is necessary to write the semi-linear equation in the right divergence form. Then the solution of the limit equation does not depend on the initial equation. This fact is also valid for infinitely narrow solitons.

Nevertheless, the generalization of the characteristics responsible for the propagation of the solution singularities of semilinear equations depends in general on the type of the equations with a small parameter from which these semi-linear equations were obtained by passing to the limit. Such limits were intensively studied in [57], [22], [58], [114]. They are closely connected with the conditions of the Hugoniot type.

3.2. *Infinitely narrow solitons.* Infinitely narrow solitons are rather unusual discontinuity solutions. They arise as limits of some solutions, for example of the Korteweg-de Vries equation with a small dispersion h

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+h^2\frac{\partial^3 u}{\partial x^3}=0,$$

where $h\to 0$. This equation has soliton solutions of the form

$$u=B+A\cosh^{-2}\left(\frac{C(x-Vt)}{h}\right), \tag{2.8}$$

where A, B, C, V are constants. As $h\to 0$ such a solution tends to the function which is equal everywhere to B, but at one point $x=Vt$ this function is equal to $B+A$. We denote this function by $A\delta_1(x-Vt)+B$ and call it the infinitely narrow soliton. Such solutions were not considered in the linear theory. We consider for the Korteweg-de Vries equation the Cauchy problem with a one-parameter family of initial data (2.8), where $t=\tau$ and $B=\Phi_0(x)\in C_0^\infty$. The solution of this problem has the form $\Phi(x,t)+A(x,t)\delta_1\big(s(x,t)-\tau\big)$ as $h\to 0$. Here $\Phi(x,t)$ satisfies the limit equation, and $s(x,t)$ and $A(x,t)$ are connected by

$$\begin{gathered}3\frac{\partial s}{\partial t}+(A+3\Phi)\frac{\partial s}{\partial x}=0,\qquad s|_{t=0}=x/V,\\ \frac{\partial}{\partial t}\big(A(x,t)+2\Phi(x,t)\big)=0.\end{gathered} \tag{2.9}$$

These conditions are analogous to those of the Hugoniot type. It is natural to consider them as the equations of characteristics for the limit equation, corresponding to the propagation of an "infinitely narrow soliton" δ_1 on a variable smooth background. The background $\Phi(x, t)$ in the rapidly oscillating case was calculated by Lax, 1979, [91]. Note that these conditions are applicable to the more general multi-soliton case [119], [118], [123]. Like conditions of the Hugoniot type, they give a good qualitative and quantitative description of the initial equation also for not very small h. Numerical experiments confirm this fact [119]. The second expression in (2.9) essentially depends on the form of the initial equation with a small parameter. Thus the second condition should be obtained from the asymptotics. On the other hand, the consideration of solution asymptotics from this point of view enables us to simplify essentially, and hence to substantiate ([119], [118], [123], [179]) the asymptotical formulas obtained, for example, in physical papers [63], [75], [45], [136], [88], [69], [159], [132], [126]. In particular, even condition (2.9) has been unknown in physical literature.

This approach can be applied also to h-pseudodifferential semi-linear equations (in particular, to the Toda lattice equation, to difference equations, etc.). Considerations from the general point of view enable us to transfer the results obtained for some equations to others. For example, the important results obtained by Rusanov and Bezmenov, 1980 [119], for difference schemes have been used in the problem on asymptotics for nonlinear oscillations of a lattice (Maslov and Mosolov, 1983).

We have discussed the propagation of the singularities of solutions of quasi-linear equations, singularities whose support is of codimension 1. For the propagation of the singularities with the support at a point one can also write conditions of the Hugoniot type, namely, the characteristics equations. They well describe qualitatively, for example, some physical phenomena which arise near the centre of the typhoon (the so-called eye of the typhoon) if we consider the typhoon as the propagation of the singularities for some complicated system of quasi-linear equations.

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