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Turbulent Dynamical Systems

Recent developments in the dynamical theory of the onset of turbulence are reviewed, and some historical and philosophical aspects of the subject are discussed.

Introduction

The phenomenon of hydrodynamic turbulence constitutes one of the great puzzles of theoretical physics, and deep studies have been devoted to it, over a long period of time. In recent years a conceptual clarification of some essential points has been obtained. Yet, other essential points remain obscure, and we do not have to-day a general theory of turbulence.

From an operational point of view it is easy enough to agree that certain flows are turbulent. Since the phenomenon is ubiquitous, and accessible to everyone for inspection, a number of theoretical interpretations have been proposed. Von Neumann [82] has given a lucid review of the early theories of turbulence. Many good ideas are contained in these theories, but they fail to explain the phenomenon. In brief, this failure can be understood as follows. The basic equations of hydrodynamics are nonlinear partial differential equations; their mathematical study is hard, and so is their numerical treatment in the turbulent regime. Experimental studies are also much more difficult than one would think, when precise quantitative information is desired.

The recent improvement of our understanding of the nature of turbulence has three different roots. The first is the injection of new mathematical ideas from the theory of dynamical systems. The second is the availability of powerful computers which permit, among other things, experimental mathematics on dynamical systems and numerical simulation of hydrodynamic equations. The third is the improvement of experimental techniques (in particular, Doppler measurements of velocities by use of a laser beam, and then numerical Fourier analysis of the time series obtained).

The discussion of hydrodynamic time evolution from the point of view of the qualitative theory of dynamical systems has had a profound impact. This was at first not due to the proof of any deep theorem, but rather to the reevaluation of accepted ideas. In the present review we shall analyze the somewhat complex evolution of ideas on the onset of turbulence, and the related problem of sensitive dependence on initial condition. Our discussion will thus emphasize historical and philosophical aspects.[1] I am aware of the difficulties of such an enterprise, but feel that the articulation of mathematics, physics and computer work in to-day's science deserves study, and that an imperfect discussion is better than no discussion at all.

Hydrodynamic time evolution as a dynamical system

A reasonable description of fluid dynamics is given by the *Navier–Stokes equation* (1) and the *incompressibility condition* (2) in d dimensions ($d = 3$ normally, but $d = 2$ is also much studied):

$$\frac{dv_i}{dt} = -\sum_{j=1}^{d} v_j \cdot \frac{\partial v_i}{\partial x_j} + \nu \Delta v_i - \frac{\partial p}{\partial x_i} + g_i, \tag{1}$$

$$\sum_j \frac{\partial v_j}{\partial x_j} = 0. \tag{2}$$

In these equations (v_i) is the velocity field of the fluid enclosed in a region Ω, ν is the viscosity coefficient, p the pressure (divided by the density), and (g_i) describes an external force. By projecting (1) on a space of divergence free vector fields, one takes (2) into account, and the pressure term is eliminated. The fluid sticks to the boundary. This boundary condition is imposed by writing $v_i = v_i^0 = v_i^0 + w_i$ where (w_i) belongs to a functional space $\mathscr{H}$ of vector fields vanishing on $\partial\Omega$. One has thus an evolution equation of the form

$$\frac{dw}{dt} = F_\mu(w), \tag{3}$$

where the right-hand side is assumed to be time independent, and μ is a parameter describing the intensity of external action exerted on the fluid through the force (g_i) or the boundary condition (v_i^0).

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[1] For reviews of the same subject from a different viewpoint see Ruelle [69], [71].

The derivation of (1) involves a "linear response" approximation, and the Navier–Stokes equation is thus essentially less exact than the equations of celestial mechanics for instance. In the two-dimensional case ($d = 2$) there is a good existence and uniqueness theorem (Leray, Ladyzhenskaya). For $d = 3$, existence and uniqueness can be proved for small times, and a "weak solution", introduced by Leray[2] may have singularities and not be unique. In fact it is not known at present if singularities actually occur.[3] One knows, however, that the set of singularities cannot be too large. The study of this point has been initiated by Leray and pursued by Scheffer [76] and Caffarelli, Kohn and Nirenberg [5]. In [5] it is shown that the set of singularities in 4 dimensional space time has Hausdorff 1-dimensional measure 0. The low dimension of the singularities (they cannot form a curve) implies that they may not be very conspicuous, if they are present. It also implies that they probably do not have much to do with the physical phenomenon of turbulence (see below the discussion of intermittency).

The physical problem of turbulence, thus, is related in ways which have not yet become clear to the mathematical problem of understanding the Navier–Stokes equation. In what follows we shall take the region Ω to be bounded, and we shall *assume* that the evolution equation (3) defines a dynamical system. By this we mean that there is a bounded open set U in the space of square integrable w's, such that (3) has a good solution $f^t w$ for $t > 0$ and initial conditions $w \in U$, and $f^t w \in U$ for sufficiently large t. In particular we have the semigroup property

$$f^s \circ f^t = f^{s+t} \tag{4}$$

If $d = 2$ the existence of U can be proved (under suitable regularity conditions for $\partial\Omega$, (g_i) and (v_i^0)). When $d = 3$ it is not known if U exists in general. (It may then be physically required to replace the Navier–Stokes equation by another evolution equation, but we do not discuss this possibility here). If the dynamical system (f^t) exists, it has nice properties: $(t, w) \to f^t w$ is real analytic, and f^t is compact (it sends bounded sets to relatively compact sets, the derivative $D_w f^t$ is a compact linear map).[4]

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[2] Leray [39] treated the case $\Omega = \boldsymbol{R}^3$, Hopf [24] later discussed the case of bounded Ω.

[3] The solution of a time evolution equation with "good" initial data may become "bad" in the sense that the evolution equation is no longer a useful physical approximation. Whether the problem occurs here or not is unknown but, as stressed by Leray, the question is of general relevance for the equations of mathematical physics.

[4] For Navier–Stokes theory see the monographs by Ladyzhenskaya [32], Lions [41], Serrin [77], Temam [81]. See also the excellent review by Foias and Temam [16], and Ruelle [70] where the dynamical systems viewpoint is discussed.

QUESTION. Can one define a dynamical system with weak solutions of the Navier–Stokes equation? In other words can (4) be made to survive when singularities are present ($d = 3$)?

Strange attractors

In equation (3) the parameter $\mu \geqslant 0$ is a measure of the external forces acting on the fluid. In practice μ is often the *Reynolds number* (or the *Rayleigh number* in convection experiments, where the temperature is introduced). If $\mu = 0$ (no external force) the fluid goes to rest, and for small μ tends to a steady state. For somewhat larger action, a periodic oscillation may appear. In terms of dynamical systems, the occurrence of periodic solutions is explained by the *Hopf bifurcation* [22]. Both Landau [33], [34] and Hopf [23] have suggested that, as μ is increased, more and more frequencies appear, and that a quasiperiodic motion is obtained:

$$w(t) = F(\omega_1 t, \dots \omega_k t)$$

where $\omega_1, \dots, \omega_k$ are the frequencies (for a certain value of μ) and F is periodic of period 2π in each argument separately. The quasiperiodic motion must occur on an attracting k-dimensional torus in the infinite dimensional phase space of the fluid (the open set U of the last section). When k is large enough, the time dependence of w is quite complicated, and Hopf and Landau propose that the fluid is turbulent. If this view is accepted, every independent frequency of the system requires one dimension of phase space. Furthermore the system does not depend sensitively on initial condition. This means that a small change δw_0 in initial condition does not grow exponentially with time.

From a mathematical point of view, there is something wrong with quasiperiodic motions on a k-torus, $k \geqslant 2$: these motions are non generic. This means that quasiperiodicity may be replaced by something else under a small perturbation of the evolution equation (3). Takens and myself [75] proposed in 1971 that "something else", which we called *strange attractors*, should describe turbulence. The prototype of strange attractors which we had in mind were the "Strange" Axiom A attractors introduced by Smale [80], and which can appear by perturbation of quasiperiodic motions on the k-torus for $k \geqslant 3$ (see [52], in [75] we needed $k \geqslant 4$). The advantage which we saw to strange attractors over quasiperiodicity is that a continuum of frequencies can be produced with a motion in finite dimensional space (thus involving a finite number of degrees of freedom only). In physical language, the nonlinear interaction between three "modes" already produces "continuous spectrum".

Although the ideas expressed in [75] were less completely new than we thought (see below), they were the first proposal, made explicitly and in print, of a general interpretation of hydrodynamic turbulence, in the framework of dynamical systems, rejecting the quasiperiodic dogma. The reaction of the scientific public to our proposal was quite cold. In particular, the notion that continuous spectrum would be associated with a few degrees of freedom was viewed as heretical by many physicists.[5] Finally, around 1974–1975, the problem was settled by the hydrodynamical experiments of Ahlers [1], and Gollub and Swinney [19], and the computer experiments of McLaughlin and Martin [48], [49]. These experiments (and many others which followed) show that continuous spectrum appears fairly suddenly hen the parameter μ (Rayleigh or Reynolds number) is increased. This is in agreement with the strange attractor picture, and contradicts the idea that new frequencies are added one after the other, as in Landau theory.

We must now come back to an important paper [42] published in 1963 by Lorenz in a meteorology journal, and which largely escaped the attention of mathematicians and physicists for a while. This paper became justly popular after a note by Guckenheimer [20] (published in 1976) brought attention to it. Lorenz does not discuss turbulence in general, but considers a simple evolution equation of the type (3), with $x \in \boldsymbol{R}^3$, which is a rough model for hydrodynamical convection. (As a meteorologist, Lorenz is interested in turbulent convection in the atmosphere). The numerical study of the Lorenz equation reveals a new strange attractor (not of Axiom A type). Sensitivity to initial condition is exhibited, and Lorenz takes argument of this to explain why meteorologists cannot accurately predict the weather long in advance.

It seems that some Russian mathematicians were also unhappy with Landau's quasiperiodic theory of turbulence. V. Arnold informs me that the subject was discussed in Moscow[6] and that he mentioned it in a seminar in Paris in 1965. Such preoccupations motivated Arnold's well-known paper [2]. Unfortunately, no non trivial result about the qualitative dynamics of Navier–Stokes was proved, and the ideas mentioned by Arnold remained unpublished. In fact, while the impact of the new mathematical

[5] As Monin [50] remarks in 1978: "it was tacitly assumed only ten years ago that only stationary points and closed or quasiperiodic orbits could be attractors for the phase paths".

[6] Letter dated 1980.

ideas on turbulence has completely changed the subject, we still have no hard theorem on the existence of strange attractors for the Navier–Stokes equation.

In the discussion (below) of sensitive dependence on initial condition we shall find other precursors of the modern ideas on turbulence.

The onset of turbulence

After the experiments of Ahlers, Gollub and Swinney, and the numerical work of McLaughlin and Martin mentioned earlier, a number of studies on the onset of turbulence followed. The onset of turbulence is the region of low values of μ (Reynolds or Rayleigh number) where the fluid starts to exhibit weak turbulence. It has now become apparent that a weakly turbulent viscous fluid behaves — as a dynamical system — like a generic dynamical system on a low dimensional manifold. Experimental studies (largely based on the frequency spectrum) exhibit periodicity, (with 2, sometimes 3 basic frequencies), strange attractors with sensitive dependence on initial condition, and some curious phenomena like the Feigenbaum bifurcation (see below).

One can prove (Mallet-Paret [43]) that, at finite μ, the Navier–Stokes time evolution is attracted asymptotically to a set of finite (Hausdorff) dimension. It is thus reasonable that weak turbulence appear finite dimensional. It may be more surprising that a viscous fluid behaves very much (from the dynamical viewpoint) like the solution of a randomly chosen equation (3) on a finite dimensional manifold.

Extensive computer studies of low dimensional dynamical systems have shown that sensitive dependence on initial condition is quite common, but mostly appears in systems for which we have no good mathematical theory (non Axiom A). We don't even have a very good mathematical definition of strange attractors. An *attractor* is a set such that the orbits of nearby points tend to it. This may be complemented by an irreducibility condition (see Ruelle [72]). The attractor is *strange* if it exhibits sensitive dependence on initial condition, i.e., exponential growth of small perturbations of initial condition. A precise definition involves the choice of an ergodic measure on the attractor (see below: ergodic theory). Unfortunately we do not know in general what ergodic measure to select.

One great success of the theory of the onset of turbulence is the observation of the *Feigenbaum bifurcation*. This is a new codimension 1-bifurcation first discovered numerically. As the bifurcation parameter μ is increased, an attracting periodic orbit of period T is successively replaced at values $\mu_1, \ldots, \mu_k, \ldots$ of μ by attracting orbits of period $\approx 2^k T$, and μ_k tends to μ_∞ so that $(\mu_k - \mu_\infty)/(\mu_{k+1} - \mu_\infty)$ tends to a universal constant. Beyond μ_∞, "chaotic" behavior with sensitive dependence on initial condition is observed (although not yet proved to exist in general). The Feigenbaum bifurcation is beautifully visible in the frequency analysis of the experimental data of Libchaber and Maurer [40] among others; it cannot possibly be mistaken for something else. The theory of the Feigenbaum bifurcation has started with the deep analysis of Feigenbaum [13], [14], [15] based on the physical idea of the renormalization group. This analysis involves looking for a fixed point in a functional space, and has been made rigorous by Landford's work [35] taken in conjunction with that of Collet, Eckmann and Koch [9] (see also Campanino, Epstein and Ruelle [7], [6]). Lanford's proof is remarkable in that it makes rigorous use of the computer to obtain numerical estimates which would be exceedingly painful to do by hand.

Many sequences of bifurcations lead to turbulence. Eckmann [12] calls them *scenarios*. Three main scenarios have been investigated. The quasiperiodic scenario (Ruelle, Takens and Newhouse [52], [75]), involves the creation of a quasiperiodic 2-torus and its destruction with appearance of a strange attractor. The period doubling cascade scenario is the Feigenbaum bifurcation. The intermittent[7] scenario of Pomeau and Manneville [60] corresponds to a saddle-node bifurcation and manifests itself as "turbulent bursts" in an apparently periodic background. The three scenarios have all been clearly recognized experimentally, but their theoretical study is still quite incomplete.

Strange attractors in general dissipative systems

Our discussion of the onset of hydrodynamic turbulence has made no use of the hydrodynamic equations. A "turbulent" behavior may therefore be expected in all kinds of natural systems. It is convenient to exclude here conservative (i.e. Hamiltonian) systems because of their special (non generic) character: conservative systems may show sensitive dependence on initial condition, but cannot have attractors because of the conservation of Liouville measure. We are thus left with the idea that dissipative (i.e. non conservative) systems exhibit turbulence. For instance, it is predicted that homogeneous chemical reactions may exhibit aperiodic

[7] The temporal intermittency which occurs here seems to be unrelated to the spacial intermittency discussed elsewhere.

oscillations (Ruelle [63]).[8] This *chemical turbulence* is indeed seen experimentally, and has provided the first example of a strange attractor reconstructed from experimental data (see Roux, Rossi, Bachelart and Vidal [62]). All kinds of electromechanical systems also exhibit turbulent time evolution, described as *deterministic noise*, which can now be correctly identified and studied, and may play a significant practical role.

Sensitive dependence on initial condition for dissipative systems has come to be called *chaos*. The chief requirement for a nonlinear system to exhibit chaos is that its phase space have at least 3 dimensions.

Strange attractors and chaotic behavior should also occur in biological systems. The case of ecological models has been discussed by May [47]. Aperiodic cyclès are also expected in macroeconomics [69]. Since the experimental conditions in ecology and economics cannot be precisely controlled, precise predictions are also not expected, but it is at least of philosophical interest to perceive the dynamical causes of chaos in these disciplines.

Sensitive dependence on initial condition

By differentiating (3) we obtain the evolution equation for tangent vectors:

$$\frac{dW}{dt} = (D_{w(t)}F_\mu)W, \tag{5}$$

where D denotes the derivative. Sensitive dependence on initial condition arises if W grows exponentially with time.

It is part of popular wisdom that, for certain particular initial conditions w_0 of the time evolutions which occur in nature, a small change may lead after a while to very different situations. (A pebble at the top of a mountain may fall on one side or the other). It is less obvious that for some dynamical systems there is an exponentially growing W for every initial w_0. That this is so for the geodesic flow on a surface of negative curvature was shown by Hadamard [21]. There are thus natural systems for which no precise prediction can be made, because any small imprecision on the initial condition will result in a large uncertainty for the future behavior. In 1906, P. Duhem [11], referring to Hadamard's work, stressed the philosophical importance of this fact for the problem of predictability

[8] While such a prediction may seem trivial in retrospect, things did not appear so at the time, and reference [63] shared with [75] the fate of not being accepted by the first journal to which it was submitted for publication. Experimentalists are, by the way, absolutely right in treating "new theoretical ideas" with great caution.

in physics.[9] In 1908, H. Poincaré [59] also emphasises the importance of sensitivity to initial condition in a discussion of chance. He already considers meteorology and recognizes the reason why weather predictions are imprecise (see [59], p. 69). He also gives the example of a gas, where a little change in the initial data for one molecule will be amplified by collisions until a molecule which should have hit another one now misses it, so that the microscopic dynamics of the gas has now become completely different. M. Berry[10] has estimated that it would take only 50 collision times to reach this result, if the initial perturbation was the gravitational action of one electron located at the limit of the known universe!

Our understanding of developed turbulence is quite imperfect, but nevertheless gives us the possibility to estimate the growth of fluctuations in a system like the earth's atmosphere. This problem has been studied by Lorenz, Kraichnan and Leith, relevant times are of the order of a week or two. In the kind of turbulence present in air above a radiator it may take of the order of one minute for molecular fluctuations to be amplified to the macroscopic level. (This estimate uses Kolmogorov's model of turbulence, see Ruelle [68]). Putting these facts together the reader is left to imagine how the gravitational effect of an electron at the edge of the universe may affect his fate and change the course of his life. Even if we suppose that the deterministic laws of classical mechanics govern the evolution of our universe, we see that the introduction of chance and probabilities is a necessity in the practice of physics and in everyday life.

It may occur to the reader that, among other things, the position of the planets in the sky may influence his life. Does this justify the claims of astrology? On the contrary, the arguments which show how the planets may influence human fate also indicate that such influences are, for all practical purposes, impredictable.

Ergodic theory of differentiable dynamical systems

Although Hadamard, Duhem and Poincaré understood the origins and implications of sensitive dependence on initial conditions, a quantitative treatment of this notion came much later. The relevant concepts belong to ergodic theory, they are the *entropy* (invariant of Kolmogorov [28] and Sinai [78]) and the *characteristic exponents* (or "Liapounov" exponents, defined in general by Oseledec [54] only in 1968). A dynamical system

[9] The relevant section of Duhem's book is entitled "Exemple de déduction mathématique à tout jamais inutilisable". It was kindly pointed out to me by R. Thom.

[10] Private communication.

known with finite precision acts as a random number generator, and the entropy describes the rate of information production by the system. The characteristic exponents describe the rate of increase of the perturbations of the initial condition. The time rates involved in these definitions are defined with respect to a measure ϱ invariant under time evolution (ϱ is the ergodic measure corresponding to time averages). The brilliant work of N. S. Krylov [31] anticipates the notions of entropy and characteristic exponents, but also confuses them as noted by Sinai.

The idea to study differentiable dynamical systems almost everywhere with respect to invariant measures was developed by Pesin [55], [56], [57] (and later Katok [25], etc.) and turned out to be remarkably fruitful. In particular, the construction of stable and unstable manifolds almost everywhere can be extended to the infinite dimensional situation of hydrodynamics (Ruelle [67], [73], Mañé [46]).

An important conceptual problem arises now: what are the invariant measures μ which describe turbulence? In classical mechanics there is a natural measure because of unique ergodicity. But a strange Axiom A attractor has uncountably many different ergodic measures. Which one is physically relevant? I.e., which one reproduces time averages? Perhaps the one which maximizes entropy? This guess is *wrong*! It is natural to look for measures which are invariant under small stochastic perturbations (Kolmogorov) but this does not solve the problem. A satisfactory answer has been obtained first in the Anosov case (Sinai [79]), then in the general Axiom A case (Ruelle [64], Bowen and Ruelle [4], Kifer [26]). There, the time averages for almost all initial conditions with respect to Lebesgue measure in the neighborhood of an attractor yields the same measure μ on the attractor, and this measure is stable under small stochastic perturbations. The measure μ has conditional measures on unstable manifolds which are absolutely continuous with respect to Lebesgue measure, and its entropy is the sum of the positive characteristic exponents. I believe that this situation has some generality (see Ruelle [65], [66]). At present there are both counterexamples (Bowen, Katok [25]) and deep positive results (Pugh and Shub [61], Ledrappier [37], [38]). General results on the Hausdorff dimension of attractors are also known (Mañé [45], Douady et Oesterlé [10], Ledrappier [36], L.-S. Young [83]).

Developed turbulence

For completeness we deal here very briefly with the vast subject of developed turbulence. Of central importance is the Kolmogorov [27] theory (discovered by Kolmogorov, Oboukhov, Onsager, etc.). This physical theory describes the energy cascade from large spatial structures (where energy is injected) to small "eddies" (where energy is dissipated by viscosity). As noted by Landau [34], Kolmogorov theory is essentially a consequence of arguments of dimensional analysis. This explains why it is robust and successful. However, this theory does not take into account the important phenomenon of intermittency, i.e., the fact that most of the vorticity and dissipation is found in a small subset of physical space. Intermittency in 2 dimensions was noted by Poincaré [58] (existence of localized vortices, visible at the surface of a river for instance). Poincaré tried to give an explanation based on stability of the motion arguments. Onsager [53] proposed an explanation based on the statistical mechanics of vortices, and using negative temperatures. This explanation has been challenged recently (Fröhlich and Ruelle [18]). Three-dimensional intermittency can be modelled (see Frisch, Sulem and Nelkin [17] and references given there) and numerical experiments give a coherent picture (see in particular Chorin [8]) of a self-similar structure (see Mandelbrot [44]). The associated self-similarity dimension is numerically ≈ 2.6, which is much larger than the dimension of the set of possible singularities of solutions of the Navier–Stokes equation. (The existence of such singularities is thus apparently unrelated to intermittency). Elements of an ergodic analysis of Navier–Stokes time evolution have been given (Ruelle [74]). However, a deductive theory of developed turbulence does not exist, and a mathematical basis for the important theoretical literature on this subject is still lacking. (See the monographs of Monin and Yaglom [51], Batchelor [2], the papers of Kraichnan [29], [30], etc.).

Conclusion

The application of non trivial mathematical ideas has given us some understanding of the onset of hydrodynamic turbulence, and changed our notions on turbulence in general. It is good to assess — without epistemological prejudice — the articulation of mathematics and physics in this example. First it must be admitted that the success obtained here did not depend on the proof of some very difficult and deep theorem (even though non trivial theorems have been proved in the course of the study). Does this mean that the important new fact was the revelation of a general philosophical principle like sensitive dependence on initial condition? In fact no: Duhem and Poincaré had a perfectly clear understanding of the principle of sensitive dependence on initial condition, and of its consequences. But the principle was not embodied in mathematical theories of turbulence.

If one looks at the conceptual framework of the modern theory of the onset of turbulence, one finds mathematically sophisticated objects like strange attractors, characteristic exponents, or the Feigenbaum bifurcation. It is important that these objects become useful before their mathematical status is completely elucidated. In other words our understanding of phenomena is made possible by the use of pieces of sophisticated mathematical reasoning, connecting experimental data, computer evidence, and physical assumptions. A purely deductive analysis starting with the Navier–Stokes equation is not attempted: it does not appear feasible at this moment, and might be inappropriate because of the approximate nature of the Navier–Stokes equation.

Looking back at the history of science, we would see that the kind of close interaction which we find here between mathematical ideas and experimental facts has been a regular feature of the evolution of physics. This interaction is of considerable mathematical and philosophical value: it gives us a glimpse, different from intuition, into mathematical reality which is not yet mathematical theory.

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