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Optimal Control of Markov Processes*

1. Introduction

The purpose of this article is to give an overview of some recent developments in optimal stochastic control theory. The field has expanded a great deal during the last 20 years. It is not possible in this overview to go deeply into any topic, and a number of interesting topics have been omitted entirely. The list of references includes several books, conference proceedings and survey articles.

The development of stochastic control theory has depended on parallel advances in the theory of stochastic processes and on certain topics in partial differential equations. On the probabilistic side one can mention decomposition and representation theorems for semimartingales, formulas for absolutely continuous change of probability measure (e.g. the Girsanov formula), and the study of Ito-sense stochastic differential equations with discontinuous coefficients. It seems fair to say that these developments in stochastic processes were in turn to an extent influenced by their applications in stochastic control. For controlled Markov diffusion processes, there is a direct connection with certain nonlinear partial differential equations via the dynamic programming equation. These equations are of second order, elliptic or parabolic, and possibly degenerate. Stochastic control gives a way to represent their solutions probabilistically. There is an unforeseen connection with differential geometry via the Monge–Ampère equation.

Broadly speaking, stochastic control theory deals with models of systems whose evolution is affected both by certain random influences

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and also by certain inputs chosen by a "controller". We are concerned here only with state-space formulations of control problems in continuous time. Moreover, we consider only Markovian control problems in which the state x_t of the process being controlled is Markov provided the controller follows a Markov control policy. We shall not discuss at all the extensive engineering literature on input-output formulations particularly for linear system models, see Åström [1].

We shall mainly discuss the case of continuously acting control, in which at each time t a control u_t is applied to the system. However, in § 8 we briefly mention impulsive control problems, in which control is applied only at discrete time instants. In optimal stochastic control theory the goal is to minimize (or maximize) some criterion depending on the states x_t and controls u_t during some finite or infinite time interval. In § 2 we formulate a class of optimal control problems for Markov processes, with criterion (2.2) to be minimized. The distinction between problems in which x_t is known to the controller, and problems with partial observations is made there. When x_t is known, the dynamic programming method can be used. In principle, this method leads directly to an optimal Markov control policy, although it rarely gives the optimal policy explicitly. In § 3, both analytical and probabilistic approaches are indicated. Associated with dynamic programming is the Nisio nonlinear semigroup (§ 4). In § 5 we discuss methods of approximate solution and special problems. In § 6 a logarithmic transformation is applied to positive solutions of the backward equation of a Markov process. There results a controlled Markov process, leading to connections between stochastic control and such topics as stochastic mechanics, large deviations and nonlinear filtering. The case of controlled, partially observed processes is mentioned in § 7, along with adaptive control of Markov processes. Finally in § 9 we indicate a few of the various difficulties encountered in seeking to implement in engineering applications the mathematically sophisticated results of the theory, and mention some newer areas of application.

2. Controlled Markov processes

We consider optimal stochastic control problems of the following kind. We are given metric spaces Σ , U called the *state space* and *control space*, respectively. For each fixed $u \in U$ there is a linear operator L^u which generates a Markov, Feller process with state space Σ . The domain of L^u contains, for each $u \in U$, a set D dense in the space $C(\Sigma)$ of bounded uniformly continuous functions on Σ . The state and control processes

x_t, u_t are defined on some probability space (Ω, \mathcal{F}, P) . The Σ -valued process x_t is adapted to some increasing family of σ -algebras $\mathcal{F}_t \subset \mathcal{F}$, and the trajectories ω are right continuous. The U -valued process u_t is predictable with respect to an increasing family of σ -algebras $\mathcal{G}_t \subset \mathcal{F}_t$. The σ -algebra \mathcal{G}_t describes in a measure theoretic way the information available to the controller at time t . The processes (x_t, u_t) are related by the requirement that

$$Mg(t) = g(x_t) - g(x_0) - \int_0^t L^{u_s} g(x_s) ds \tag{2.1}$$

is a (\mathcal{F}_t, P) martingale for every $g \in D$. We consider a fixed, finite time interval $0 \leq t \leq T$, and the objective to minimize a criterion of the form of an expectation

$$J = E \left\{ \int_0^T k(x_t, u_t) dt + G(x_T) \right\}. \tag{2.2}$$

Example 1. Controlled finite-state Markov chain, with $\Sigma = \{1, 2, \dots, N\}$. In this case L^u is identified with the infinitesimal matrix (q_{ij}^u) of the chain. When the control u_t is applied, the jumping rate of x_t from state i to j is $q_{ij}^{u_t}$.

Example 2. Controlled diffusion process with $\Sigma = \mathbf{R}^n$,

$$x_t = x_0 + \int_0^t f(x_s, u_s) ds + \int_0^t \sigma(x_s, u_s) dw_s, \tag{2.3}$$

with w_t a brownian motion (of some dimension d) independent of the initial state x_0 . In this case

$$L^u = \frac{1}{2} \sum_{i,j=1}^n a_{ij}(x, u) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n f_i(x, u) \frac{\partial}{\partial x_i} \tag{2.4}$$

with $a = \sigma\sigma'$ and $D = \{g: g, g_{x_i}, g_{x_i x_j} \in C(\mathbf{R}^n), i, j = 1, \dots, n\}$. The diffusion is called *nondegenerate* if the eigenvalues of $a(x, u)$ are bounded below by $c > 0$.

Further assumptions, which vary from author to author in the literature, need to be made. To avoid undue complication, in the discussion to follow we take a compact control space U , and $k(x, u), G(x)$ bounded,

uniformly continuous. In (2.3), $f(x, u)$, $\sigma(x, u)$ are bounded and as smooth as necessary. The σ -algebras $\mathcal{F}_t, \mathcal{G}_t$ are right continuous and completed.

If x_t is \mathcal{G}_t -measurable, then the controller can observe the state x_t . In this case, one may as well take $\mathcal{G}_t = \mathcal{F}_t$ and known initial state x_0 . This is the situation in Sections 3–6 to follow. If (2.1) holds, we call

$$\alpha = (\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}, x_0, u)$$

an *admissible system* for the control problem with completely observed states.

A *Markov control policy* is a Borel measurable function from $[0, T] \times \Sigma$ into U . An admissible system α is obtained via a Markov control policy \underline{u} if

$$u_t = \underline{u}(t, x_{t-}). \quad (2.5)$$

Given \underline{u} and $x_0 \in \Sigma$, one would like to know whether a corresponding admissible system exists, with x_t a Markov process. Under sufficiently strong restrictions this is well known. For instance, in case of controlled diffusions a Lipschitz condition on $\underline{u}(t, x)$ would imply the classical Ito conditions. For nondegenerate controlled diffusions, existence follows from Krylov [8, p. 87] for any bounded \underline{u} . The Markov property of x_t can be obtained under stronger hypotheses. For instance, for nondegenerate diffusions it holds if in (2.3) $\sigma = \sigma(x)$. A martingale method for obtaining the Markov property is to show that the probability distribution $P_{x_0}^{\underline{u}}$ of the state trajectory x is unique and depends continuously on the initial state x_0 [59]. In general x_t is only a weak-sense solution to (2.3), since neither the probability space nor the brownian motion w_t are given in advance. However, in the nondegenerate case with $\sigma = \sigma(x)$ a result of Veretennikov [61] gives a strong solution.

3. Dynamic programming

The dynamic programming approach to the control problem with completely observed states x_t can be described in a purely formal way, as follows. For initial state $x_0 \in \Sigma$ and admissible system α , write $J = J(T, x_0, \alpha)$ in (2.2). Let

$$W(T, x_0) = \inf_{\alpha} J(T, x_0, \alpha). \quad (3.1)$$

Formal reasoning indicates that $W(T, x)$ should satisfy the dynamic programming equation

$$\frac{\partial W}{\partial T} = AW, \quad T > 0, \quad (3.2)$$

with initial data $W(0, x) = G(x)$, where

$$Ag(x) = \min_{u \in U} [L^u g(x) + k(x, u)]. \quad (3.3)$$

Formally, an optimal Markov policy \underline{u}^* is found by requiring $\underline{u}^*(t, x)$ to minimize $L^u W(T-t, x) + k(x, u)$ among all $u \in U$. Instead of the finite time control problem, control until x_t exits a given open set $\mathcal{O} \subset \Sigma$ can be considered. In that case the dynamic programming equation becomes the autonomous form of (3.2) in \mathcal{O} , with $W(x) = G(x)$ for $x \in \partial\mathcal{O}$. There are also autonomous dynamic programming equations associated with the infinite time control problem, with discounted cost or average cost per unit time criteria to be minimized.

In the rigorous mathematical treatment of dynamic programming there is one easy result, the so-called *Verification Theorem* [7, p. 159]. Roughly speaking, it states that if $W(T, x)$ satisfying (3.2) with the initial data and the associated Markov policy \underline{u}^* are both "sufficiently regular", then \underline{u}^* is indeed optimal and $W(T, x)$ is the minimum performance in (3.1). The Verification Theorem is used to obtain explicit solutions, in those cases where such a solution is known. Much more difficult are the questions of existence of sufficiently regular W and \underline{u}^* , and there is a large literature dealing with various aspects of them. One approach is analytical with the stochastic interpretation made afterward. In this approach, existence of solutions to the dynamic programming equation and their regularity properties are studied, using non-probabilistic methods. It is then proved that optimal (or at least ε -optimal) Markov control policies exist. A second approach is probabilistic. In this approach, one starts with the minimum cost function W in (3.1) and develops stochastic counterparts to the dynamic programming conditions for a minimum. A third approach is to consider an associated nonlinear semigroup (§ 4). While this approach leads to fewer technical difficulties than either of the other two, it also leads to weaker results.

For controlled diffusions the analytical approach is remarkably well developed (see Krylov [8], Lions [45]). In the nondegenerate case the dynamic programming equation is a second order nonlinear partial differential equation of parabolic type also called a Hamilton–Jacobi–Bellman

equation. In various other formulations, with x_t controlled for all time $t \geq 0$ or until exit from an open set \mathcal{O} , the Hamilton–Jacobi–Bellman equation is elliptic rather than parabolic. Under reasonable assumptions the problem, the solution S has generalized second derivatives which are locally bounded. In the elliptic case a deeper regularity result of Evans ([26], [60]) gives a classical solution. In the degenerate case W is less regular with locally bounded generalized first derivatives W_{x_i} . The dynamic programming equation (3.2), suitably interpreted in terms of Schwartz distributions, still holds ([8], [45]). For the case of controlled jump Markov processes, results on existence, uniqueness and regularity of solutions to (3.2) were obtained by Pragarauskas [52].

A large class of nonlinear elliptic or parabolic equations, satisfying appropriate convexity conditions, can be represented as Hamilton–Jacobi–Bellman equations. As Gaveau [35] pointed out, the Monge–Ampère equation has such a representation.

In the probabilistic approach, the starting point is to rewrite the dynamic programming principle in the following martingale form. Given an admissible system α let

$$m_t = \int_0^t k(x_s, u_s) ds + W(T-t, x_t).$$

Then m_t is a (\mathcal{F}_t, P) submartingale, and α is optimal if and only if m_t is a (\mathcal{F}_t, P) martingale. With the aid of the Doob–Meyer decomposition for submartingales and some martingale representation theorems, conditions for optimality are obtained (see Bismut [21], Davis [16], Elliott [25], El Karoui [5]). These conditions are probabilistic counterparts of those expressed analytically by the dynamic programming equation (3.2). With the probabilistic approach difficult questions of regularity of solutions to (3.2) are avoided. The probabilistic techniques give results about existence of optimal Markov policies ([21], [5, p. 218]). These methods also give conditions for a minimum for optimal control under partial observations.

A different kind of Markovian control problem for diffusions, in which the control acts only on the boundary of a region $\mathcal{O} \subset \mathbf{R}^n$ was considered by Vermes [62].

4. The Nisio nonlinear semigroup

The dynamic programming principle can be restated in another form, in terms of a semigroup of nonlinear operators. In purely formal way, this is done as follows. In (2.2) we fix k but consider various G . We rewrite

the infimum in (3.1) as $W(T, x) = S_T G(x)$. The dynamic programming principle is formally equivalent to the semigroup property

$$S_{T_1+T_2} = S_{T_1} \circ S_{T_2} \quad (4.1)$$

of the family $\{S_T\}$ of nonlinear operators. In addition, for "sufficiently regular" G , one should have

$$\frac{d}{dT} S_T G|_{T=0} = AG. \quad (4.2)$$

This formal procedure was put on a rigorous basis by Nisio [10], who obtained $\{S_T\}$ as a semigroup on the space $\mathcal{C}(\mathcal{X})$ and showed under some mild additional conditions that (4.2) holds for $G \in \mathcal{D}$ (notation of § 2). Equations (4.1), (4.2) would imply the dynamic programming equation (3.2) if we knew that $W(T, \cdot) = S_T G$ is sufficiently regular (in particular, if S_T maps \mathcal{D} into \mathcal{D}). However, W does not generally have the desired regularity. In such instances (4.2) is a kind of weaker substitute for (3.2).

Nisio's treatment is analytical. She obtains S_T as the lower envelope of the family of linear semigroups S_T^u , where for constant control $u \in U$ the generator of S_T^u coincides on \mathcal{D} with the operator $L^u + k(\cdot, u)$. A stochastic treatment of the Nisio semigroup is given in Bensoussan and Lions [2], and a uniqueness result in case of nondegenerate diffusions in Nisio [51]. El Karoui, Lepeltier, and Marchal [24] used another procedure, and obtained a nonlinear semigroup on a larger space of bounded functions \mathcal{G} which are measurable in a suitable sense.

5. Explicit and approximate solutions

In a few instances the dynamic programming equation (3.2) can be solved explicitly. Examples are the well known stochastic linear regulator and Merton's optimal portfolio selection problem [7, pp. 160, 165]. For other special problems the solution can be reduced to a free boundary problem. The boundaries to be determined separate regions where some control constraint holds or not. See for example Karatzas and Benes [40].

When a solution cannot be found by special methods, one can seek an approximate solution to (3.2). One class of approximate methods involve discretizations of (3.2). Among such methods the algorithm of Kushner [9] has a natural stochastic control interpretation. The difference equations associated with the algorithm correspond to the dynamic programming equation for an approximating controlled Markov chain.

For the special case of controlled one-dimensional diffusions, Borkar and Varaiya [22] used a procedure in which piece-wise constant approximating Markov control policies are allowed.

Other results give approximate solutions to (3.2) when the state process x_t is a nearly-deterministic controlled diffusion. In (2.3) let $\sigma = \varepsilon^{1/2} \tilde{\sigma}$. The solution is sought in the form of an asymptotic series in ε . In [29] this is done by expanding the solution $W^\varepsilon(T, x)$ in an asymptotic series. The expansion is valid in regions where the solution $W^0(T, x)$ of the corresponding Hamilton–Jacobi equation is smooth. In [20] Bensoussan obtains an asymptotic expansion, using a stochastic maximum principle instead of (3.2).

6. A logarithmic transformation

Consider a linear operator of the form $L + V(x)$, where L is the generator of a Markov process ξ_t with state space Σ . The initial value problem

$$\frac{d\varphi}{dT} = L\varphi + V(x)\varphi \quad (6.1)$$

with data $\varphi(0, x) = \Phi(x)$ has a probabilistic solution by a well known formula of Feynman–Kac type. For positive solutions of (6.1) another probabilistic representation for $\varphi(T, x)$ can often be found in the following way. The logarithmic transformation $I = -\log\varphi$ changes (6.1) into the nonlinear equation

$$\frac{dI}{dT} = H(I) - V(x), \quad (6.2)$$

$$H(I) = -e^I L(e^{-I}). \quad (6.3)$$

If one can find a control problem of the kind in § 2 such that

$$H(I) = \min_{u \in U} [L^u I + k(x, u)], \quad (6.4)$$

then (6.2) is the dynamic programming equation (3.2). The stochastic control interpretation of $I(T, x)$ is as the minimum of the criterion J in (2.2). Thus, in (3.1) we have $W = I$. For a nondegenerate diffusion obeying the stochastic differential equation

$$d\xi_t = b(\xi_t)dt + \sigma(\xi_t)dw_t, \quad (6.5)$$

a Markov control policy $\underline{u}(t, x)$ changes the generator L to $L^{\underline{u}}$, corresponding to change of drift from $b(x)$ to $\underline{u}(t, x)$ in (6.5). In (2.2) one takes

$$k(x, u) = \frac{1}{2}(b(x) - u)' a^{-1}(x) (b(x) - u),$$

with $a = \sigma\sigma'$. An appropriate control problem for the case of ξ_t a jump Markov process is described in [31], and for a general class of Markov ξ_t in Sheu's thesis [58]. The change of generator from L to $L^{\underline{u}}$ corresponds to a change of probability measure. It was pointed out by M. Day that this change of measure results by conditioning with respect to $\Phi(x_T)$ (see [31, (4.5)]).

In case $L = \frac{1}{2}\Delta$, corresponding to ξ_t a brownian motion (6.1) is the heat equation with a potential term. The stochastic control interpretation of $S = -\log\varphi$ is as least average action. Upon rescaling, taking $L = \frac{1}{2}\lambda\Delta$ and replacing V by $\lambda^{-1}V$, the usual least action is obtained as a "classical mechanical limit" as $\lambda \rightarrow 0$ [28]. The heat equation with potential is the "imaginary time" analogue of the Schrödinger equation of quantum mechanics. There is an intriguing connection between stochastic control and the Schrödinger equation, whose implications are not as yet well understood [36]. This work is in the framework of Nelson's stochastic mechanics. An apparently different theory of "stochastic mechanics" was developed by Bismut [4].

Holland [39] gave a stochastic control interpretation of the dominant eigenvalue of the Schrödinger equation as minimum mean total energy of a particle in equilibrium. The approach was again based on a logarithmic transformation and subsequently led to Sheu's treatment [58] of the Donsker-Varadhan formula for the dominant eigenvalue of the operator $L + V$ appearing in (6.1).

The Ventsel-Freidlin theory of large deviations deals with asymptotic probabilities of rare events associated with nearly deterministic Markov processes. The logarithmic transform gives another approach to results of this kind. As an illustration we consider the problem of exit from an open set $D \subset \Sigma$ during the time interval $0 \leq t \leq T$. Let w_t^ε be a Markov process tending to a deterministic limit w_t^0 as $\varepsilon \rightarrow 0$. Let $I^\varepsilon = -\varepsilon \log P_x(\tau^\varepsilon \leq T)$, where τ^ε is the exit time of w_t^ε from D . Under various assumptions (including a suitable scaling of ε), I^ε tends to a limit I^0 , where $I^0(T, \omega)$ is the minimum of a certain "action functional" among curves starting at $x \in D$ and leaving D by time T . In the stochastic control approach $I^\varepsilon(T, \omega)$ is the minimum performance in a corresponding stochastic

control problem [27], [31], [58]. In this approach a minimum principle is associated with the large deviation problem for $\varepsilon > 0$, not just in the limit as $\varepsilon \rightarrow 0$.

In [32], the logarithmic transformation was applied to solutions to the pathwise equation of nonlinear filtering, making a connection between filtering and stochastic control.

7. Partial observations; adaptive control

The states x_t of a stochastic system often cannot in practice be measured directly, or perhaps can only be measured with random errors. This has led to an extensive literature on nonlinear filtering and on optimal control under partial observations. For controlled diffusions, a standard model is to take state dynamics (2.3) and an observation process y_t governed by

$$y_t = \int_0^t h(x_s) ds + W_t, \quad (7.1)$$

with W a brownian motion independent of w . The information available to the controller at time t is usually assumed to be described by the σ -algebra \mathcal{G}_t generated by observations y_s for $s \leq t$. However, existence of optimal controls has been proved only with a somewhat wider class of admissible controls than those adapted to this family $\{\mathcal{G}_t\}$.

Several good survey articles on controlled partially observed diffusions have recently appeared [15], [16], [17]. Hence, we shall not try to summarize the various results here. In studying partially observed control problems it is useful to introduce an auxiliary "separated" control problem. In the separated problem the role of "state" process is taken by a measure-valued stochastic process σ_t [34]. The measure σ_t represents an unnormalized conditional distribution of x_t given observations and controls $y_s, u_s, 0 \leq s \leq t$. A nonlinear semigroup for the controlled, measure-valued process σ_t has been constructed [19], [30], [33]. Among other recent work, we mention that of Rishel [53] on partially observed jump processes, and of Mazziotto and Szpirglas [49] on impulsive control under partial information.

In *adaptive control* the objective is the simultaneous control and identification of unknown system parameters. Common techniques in discrete-time adaptive control involve sequential techniques, based on maximum likelihood or least squares, for updating esti-

mates of unknown parameters. In the context of adaptive control of Markov chains see the pioneering work of Mandl [48], also Borkar and Varaiya [23], Kumar and Lin [41]. Another (Bayesian) viewpoint is to treat adaptive control of Markov processes as a special case of stochastic control under partial observations. This is done by simply regarding the unknown parameters as additional (nontime-varying) components of the system state. From a practical standpoint this approach encounters well known difficulties, in that effective solutions to partially observed stochastic problems are difficult to obtain. Nevertheless, special cases in which the problem becomes finite-dimensional have been treated by Hijab [38] and Rishel [54].

8. Impulse control; problems with switching costs

In impulse control problems the control actions are taken at discrete (random) time instants, and each control action leads to an instantaneous change in the state x_i . Typical impulse control problems are those of stock inventory management, in which a control action is to reorder with immediate delivery of the order.

The analytic treatment of impulse control was initiated and developed systematically by Bensoussan and Lions [3], with emphasis on the control of nondegenerate diffusions. The dynamic programming equation is replaced by a set of inequalities which take the form of a quasivariational inequality. For the case of degenerate diffusions see Menaldi [50], and for impulsive control for Markov-Feller process see Robin [55], [56]. Lepeltier and Marchal [43] gave a probabilistic treatment.

Another class of stochastic control problems of recent interest are those in which control actions are taken at discrete time instants, with no instantaneous change in x_i but with a cost of switching control actions. Such problems arise in the theory of controlled queues (see Sheng [57]) and in control of energy generating systems under uncertain demand. The analytical treatment again is to reduce the problem to a quasivariational inequality (see Lenhart and Belbas [42], Liao [44]).

9. Applications

Optimal stochastic control theory was initially motivated by problems of control of physical devices. More recent influences have come from management science, economics, and information systems. Until now,

the impact on engineering practice of much of the sophisticated mathematical theory has been small. The stochastic linear regulator is a standard tool, because the optimal Markov control policies turn out to be linear in the state x . If the Markov policy is nonlinear, it is difficult to implement. Moreover, other issues may be considered in practice more important than optimality of system performance as predicted by the stochastic control model. The model is generally a simplification of nature, through linearizations, reductions of dimensionality, assumptions that noises are white, etc. A control which performs well (even optimally) according to the model may behave poorly in a real control system. The question of robustness of controls with respect to unmodelled system dynamics is of current interest in the engineering control literature (see for example [63]). A different sort of question is that of stochastic controllability [64].

We conclude by mentioning two novel applications of stochastic control. One is Arrow's model of exploration consumption, and pricing of a randomly distributed natural resource. This model was analyzed in detail by Hagan, Oaflich and Keller [37]. They determined approximately the free boundary between portions of the state space where new exploration should or should not be undertaken.

Ludwig and associates have applied a stochastic control method to fishery management problems [47]. The fishery resource is controlled through the rate at which fish are harvested. This work has an important statistical aspect as well as the control aspect, since errors in measuring unknown parameters in the fishery model can be important.

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