Proceedings of the International Congress of Mathematicians
Berkeley, California, USA, 1986

Topics in Quasiconformal Mappings

F. W. GEHRING

I. Introduction.

1. *Notation.* For $n \geq 1$ we let $\mathbb{R}^n$ denote euclidean n-space, and for $x \in \mathbb{R}^n$ and $0 < r < \infty$ we let $\mathbb{B}^n(x,r)$ denote the open n-ball with center x and radius r, $\mathbb{S}^{n-1}(x,r) = \partial\mathbb{B}^n(x,r)$, $\mathbb{B}^n = \mathbb{B}^n(0,1)$, and $\mathbb{S}^{n-1} = \mathbb{S}^{n-1}(0,1)$. We also denote by $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ the one point compactification of $\mathbb{R}^n$ equipped with the chordal metric

$$q(x,y) = |p(x) - p(y)|, \tag{1.1}$$

where p denotes stereographic projection of $\overline{\mathbb{R}}^n$ onto the sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$. Throughout this paper, all notions of topology and convergence will be taken with respect to this metric.

Suppose that D and D' are domains in $\overline{\mathbb{R}}^n$ and that $f\colon D \to D'$ is a homeomorphism. We let

$$H_f(x) = \limsup_{r \to 0} H_f(x,r) \tag{1.2}$$

for $x \in D\backslash\{\infty, f^{-1}(\infty)\}$, where for $0 < r < \operatorname{dist}(x, \partial D)$

$$H_f(x,r) = \frac{\max\{|f(x) - f(y)|\colon |x-y| = r\}}{\min\{|f(x) - f(z)|\colon |x-z| = r\}}, \tag{1.3}$$

and we extend $H_f(x)$ to the points ∞ and $f^{-1}(\infty)$ by setting $H_f(\infty) = H_{f\circ g}(0)$ and $H_f(f^{-1}(\infty)) = H_{g\circ f}(f^{-1}(\infty))$, where $g(x) = x/|x|^2$. When $n \geq 2$, we call

$$K(f) = \begin{cases} \infty & \text{if } \sup_{x\in D} H_f(x) = \infty, \\ \operatorname{ess\,sup}_{x\in D} H_f(x) & \text{if } \sup_{x\in D} H_f(x) < \infty \end{cases} \tag{1.4}$$

the *linear dilatation* of f in D. For the purposes of this lecture, we say that f is *quasiconformal* if $K(f) < \infty$ and *K-quasiconformal* if $K(f) \leq K$, $1 \leq K < \infty$. Thus a homeomorphism is quasiconformal if it distorts the shape of an infinitesimal $(n-1)$-sphere about each point by at most a bounded factor; it is K-quasiconformal if, in addition, this factor does not exceed K at almost every point.

The following result shows that the class of quasiconformal mappings is, as the name suggests, an extension of the family of conformal mappings.

This research was supported in part by grants from the National Science Foundation.

1.5. THEOREM. *Suppose that D, D' are domains in $\overline{\mathbb{R}}^n$ and that $f\colon D \to D'$ is a homeomorphism. If $n = 2$, then f is 1-quasiconformal if and only if f or its complex conjugate is a meromorphic function of a complex variable in D. If $n \geq 3$, then f is 1-quasiconformal if and only if f is the restriction to D of a Möbius transformation, i.e., the composition of a finite number of reflections in $(n-1)$-spheres and planes.*

When $n = 2$, Theorem 1.5 is simply a restatement of a theorem due to Menchoff [**M3**]. When $n \geq 3$, Theorem 1.5 is an extension of a well-known result of Liouville to a context which requires *no a priori* hypotheses on the smoothness of f [**G3, R3**].

2. *Historical remarks.* Plane quasiconformal mappings have been studied for almost sixty years. They appear in the late 1920s in papers by Gröztsch, who considered the problem of determining the *most nearly conformal* homeomorphisms between pairs of topologically equivalent plane configurations with one conformal invariant [**G14**]. They occur later under the name *quasiconformal* in a paper by Ahlfors on covering surfaces [**A1**].

In the late 1930s Teichmüller vastly extended the study of Grötzsch to mappings between closed Riemann surfaces and obtained a very natural parameter space for surfaces of fixed genus g, a space which is homeomorphic to $\mathbb{R}^{6g-6}$ [**T1**]. At about the same time, Lavrentieff and Morrey generalized a classical result due to Gauss on the existence of isothermal coordinates by establishing versions of what is now known as the *measurable Riemann mapping theorem* for quasiconformal mappings [**L1, M4**].

In recent years, Ahlfors, Bers, and their school have greatly expanded the results of Teichmüller and applied plane quasiconformal mappings with success to a variety of areas in complex analysis, including kleinian groups and surface topology [**A5, B6, E1, K2**]. Sullivan's recent solution of the Fatou-Julia problem shows that this class can also be used very effectively to study problems on the iteration of rational functions [**S3, S4**].

Higher dimensional quasiconformal mappings were already considered by Lavrentieff in the 1930s [**L2**]. However, no systematic tool for studying this class was available until 1959 when Loewner introduced the notion of *conformal capacity* to show that $\mathbb{R}^n$ cannot be mapped quasiconformally onto a proper subset of itself [**L7**].

Subsequently, Gehring, Väisälä, and many others applied Loewner's method and its equivalent extremal length formulation to develop the initial results for quasiconformal mappings in $\mathbb{R}^n$ [**G3, V1**]. Then in the late 1960s, Reshetnyak and the Finnish school initiated a series of papers which extended the higher dimensional theory to noninjective quasiconformal, or *quasiregular*, mappings [**M1, R2, V4**], a study which recently resulted in Rickman's remarkable extension of the Picard theorem [**R6**].

3. *Role played by quasiconformal mappings.* Plane quasiconformal mappings constitute an important tool in complex analysis and they are particularly

valuable in the study of Riemann surfaces and discontinuous groups. Bers's theorem on simultaneous uniformization [**B3**] is a beautiful application of the measurable Riemann mapping theorem, while Drasin's solution of the inverse problem of Nevanlinna theory [**D2**] illustrates how this theorem can be used to attack problems of complex analysis in a manner similar to the way the $\overline{\partial}$-equation has been applied in harmonic analysis and several complex variables.

The geometric proofs usually required to establish quasiconformal analogues of results for conformal mappings sometimes yield new insight into classical theorems and methods of complex function theory [**L3**]. Quasiconformal mappings also arise in exciting and unexpected ways in other parts of mathematics, for example, in harmonic analysis in connection with functions of bounded mean oscillation and singular integrals [**B1**], and in geometry and elasticity in connection with the injectivity and extension of quasi-isometries.

Higher dimensional quasiconformal mappings offer a new and nontrivial extension of complex analysis to $\mathbb{R}^n$ which is distinct from [**N2**] and perhaps more geometric and flexible than the analytic theory through several complex variables. These mappings have been applied to solve problems in differential geometry, and they constitute a closed class of mappings, interpolating between homeomorphisms and diffeomorphisms, for which many results of geometric topology hold regardless of dimension. Finally, some of the methods developed to study higher dimensional quasiconformal mappings have found important applications in other branches of mathematics, for example, reverse Hölder inequalities in partial differential equations [**G13**].

4. *Comments on the above definition.* The quasiconformal mappings studied by Grötzsch and Teichmüller were assumed to be continuously differentiable at all but a finite number of points. Later Ahlfors [**A2**] and Bers [**B2**] observed that it was more natural to work with mappings $f\colon D \to D'$ for which one has the important inequality

$$K(f) \le \liminf_{j\to\infty} K(f_j), \tag{4.1}$$

when $\{f_j\}$ is a sequence of homeomorphisms which converge to f locally uniformly in D. Indeed, we defined $K(f)$ as in (1.4), rather than by means of the simpler formula

$$K(f) = \sup_{x\in D} H_f(x), \tag{4.2}$$

just so that (4.1) would hold.

Inequality (4.1) implies that the class of K-quasiconformal mappings is closed with respect to locally uniform convergence. Moreover, when $n = 2$, the measurable Riemann mapping theorem implies that every homeomorphism f with $K(f) \le K$ is the locally uniform limit of continuously differentiable homeomorphisms f_j with $K(f_j) \le K$ [**L3**]. When $n = 3$, a quite different argument yields the same conclusion with $K(f_j) \le \tilde{K}$ where $\tilde{K}$ depends only on K [**K1**]. The situation when $n > 3$ appears to be open.

If $f\colon D \to D'$ is a homeomorphism with $K(f) < \infty$, then the Rademacher-Stepanoff theorem and an argument similar to that used by Menchoff imply that f is differentiable with Jacobian $J_f \neq 0$ a.e. in D, that f belongs to the Sobolev class $W^n_{1,\mathrm{loc}}(D)$, and that $K(f^{-1}) = K(f)$ [**G3**]. Thus the inverse of a K-quasiconformal mapping is K-quasiconformal; similarly, the composition of a K_1- and a K_2-quasiconformal mapping is K_1K_2-quasiconformal. Though 1-quasiconformal mappings are real analytic, there exists for each $K > 1$ a K-quasiconformal self mapping f of $\mathbb{R}^n$ which is not differentiable in a set of Hausdorff dimension n.

5. *Remark.* Since there are several excellent expository articles on plane quasiconformal mappings and their connections with Teichmüller spaces [**A6, B4, B5, B7**], the remainder of this lecture will emphasize the less developed theory in higher dimensions. In Chapter II we consider some basic results and open problems for quasiconformal mappings, comparing what is known for $n = 2$ and for $n > 2$. Then in Chapter III we mention several instances where these mappings arise naturally in other areas of mathematics.

II. Some results and open problems.

6. *Tools for studying quasiconformal mappings.* A homeomorphism $f\colon D \to D'$ is quasiconformal if the distortion function H_f in (1.2) is bounded. This is a local restriction and we must find some way to integrate it over D in order to obtain global properties of f. In classical complex function theory, this is accomplished by means of the Cauchy integral formula. Though Pompeiu's analogue is sometimes useful in treating plane quasiconformal mappings, the tool most often used to replace the Cauchy formula is the method of extremal length, formulated by Ahlfors and Beurling [**A8**], and its extension to higher dimensions [**F3, V3**].

7. *Modulus of a curve family.* Suppose that Γ is a family of curves in $\overline{\mathbb{R}}^n$ and let $\mathrm{adm}(\Gamma)$ denote the collection of all Borel measurable functions $\rho\colon \mathbb{R}^n \to [0,\infty]$ such that $\int_\gamma \rho\, ds \geq 1$ for each locally rectifiable curve γ in Γ. Then

$$\mathrm{mod}(\Gamma) = \inf_{\rho\in\mathrm{adm}(\Gamma)} \int_{\mathbb{R}^n} \rho^n\, dm \quad \text{and} \quad \lambda(\Gamma) = \mathrm{mod}(\Gamma)^{1/(1-n)} \tag{7.1}$$

are the *conformal modulus* and *extremal length*, respectively, of Γ.

It is not difficult to see that $\mathrm{mod}(\Gamma)$ is an outer measure on the space of all curve families in $\overline{\mathbb{R}}^n$. Alternatively, if we regard the curves in Γ as homogeneous wires, then we may think of $\lambda(\Gamma)$ as the resistance of the family Γ. In particular, $\mathrm{mod}(\Gamma)$ is large if the curves in Γ are short and plentiful, and small otherwise.

The importance of the conformal modulus in the present context is due to its quasi-invariance with respect to quasiconformal mappings.

7.2. THEOREM. *If $f\colon D \to D'$ is K-quasiconformal and if Γ is a family of curves which lie in D, then*

$$K^{1-n}\,\mathrm{mod}(\Gamma) \leq \mathrm{mod}(f(\Gamma)) \leq K^{n-1}\,\mathrm{mod}(\Gamma). \tag{7.3}$$

Inequality (7.3) plays a key role in the study of quasiconformal mappings. For this reason it is customary to refer to

$$K^*(f) = \max\left(\sup_\Gamma\left(\frac{\operatorname{mod}(f(\Gamma))}{\operatorname{mod}(\Gamma)}\right), \sup_\Gamma\left(\frac{\operatorname{mod}(\Gamma)}{\operatorname{mod}(f(\Gamma))}\right)\right) \tag{7.4}$$

as the *maximal dilatation* of f and say that f is *K-quasiconformal* if $K^*(f) \le K$; here the supremum in (7.4) is taken over all curve families Γ in D for which $\operatorname{mod}(\Gamma)$ and $\operatorname{mod}(f(\Gamma))$ are not simultaneously 0 or ∞. The inequality

$$K(f)^{n/2} \le K^*(f) \le K(f)^{n-1} \tag{7.5}$$

shows that this definition yields the same class of quasiconformal mappings and that $K^*(f) = K(f)$ whenever $n = 2$ or $K(f) = 1$.

A homeomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is quasiconformal if and only if there exists a constant c such that

$$\limsup_{r\to 0} H_f(x,r) \le c \tag{7.6}$$

for all $x \in \mathbb{R}^n$. We illustrate the use of (7.3) by establishing a global form of this inequality.

7.7. THEOREM. *If $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is K-quasiconformal, then*

$$H_f(x,r) \le c \tag{7.8}$$

for all $x \in \mathbb{R}^n$ and $0 < r < \infty$, where $c = c(K,n)$.

The proof depends on two estimates for the conformal moduli of certain curve families [**G2**, **G12**, **V1**].

7.9. LEMMA. *If $0 < a < b < \infty$ and if Γ is a family of open arcs in $\overline{\mathbb{R}}^n$ which join $\mathbb{S}^{n-1}(0,a)$ to $\mathbb{S}^{n-1}(0,b)$, then*

$$\operatorname{mod}(\Gamma) \le \omega_{n-1}(\log \tfrac{b}{a})^{1-n},$$

where ω_{n-1} denotes the $(n-1)$-measure of $\mathbb{S}^{n-1}$.

7.10. LEMMA. *If C_1 and C_2 are disjoint continua in $\overline{\mathbb{R}}^n$ which join 0 to $\mathbb{S}^{n-1}(0,a)$ and ∞ to $\mathbb{S}^{n-1}(0,b)$, respectively, and if Γ is the family of all open arcs which join C_1 to C_2 in $\overline{\mathbb{R}}^n\backslash(C_1 \cup C_2)$, then*

$$\operatorname{mod}(\Gamma) \ge \omega_{n-1}(\log(\lambda_n(\tfrac{b}{a}+1)))^{1-n},$$

where λ_n depends only on n.

7.11. COROLLARY. *If $n > 2$, if C_1 and C_2 are disjoint, linked continua in $\overline{\mathbb{R}}^n$ and if Γ is the family of all open arcs which join C_1 and C_2 in $\overline{\mathbb{R}}^n\backslash(C_1 \cup C_2)$, then* $\operatorname{mod}(\Gamma) \ge c$ *where $c = c(n) > 0$.*

PROOF OF THEOREM 7.7. By performing preliminary translations, we may assume that $x = 0$ and $f(0) = 0$. Let m and M denote the minimum and maximum values assumed by $|f|$ on $\mathbb{S}^{n-1}(0,r)$ and suppose that $m < M$. Next set

$$C_1 = \{x \in \mathbb{R}^n\colon |f(x)| \le m\}, \qquad C_2 = \{x \in \mathbb{R}^n\colon |f(x)| \ge M\} \cup \{\infty\},$$

and let Γ be the family of open arcs which join C_1 and C_2 in $\overline{\mathbb{R}}^n \backslash (C_1 \cup C_2)$. Then the above estimates and (7.3) imply that

$$\omega_{n-1}(\log 2\lambda_n)^{1-n} \le \operatorname{mod}(\Gamma) \le K^{n-1}\operatorname{mod}(f(\Gamma)) \le K^{n-1}\omega_{n-1}(\log(M/m))^{1-n}$$

and we obtain (7.8) with $c = (2\lambda_n)^K$.

8. *Mapping problems.* A basic question in this area is to decide when two domains in $\overline{\mathbb{R}}^n$ are quasiconformally equivalent, i.e., if one can be mapped quasiconformally onto the other. Since the general case is quite difficult even when $n = 2$, we consider here the simpler problem of characterizing the domains D in $\overline{\mathbb{R}}^n$ which are quasiconformally equivalent to the unit ball $\mathbb{B}^n$. The Riemann mapping theorem and the estimates in Lemmas 7.9 and 7.10 yield a complete answer when $n = 2$.

8.1. THEOREM. *A domain D in $\overline{\mathbb{R}}^2$ < is quasiconformally equivalent to $\mathbb{B}^2$ if and only if ∂D is a nondegenerate continuum.*

No such characterization exists in higher dimensions. Indeed, the domains D_3 and D_4 in (8.6) below show that when $n > 2$, there is no way to decide whether the image of $\mathbb{B}^n$ under a self homeomorphism of $\overline{\mathbb{R}}^n$ is quasiconformally equivalent to $\mathbb{B}^n$ by looking only at its boundary.

The following sufficient condition is a consequence of methods used to treat the higher dimensional Schoenflies problem [**G5**, **M2**].

8.2. THEOREM. *A domain D in $\overline{\mathbb{R}}^n$ is quasiconformally equivalent to $\mathbb{B}^n$ if there exist closed sets $E \subset D$, $E' \subset \mathbb{B}^n$ and a quasiconformal mapping $g\colon D\backslash E \to \mathbb{B}^n \backslash E'$ such that $|g(x)| \to 1$ as $x \to \partial D$ in D.*

As in the topological case, localized versions of Theorem 8.2 can be established when D is a Jordan domain in $\overline{\mathbb{R}}^n$, i.e., when ∂D is homeomorphic to $\mathbb{S}^{n-1}$ [**B10**, **G1**].

8.3. COROLLARY. *If D is a domain in $\mathbb{R}^n$ and if D is diffeomorphic to $\mathbb{S}^{n-1}$, then D is quasiconformally equivalent to $\mathbb{B}^n$.*

It is easy to construct a domain in $\mathbb{R}^n$ which is quasiconformally equivalent to $\mathbb{B}^n$ and does not have a tangent plane at any point of its boundary [**G12**]. Thus the sufficient condition in Corollary 8.3 is far from necessary.

A necessary condition for quasiconformal equivalence to $\mathbb{B}^n$ depends on the following refinement of the notion of local connectivity. A set $E \subset \overline{\mathbb{R}}^n$ is said to be *linearly locally connected* if there exists a constant c, $1 \le c < \infty$, such that for each $x \in \mathbb{R}^n$ and $0 < r < \infty$

$$\begin{array}{ll} E \cap \overline{\mathbb{B}}^n(x,r) & \text{lies in a component of } E \cap \overline{\mathbb{B}}^n(x,cr), \\ E \backslash \mathbb{B}^n(x,r) & \text{lies in a component of } E \backslash \mathbb{B}^n(x,r/c). \end{array} \tag{8.4}$$

Then an argument based again on inequality (7.3) and the estimates in Lemma 7.9 and Corollary 7.11 implies the following result [**G7**, **G12**].

8.5. THEOREM. *If $n > 2$ and if D in $\overline{\mathbb{R}}^n$ is quasiconformally equivalent to $\mathbb{B}^n$, then $\overline{\mathbb{R}}^n \backslash D$ is linearly locally connected.*

Theorem 8.5 yields many simple domains in $\mathbb{R}^n$ which are homeomorphic, but not quasiconformally equivalent, to $\mathbb{B}^n$. For example, let

$$\begin{aligned} D_1 &= \{x \in \mathbb{R}^n \colon r < 1, |x_n| < \infty\}, \\ D_2 &= \{x \in \mathbb{R}^n \colon r < \infty, |x_n| < 1\}, \\ D_3 &= \{x \in \mathbb{R}^n \colon x_n > \min(r^{1/2}, 1)\}, \\ D_4 &= \{x \in \mathbb{R}^n \colon x_n < \min(r^{1/2}, 1)\}, \end{aligned} \tag{8.6}$$

where $x = (x_1, \ldots, x_n)$ and $r = (x_1^2 + \cdots + x_{n-1}^2)^{1/2}$. Then explicit constructions yield homeomorphisms which map D_1 and D_3 quasiconformally onto $\mathbb{B}^n$. On the other hand when $n > 2$, $\overline{\mathbb{R}}^n \backslash D_2$ and $\overline{\mathbb{R}}^n \backslash D_4$ are not linearly locally connected and hence D_2 and D_4 are not quasiconformally equivalent to $\mathbb{B}^n$.

The necessary condition in Theorem 8.5 is not sufficient and the problem of finding sharp geometric criteria for testing quasiconformal equivalence to $\mathbb{B}^n$ remains a most interesting open question.

9. *Homeomorphic and quasiconformal extensions.* Suppose that D and D' are domains in $\overline{\mathbb{R}}^n$ and that $f \colon D \to D'$ is quasiconformal. We consider next under what circumstances f admits a homeomorphic extension to $\overline{D}$ or a quasiconformal extension to $\overline{\mathbb{R}}^n$.

9.1. THEOREM. *If D and D' are simply-connected domains of hyperbolic type in $\overline{\mathbb{R}}^2$, then each quasiconformal $f \colon D \to D'$ has a homeomorphic extension to $\overline{D}$ if and only if D and D' are Jordan domains.*

The sufficiency in Theorem 9.1 follows from a theorem of Ahlfors [**A2**] and the necessity from [**E3**]. In higher dimensions we have the following result [**V2**].

9.2. THEOREM. *If D and D' are Jordan domains in $\overline{\mathbb{R}}^n$ and if D is quasiconformally equivalent to $\mathbb{B}^n$, then each quasiconformal $f \colon D \to D'$ has a homeomorphic extension to $\overline{D}$.*

When $n = 2$, the second hypothesis in Theorem 9.2 is superfluous since every Jordan domain is conformally equivalent to $\mathbb{B}^2$. When $n > 2$, this is not the case as seen by the examples in (8.6), and Theorem 9.2 does not hold without this additional restriction [**K3**].

As to the problem of quasiconformal extension to $\overline{\mathbb{R}}^n$, we say that a set E in $\overline{\mathbb{R}}^2$ is a *K-quasidisk* or *K-quasicircle* if it is the image of $\mathbb{B}^2$ or $\mathbb{S}^1$, respectively, under a K-quasiconformal self mapping of $\overline{\mathbb{R}}^2$. By a theorem of Ahlfors, a Jordan domain D is a quasidisk if and only if there exists a constant c such that

$$\min_{j=1,2} \operatorname{dia}(\gamma_j) \leq c|z_1 - z_2| \tag{9.3}$$

for each $z_1, z_2 \in \partial D$, where γ_1 and γ_2 denote the components of $\partial D \backslash \{z_1, z_2\}$ [**A3**].

9.4. THEOREM. *If D and D' are Jordan domains in $\overline{\mathbb{R}}^2$, then each quasiconformal $f\colon D \to D'$ has a quasiconformal extension to $\overline{\mathbb{R}}^2$ if and only if D and D' are quasidisks.*

A simply-connected domain D in $\overline{\mathbb{R}}^2$ is a quasidisk if and only if it is linearly locally connected. Hence this notion also arises in connection with quasiconformal extension.

The sufficiency in Theorem 9.4 is due to Ahlfors [**A2**] and the necessity to Rickman [**R5**]. A higher dimensional analogue of this result is as follows [**G4**, **V5**].

9.5. THEOREM. *If $n > 2$ and if D is a Jordan domain in $\overline{\mathbb{R}}^n$, then each quasiconformal $f\colon D \to \mathbb{B}^n$ has a quasiconformal extension to $\overline{\mathbb{R}}^n$ if and only if $D^* = \overline{\mathbb{R}}^n \backslash \overline{D}$ is quasiconformally equivalent to $\mathbb{B}^n$.*

Thus the problem of quasiconformal extension in higher dimensions differs from the plane case in two respects. First, when $n = 2$, the exterior D^* of every Jordan domain D is quasiconformally equivalent to $\mathbb{B}^2$; this is not true when $n > 2$ as we observed above. Second, when $n > 2$, each quasiconformal $f\colon D \to \mathbb{B}^n$ has a quasiconformal extension to $\overline{\mathbb{R}}^n$ whenever D^* is quasiconformally equivalent to $\mathbb{B}^n$; this is not true when $n = 2$ since there exist Jordan domains D which do not satisfy condition (9.3) and hence are not quasidisks.

10. *Boundary correspondence.* We turn to the problem of characterizing the boundary mappings induced by quasiconformal self mappings of balls and half-spaces. For $n \geq 2$ let $\mathbb{H}^n$ denote the upper halfspace $\{x \in \mathbb{R}^n\colon x_n > 0\}$. Then each quasiconformal $f\colon \mathbb{H}^n \to \mathbb{H}^n$ has a quasiconformal extension $\tilde{f}$ to $\overline{\mathbb{R}}^n$ whose restriction to $\partial\mathbb{H}^n$ is a self homeomorphism φ of $\overline{\mathbb{R}}^{n-1}$. The problem of studying such boundary correspondences was initiated by Beurling and Ahlfors [**B8**].

10.1. THEOREM. *A homeomorphism $\varphi\colon \overline{\mathbb{R}}^1 \to \overline{\mathbb{R}}^1$ with $\varphi(\infty) = \infty$ is the boundary correspondence for a quasiconformal self mapping f of $\mathbb{H}^2$ with $\tilde{f}(\infty) = \infty$ if and only if there exists a constant c such that*

$$\frac{1}{c} \leq \frac{\varphi(x+r) - \varphi(x)}{\varphi(x) - \varphi(x-r)} \leq c \tag{10.2}$$

for all $x \in \mathbb{R}^1$ and $0 < r < \infty$.

Inequality (10.2) is equivalent to the requirement that $H_\varphi(x,r) \leq c$. This condition is replaced by its local form $H_\varphi(x) \leq c$, or that φ is quasiconformal, in the higher dimensional analogue of Theorem 10.1.

10.3. THEOREM. *When $n > 2$, a homeomorphism $\varphi\colon \overline{\mathbb{R}}^{n-1} \to \overline{\mathbb{R}}^{n-1}$ is the boundary correspondence of a quasiconformal self mapping f of $\mathbb{H}^n$ if and only if φ is itself quasiconformal.*

The necessity in Theorems 10.1 and 10.3 follows, respectively, from inequalities (7.8) and (7.6) and the fact that

$$H_\varphi(x,r) \leq H_{\tilde{f}}(x,r) \quad \text{and} \quad H_\varphi(x) \leq H_{\tilde{f}}(x) \tag{10.4}$$

for relevant x and r.

Beurling and Ahlfors established the sufficiency in Theorem 10.1 by showing that the formula

$$\begin{aligned} f(x_1, x_2) = &\frac{1}{2x_2}\int_0^{x_2} (\varphi(x_1+t)+\varphi(x_1-t))\,dt \\ &+ \frac{i}{2x_2}\int_0^{x_2} (\varphi(x_1+t)-\varphi(x_1-t))\,dt \end{aligned} \tag{10.5}$$

defines a quasiconformal extension of φ to $\mathbb{H}^2$.

Ahlfors [**A4**] modified this construction and used the fact that each quasiconformal $\varphi\colon \overline{\mathbb{R}}^2 \to \overline{\mathbb{R}}^2$ can be written as the composition of mappings with small dilatation (see Corollary 11.4) to obtain a quasiconformal extension of φ to $\mathbb{H}^3$ and thus prove the sufficiency in Theorem 10.3 when $n = 3$. Next Carleson [**C1**] employed quite different methods from three-dimensional topology to extend each quasiconformal $\varphi\colon \overline{\mathbb{R}}^3 \to \overline{\mathbb{R}}^3$ to $\mathbb{H}^4$. Finally, Tukia and Väisälä [**T5**] started from an idea of Carleson's and applied results of Sullivan's [**S1**] to establish the sufficiency in Theorem 10.3 for general n.

After composition with suitable Möbius transformations, (10.2) yields a cross ratio characterization for the boundary mappings $\varphi\colon \partial D \to \partial D$ induced by arbitrary quasiconformal self mappings of a disk or halfplane D in $\mathbb{R}^2$, and (10.5) gives an explicit quasiconformal extension $T(\varphi)\colon D \to D$ of each such correspondence φ. Tukia [**T4**] recently settled an important problem in Teichmüller theory by showing that if G is a subgroup of $\text{Möb}(D)$, the group of all Möbius self mappings of D, then each *G-compatible* boundary correspondence $\varphi\colon \partial D \to \partial D$ has a G-compatible quasiconformal extension to D. Douady and Earle [**D1**] extended this work by exhibiting a *conformally natural* quasiconformal extension operator T_0 such that

$$g \circ T_0(\varphi) \circ h = T_0(g \circ \varphi \circ h) \tag{10.6}$$

for each homeomorphism $\varphi\colon \partial D \to \partial D$ and all $g, h \in \text{Möb}(D)$. This beautiful operator should yield many new results in the area; see [**E2**].

If D is a ball or halfspace in $\mathbb{R}^n$ where $n > 2$, then the method of Douady and Earle assigns to each homeomorphism $\varphi\colon \partial D \to \partial D$ a continuous extension $T_0(\varphi)\colon D \to D$ for which (10.6) holds. However, $T_0(\varphi)$ will, in general, be neither quasiconformal nor injective except when $K(\varphi)$ is small, i.e., $K(\varphi) \le K_n$ where K_n depends only on n. It would be interesting to know if every quasiconformal φ has a conformally natural quasiconformal extension.

11. *Measurable Riemann mapping theorem and decomposition.* If $f\colon D \to D'$ is quasiconformal, then f has a nonsingular differential $df\colon \mathbb{R}^n \to \mathbb{R}^n$ at almost all $x \in D$. At each such x, $df = df(x)$ maps an ellipsoid $E_f = E_f(x)$ about 0 with minimum axis length 1 onto an $(n-1)$-sphere about 0. Then $H_f(x)$ is the maximum axis length of $E_f(x)$ and the maximum stretching under f at x occurs in the directions of the smallest axes of $E_f(x)$. If $g\colon D' \to D''$ is quasiconformal, then g is conformal if and only if $E_{g\circ f} = E_f$ a.e. in D by Theorem 1.5, and E_f determines f up to postcomposition with a conformal mapping.

When $n = 2$ and f is sense preserving, E_f is determined by the *Beltrami coefficient* or *complex dilatation*

$$\mu_f(x) = f_{\bar{x}}/f_x, \qquad x = x_1 + ix_2, \tag{11.1}$$

of f at x. In particular, μ_f is measurable with

$$|\mu_f(x)| = \frac{H_f(x) - 1}{H_f(x) + 1}, \qquad \|\mu_f\|_{L^\infty} = \frac{K(f) - 1}{K(f) + 1} < 1, \tag{11.2}$$

and $\mu_{g \circ f} = \mu_f$ a.e. in D if and only if $g\colon D' \to D''$ is conformal. Moreover, in dimension two it is possible to prescribe the dilatation μ_f, and hence the ellipse E_f, at almost every $x \in D$ [**A7**].

11.3. MEASURABLE RIEMANN MAPPING THEOREM. *If μ is measurable with $\|\mu\|_{L^\infty} < 1$ in $\overline{\mathbb{R}}^2$, then there exists a quasiconformal self mapping $f = f_\mu$ of $\overline{\mathbb{R}}^2$ with $\mu_f = \mu$ a.e. If f is normalized to fix three points, then f is unique and depends holomorphically on μ.*

Theorem 11.3 is of fundamental importance in studying the complex structure on Teichmüller space. It can also be a powerful tool for attacking other problems of complex analysis. One example is the solution of the inverse problem of Nevanlinna theory [**D2**] where Drasin first constructed a locally quasiconformal function g with prescribed defects, and then applied Theorem 11.3 to obtain a quasiconformal self mapping f of $\mathbb{R}^2$ so that $g \circ f$ was meromorphic with the same defects as g. A second example is Sullivan's recent solution [**S3**] of the Fatou-Julia problem on wandering domains where Theorem 11.3 was used to construct a large real analytic family of quasiconformal deformations of a given rational function.

The following is an important consequence of Theorem 11.3.

11.4. COROLLARY. *If $n = 2$ and $\varepsilon > 0$, then each quasiconformal $f\colon D \to D'$ can be written in the form $f = f_1 \circ \cdots \circ f_m$ where $K(f_j) < 1 + \varepsilon$ for $j = 1, \ldots, m$ and $m = m(\varepsilon, K(f))$.*

There is no analogue of Theorem 11.3 in higher dimensions. Moreover, when $n > 2$, examples show that Corollary 11.4 is almost certainly not true without further restrictions on the domain D. It is an important open problem to decide if some higher dimensional form of this result holds, even for the case where $D = D' = \overline{\mathbb{R}}^n$.

12. *Quasiconformal groups.* A group G of self homeomorphisms of $\overline{\mathbb{R}}^n$ is said to be *discrete* if G contains no sequence of elements which converge to the identity uniformly in $\overline{\mathbb{R}}^n$, and *K-quasiconformal* if $K(g) \le K$ for each g in G. Though the family of quasiconformal groups contains all Möbius groups, Theorem 11.3 can be used to show that this larger family does not exhibit new phenomena when $n = 2$ [**S2, T2**].

12.1. THEOREM. *When $n = 2$, each quasiconformal group G can be written in the form $G = f^{-1} \circ H \circ f$, where H is a Möbius group and f a quasiconformal self mapping of $\overline{\mathbb{R}}^2$.*

The situation is different in higher dimensions, and for each $n > 2$ there exists a quasiconformal group which is not even isomorphic as a topological group to a Möbius group [**T3**]. Nevertheless, the following convergence property allows one to establish quasiconformal analogues of many basic properties of Möbius groups [**G10**].

12.2. THEOREM. *If G is a discrete quasiconformal group, then for each infinite sequence of distinct elements in G there exists a subsequence $\{g_j\}$ and points x_0, y_0 in $\overline{\mathbb{R}}^n$ such that $g_j \to y_0$ locally uniformly in $\overline{\mathbb{R}}^n \setminus \{x_0\}$ and $g_j^{-1} \to x_0$ locally uniformly in $\overline{\mathbb{R}}^n \setminus \{y_0\}$.*

Suppose that G is a group of self homeomorphisms of $\overline{\mathbb{R}}^n$. We say that G is a *discrete convergence* group if it satisfies the conclusion of Theorem 12.2, and that an element g of G is *elliptic* if it is of finite order or periodic, and *parabolic* or *loxodromic* if it has infinite order and one or two fixed points, respectively. The *limit set* $L(G)$ is the complement of the ordinary set $O(G)$, the set of $x \in \overline{\mathbb{R}}^n$ which have a neighborhood U such that $g(U) \cap U \neq \emptyset$ for at most finitely many $g \in G$. Finally, G is *properly discontinuous* in an open set O if for each compact $F \subset O$, $g(F) \cap F \neq \emptyset$ for at most finitely many $g \in G$ [**G10**].

12.3. THEOREM. *Suppose that G is a discrete convergence group. Then each element of G is elliptic, parabolic, or loxodromic, and the limit set $L(G)$ is nowhere dense or equal to $\overline{\mathbb{R}}^n$. Moreover if* $\operatorname{card}(L(G)) > 2$, *then $L(G)$ is perfect, $L(G)$ lies in the closure of each nonempty G-invariant set, and the set of fixed point pairs of loxodromic elements in G is dense in $L(G) \times L(G)$.*

Though discrete convergence groups resemble Möbius groups in many respects, examples exist which show that they need not be topologically conjugate to Möbius groups [**F2**, **G10**]. They also occur quite naturally in situations which have nothing to do with Möbius or quasiconformal groups.

12.4. THEOREM. *A group G of self homeomorphisms of $\overline{\mathbb{R}}^n$ is a discrete convergence group if it is properly discontinuous in $\overline{\mathbb{R}}^n \setminus E$, where E is closed and totally disconnected.*

It will be interesting to see how much of the classical theory of kleinian groups carries over for this general class of groups.

13. *Hölder continuity and integrability.* Theorem 12.2 can be deduced from (4.1) and the following estimate for change in the chordal distance q in (1.1) under a quasiconformal mapping [**G7**].

13.1. THEOREM. *If $f\colon D \to D'$ is K-quasiconformal and if $\overline{\mathbb{R}}^n \setminus D \neq \emptyset$, then*

$$q(f(x), f(y))\, q(\overline{\mathbb{R}}^n \setminus D') \leq c(q(x,y)/q(x, \partial D))^{1/K} \tag{13.2}$$

for x, y in D, where $q(E)$ denotes the chordal diameter of E and $c = c(n)$.

Theorem 13.1 is a consequence of (7.3) and Lemmas 7.9 and 7.10. When $D, D' \subset \mathbb{R}^n$, it implies that each K-quasiconformal $f\colon D \to D'$ is locally Hölder continuous with exponent $1/K$ and hence that these mappings interpolate between diffeomorphisms and homeomorphisms for $1 \le K < \infty$. This fact is also reflected in the integrability of the Jacobian J_f of f.

13.3. THEOREM. *If $f\colon D \to D'$ is K-quasiconformal where $D, D' \subset \mathbb{R}^n$, then J_f is locally L^p-integrable in D for $1 \le p < p(K,n)$, where*

$$p(K,n) \le K/(K-1), \qquad \lim_{K\to 1} p(K,n) = \infty. \tag{13.4}$$

Bojarski [**B9**] established the existence of the exponent $p(K,n)$ for $n = 2$ by applying the Calderón-Zygmund inequality to the Beurling transform in (14.7). The proof for $n > 2$ was based on the fact that $g = |J_f|$ satisfies the reverse Hölder inequality

$$\frac{1}{m(Q)}\int_Q g\,dm \le c\left(\frac{1}{m(Q)}\int_Q g^{1/n}\,dm\right)^n, \qquad c = c(K,n), \tag{13.5}$$

for each n-cube Q in D with $\operatorname{dia}(f(Q)) < d(f(Q), \partial D')$, and on a lemma to the effect that if (13.5) holds for all n-cubes Q contained in an n-cube Q', then g belongs to $L^p(Q')$ for $1 \le p < p(c,n)$ [**G6**]. These results were sharpened in [**I2**, **R4**] to obtain the second part of (13.4); the example

$$f(x) = |x|^{-a}x, \qquad a = (K-1)/K, \tag{13.6}$$

gives the first part of (13.4). There is reason to suspect that one can take $p(K,n) = K/(K-1)$ in Theorem 13.3. However, this has not been established even for the case $n = 2$.

III. Connections with other areas of mathematics.

14. *Harmonic and functional analysis.* Quasiconformal mappings are encountered in harmonic analysis through their connections with functions of bounded mean oscillation and singular integrals.

A function u is said to be of *bounded mean oscillation* in a domain $D \subset \mathbb{R}^n$, or in $\mathrm{BMO}(D)$, if u is locally integrable and

$$\|u\|_{\mathrm{BMO}(D)} = \sup_B \frac{1}{m(B)}\int_B |u - u_B|\,dm < \infty, \tag{14.1}$$

where the supremum is taken over all n-balls B with $\overline{B} \subset D$ and

$$u_B = \frac{1}{m(B)}\int_B u\,dm. \tag{14.2}$$

The class BMO was introduced by John and Nirenberg [**J3**] in connection with John's work in elasticity [**J1**], and it gained great prominence when Fefferman showed that $\mathrm{BMO}(\mathbb{R}^n)$ is the dual of the Hardy space $H^1(\mathbb{R}^n)$ [**F1**].

The following relations between the class BMO and quasiconformal mappings are due to Reimann [**R1**].

14.3. THEOREM. *If $f\colon D \to D'$ is K-quasiconformal where $D, D' \subset \mathbb{R}^n$, then $\|\log J_f\|_{\mathrm{BMO}(D)} \le c$ where $c = c(K,n)$.*

14.4. THEOREM. *Suppose that $f\colon D \to D'$ is a homeomorphism where D, $D' \subset \mathbb{R}^n$. Then f is quasiconformal if and only if there exists a constant c such that*

$$\frac{1}{c}\|u\|_{\mathrm{BMO}(G')} \le \|u \circ f\|_{\mathrm{BMO}(G)} \le c\|u\|_{\mathrm{BMO}(G')} \tag{14.5}$$

for each subdomain G of D and each u continuous in $G' = f(G)$.

Theorem 14.3 and the necessity in Theorem 14.4 follow from the fact that $g = |J_f|$ satisfies the reverse Hölder inequality in (13.5). The sufficiency in Theorem 14.4 is a variant, due to Astala [**A9**], of Reimann's original result.

Theorem 14.4 characterizes quasiconformal mappings as the homeomorphisms which preserve the class BMO. The following result [**J4**] characterizes quasidisks in terms of extension properties for BMO.

14.6. THEOREM. *If D is a simply-connected domain of hyperbolic type in $\mathbb{R}^2$, then each function u in* BMO(D) *has a* BMO *extension to $\mathbb{R}^2$ if and only if D is a quasidisk.*

Next the best possible exponents for Jacobian integrability and area distortion for plane quasiconformal mappings are closely connected with sharp constants in two inequalities for the Beurling transform

$$Tg(x) = -\frac{1}{\pi}\int_{\mathbb{R}^2} \frac{g(y)}{(x-y)^2}\,dm. \tag{14.7}$$

For example, T is a bounded operator on $L^p(\mathbb{R}^2)$ with

$$\|T\|_p = \sup_g \frac{\|Tg\|_{L^p(\mathbb{R}^2)}}{\|g\|_{L^p(\mathbb{R}^2)}} \ge \max\left(p-1, \frac{1}{p-1}\right) \tag{14.8}$$

for $1 < p < \infty$ and $\|T\|_2 = 1$, and there is reason to believe that

$$\liminf_{p\to\infty} \frac{1}{p}\|T\|_p = 1. \tag{14.9}$$

If true, this would yield the sharp upper bound $p(K,2) = K/(K-1)$ for the integrability of the Jacobian of a plane quasiconformal mapping discussed in §13 [**I1**].

Next, one can show that there exist constants a and b such that

$$\int_{\mathbb{B}^2} |T\chi_E(x)|\,dm \le a m(E)\log(\pi/m(E)) + b m(E) \tag{14.10}$$

for each measurable $E \subset \mathbb{B}^2$. This inequality can be combined with Theorem 11.3 to prove that

$$\frac{m(f(E))}{\pi} \le c\left(\frac{m(E)}{\pi}\right)^{K^{-a}}, \tag{14.11}$$

$c = c(K) = 1 + O(K-1)$ as $K \to 1$, for each K-quasiconformal $f\colon \mathbb{B}^2 \to \mathbb{B}^2$ with $f(0) = 0$ and each measurable set $E \subset \mathbb{B}^2$ [**G11**]. Moreover, the above reasoning can be reversed to show that if (14.11) holds for a given constant a, then so does (14.10). It is conjectured that both hold with $a = 1$. If so, this would again imply that $p(K, 2) = K/(K-1)$.

Finally, the problem of quasiconformal equivalence of domains can be reformulated in terms of function algebras. Given a domain D in $\mathbb{R}^n$, we let $A(D)$ denote the algebra of functions $u \in C(D) \cap W_n^1(D)$ with norm

$$\|u\| = \|u\|_{L^\infty(D)} + \|\nabla u\|_{L^n(D)}, \tag{14.12}$$

the so-called *Royden algebra* of D. We then have the following result [**L5**, **L6**].

14.13. THEOREM. *Two domains D and D' in $\mathbb{R}^n$ are quasiconformally equivalent if and only if $A(D)$ and $A(D')$ are isomorphic as algebras.*

Little is known about the structure of these algebras and it may be that geometric methods used to determine quasiconformal equivalence will yield more information about them than vice versa.

15. *Quasi-isometries and elasticity.* A mapping $f\colon E \subset \mathbb{R}^n \to \mathbb{R}^n$ is an *L-quasi-isometry* in E if

$$\frac{1}{L}|x_1 - x_2| \le |f(x_1) - f(x_2)| \le L|x_1 - x_2| \tag{15.1}$$

for $x_1, x_2 \in E$; f is a *local L-quasi-isometry* in E if for each $L' > L$, each $x \in E$ has a neighborhood U such that f is an L'-quasi-isometry in $E \cap U$.

If f is quasi-isometric in a domain D, then f is quasiconformal by (1.2) and (1.4); the mapping in (13.6) shows that the converse is false. Nevertheless, quasiconformal homeomorphisms arise in questions concerning extension and injectivity of these mappings.

15.2. THEOREM. *If $n \neq 4$, then a quasi-isometry f of E has a quasi-isometric extension to $\mathbb{R}^n$ if and only if f has a quasiconformal extension to $\mathbb{R}^n$.*

Theorem 15.2 [**T6**] gives a criterion for extension in terms of the mapping f. There is also a criterion in terms of the set E when E is a Jordan curve [**G9**].

15.3. THEOREM. *If C is a Jordan curve in $\mathbb{R}^2$, then each quasi-isometry f of C has a quasi-isometric extension to $\mathbb{R}^2$ if and only if C is a quasicircle.*

For each domain $D \subset \mathbb{R}^n$ let $L(D)$ denote the supremum of the numbers $L \ge 1$ with the property that each local L-quasi-isometry f in D is injective there. The constant $L(D)$ has a physical interpretation if we think of D as an elastic body and f as the deformation experienced by D when subjected to a force field. Requiring that f be a local L-quasi-isometry bounds the strain in D under the force field and $L(D)$ measures the critical strain in D before D collapses onto itself.

Little is known about this constant except that $2^{1/4} \le L(D) \le 2^{1/2}$ whenever D is a ball or halfspace [**J2**]. However, we can characterize a large class of plane domains for which $L(D) > 1$ [**G8**].

15.4. THEOREM. *If D is a simply-connected proper subdomain of $\mathbb{R}^2$, then $L(D) > 1$ if and only if D is a quasidisk.*

15.5. COROLLARY. *If f is a local L-quasi-isometry of a bounded simply-connected domain in $\mathbb{R}^2$ and if $L < L(D)$, then f has an M-quasi-isometric extension to $\mathbb{R}^2$ where $M = M(L, L(D))$.*

Corollary 15.5 says that the shape of a deformed simply-connected plane elastic body is roughly the same as that of the original provided the strain does not attain the critical value. It would be interesting to obtain a higher dimensional analogue of this result.

16. *Complex analysis.* Quasiconformal mappings sometimes arise in function-theoretic problems which appear to be completely unrelated to this class. An excellent example is Teichmüller's theorem [**T1**] which relates the extremal quasiconformal mappings between two Riemann surfaces with the quadratic differentials on these surfaces.

For a more elementary example, suppose that f is meromorphic in a simply-connected domain D of hyperbolic type in $\overline{\mathbb{R}}^2$ and let

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2. \tag{16.1}$$

By a theorem of Nehari [**N1**], f is injective whenever D is a disk or halfplane and $|S_f| \le 2\rho_D^2$ in D. Here ρ_D is the hyperbolic metric in D given by

$$\rho_D(z) = |g'(z)|(1 - |g(z)|^2)^{-1} \tag{16.2}$$

where $g\colon D \to \mathbb{B}^2$ is conformal. It is natural to ask: for which other domains D does such a result hold? That is, for which D is $\sigma(D) > 0$, where $\sigma(D)$ denotes the supremum of the numbers $a \ge 0$ such that f is injective whenever f is meromorphic with $|S_f| \le a\rho_D^2$ in D?

The answer involves quasiconformal mappings and yields a new characterization of Bers's universal Teichmüller space [**B5**].

16.3. THEOREM. *$\sigma(D) > 0$ if and only if D is a quasidisk.*

17. *Differential geometry and topology.* Some of the results mentioned in Chapter II have important applications in differential geometry. For example, Theorem 1.5 and the necessity in Theorem 10.3 are key steps in the original proof of Mostow's rigidity theorem [**M5**].

17.1. THEOREM. *If $n > 2$ and if M and M' are diffeomorphic compact Riemannian n-manifolds with constant negative curvature, then M and M' are conformally equivalent.*

Similarly, the equicontinuity property for quasiconformal mappings implied by Theorem 13.1 is an important tool in establishing the following conjecture of Lichnerowicz [**L4**].

17.2. THEOREM. *If $n \geq 2$ and if M is a compact Riemannian n-manifold not conformally equivalent to a sphere, then the group $C(M)$ of conformal self mappings of M is compact in the topology of uniform convergence.*

The work of Earle and Eells [**E1**] on the diffeomorphism group of a surface and Bers's proof [**B6**] of Thurston's theorem on the classification of self mappings of surfaces illustrate how quasiconformal mappings can be applied to problems in surface topology. Sullivan showed [**S1**] that the Schoenflies theorem, the annulus conjecture and the component problem hold for quasiconformal mappings in all dimensions. The results of this fundamental paper suggest that quasiconformal mappings may prove to be an important, intermediate category of maps between homeomorphisms and diffeomorphisms.

REFERENCES

[**A1**] L. V. Ahlfors, *Zur theorie der Überlagerungsflächen*, Acta Math. **65** (1935), 157–194.

[**A2**] ____, *On quasiconformal mappings*, J. Analyse Math. **3** (1953/54), 1–58.

[**A3**] ____, *Quasiconformal reflections*, Acta Math. **109** (1963), 291–301.

[**A4**] ____, *Extension of quasiconformal mappings from two to three dimensions*, Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 768–771.

[**A5**] ____, *Finitely generated Kleinian groups*, Amer. J. Math. **86** (1964), 413–429.

[**A6**] ____, *Quasiconformal mappings, Teichmüller spaces, and Kleinian groups*, Proc. Internat. Congr. Math. (Helsinki, 1978), Acad. Sci. Fennica, Helsinki, 1980, pp. 71–84.

[**A7**] L. V. Ahlfors and L. Bers, *Riemann's mapping theorem for variable metrics*, Ann. of Math. **72** (1960), 385–404.

[**A8**] L. V. Ahlfors and A. Beurling, *Conformal invariants and function-theoretic null sets*, Acta Math. **83** (1950), 101–129.

[**A9**] K. Astala, *A remark on quasi-conformal mappings and* BMO*-functions*, Michigan Math. J. **30** (1983), 209–212.

[**B1**] A. Baernstein II and J. J. Manfredi, *Topics in quasiconformal mapping*, Topics in Modern Harmonic Analysis, Istituto Nazionale di Alta Mathematica, Roma, 1983, pp. 819–862.

[**B2**] L. Bers, *Quasiconformal mappings and Teichmüller's theorem*, Analytic Functions, Princeton Univ. Press, Princeton, N.J., 1960, pp. 89–119.

[**B3**] ____, *Uniformization by Beltrami equations*, Comm. Pure Appl. Math. **14** (1961), 215–228.

[**B4**] ____, *Uniformization, moduli, and Kleinian groups*, Bull. London Math. Soc. **4** (1972), 257–300.

[**B5**] ____, *Quasiconformal mappings, with applications to differential equations, function theory and topology*, Bull. Amer. Math. Soc. **83** (1977), 1083–1100.

[**B6**] ____, *An extremal problem for quasiconformal mappings and a problem of Thurston*, Acta Math. **141** (1978), 73–98.

[**B7**] ____, *Finite dimensional Teichmüller spaces and generalizations*, Bull. Amer. Math. Soc. **5** (1981), 131–172.

[**B8**] A. Beurling and L. V. Ahlfors, *The boundary correspondence under quasiconformal mappings*, Acta Math. **96** (1956), 125–142.

[**B9**] B. Bojarski, *Generalized solutions of a system of first order differential equations of elliptic type with discontinuous coefficients*, Mat. Sb. **43** (1957), 451–503. (Russian)

[**B10**] M. Brown, *Locally flat embeddings of topological manifolds*, Ann. Math. **75** (1962), 331–341.

[**C1**] L. Carleson, *The extension problem for quasiconformal mappings*, Contributions to Analysis, Academic Press, New York, 1974, pp. 39–47.

[**D1**] A. Douady and C. J. Earle, *Conformally natural extension of homeomorphisms of the circle*, Acta Math. (to appear).

[D2] D. Drasin, *The inverse problem of the Nevanlinna theory*, Acta Math. **138** (1977), 83–151.

[E1] C. J. Earle and J. Eells, *A fibre bundle description of Teichmüller theory*, J. Differential Geom. **3** (1969), 19–43.

[E2] C. J. Earle and S. Nag, *Conformally natural reflections in Jordan curves with applications to Teichmüller spaces* (to appear).

[E3] T. Erkama, *Group actions and extension problems for maps of balls*, Ann. Acad. Sci. Fenn. Ser. A I Math. **556** (1973), 1–31.

[F1] C. Fefferman, *Characterizations of bounded mean oscillation*, Bull. Amer. Math. Soc. **77** (1971), 587–588.

[F2] M. H. Freedman and R. Skora, *Strange actions of groups on spheres*, J. Differential Geom. (to appear).

[F3] B. Fuglede, *Extremal length and functional completion*, Acta Math. **98** (1957), 171–219.

[G1] D. B. Gauld and J. Väisälä, *Lipschitz and quasiconformal flattening of spheres and cells*, Ann. Acad. Sci. Fenn. Ser. A I Math. **4** (1978/79), 371–382.

[G2] F. W. Gehring, *Symmetrization of rings in space*, Trans. Amer. Math. Soc. **101** (1961), 499–519.

[G3] ——, *Rings and quasiconformal mappings in space*, Trans. Amer. Math. Soc. **103** (1962), 353–393.

[G4] ——, *Extension of quasiconformal mappings in three space*, J. Analyse Math. **14** (1965), 171–182.

[G5] ——, *Extension theorems for quasiconformal mappings in n-space*, J. Analyse Math. **19** (1967), 149–169.

[G6] ——, *The L^p-integrability of the partial derivatives of a quasiconformal mapping*, Acta Math. **130** (1973), 265–277.

[G7] ——, *Quasiconformal mappings*, Complex Analysis and its Applications. II, International Atomic Energy Agency, Vienna, 1976, pp. 213–268.

[G8] ——, *Injectivity of local quasi-isometries*, Comment. Math. Helv. **57** (1982), 202–220.

[G9] ——, *Extension of quasiisometric embeddings of Jordan curves*, Complex Variables Theory Appl. **5** (1986), 245–263.

[G10] F. W. Gehring and G. J. Martin, *Discrete quasiconformal groups* I, Proc. London Math. Soc. (to appear).

[G11] F. W. Gehring and E. Reich, *Area distortion under quasiconformal mappings*, Ann. Acad. Sci. Fenn Ser. A I Math. **388** (1966), 1–15.

[G12] F. W. Gehring and J. Väisälä, *The coefficients of quasiconformality of domains in space*, Acta Math. **114** (1965), 1–70.

[G13] M. Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Ann. of Math. Studies, Vol. 105, Princeton Univ. Press, Princeton, N.J., 1983.

[G14] H. Grötzsch, *Über möglichst konforme Abbildungen von schlichten Bereichen*, Ber. Verh. Sächs. Akad. Wiss. Leipzig **84** (1932), 114–120.

[I1] T. Iwaniec, *Extremal inequalities in Sobolev spaces and quasiconformal mappings*, Z. Anal. Anwendungen **1** (1982), 1–16.

[I2] ——, *On L^p-integrability in PDE's and quasiregular mappings for large exponents*, Ann. Acad. Sci. Fenn. Ser. A I Math. **7** (1982), 301–322.

[J1] F. John, *Rotation and strain*, Comm. Pure Appl. Math. **14** (1961), 391–413.

[J2] ——, *On quasi-isometric mappings*, II, Comm. Pure Appl. Math. **22** (1969), 265–278.

[J3] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (1961), 415–426.

[J4] P. W. Jones, *Extension theorems for* BMO, Indiana Univ. Math. J. **29** (1980), 41–66.

[K1] M. Kiikka, *Diffeomorphic approximation of quasiconformal and quasisymmetric homeomorphisms*, Ann. Acad. Fenn. Sci. Ser. A I Math. **8** (1983), 251–256.

[K2] I. Kra, *On the Nielsen-Thurston-Bers type of some self-maps of Riemann surfaces*, Acta Math. **146** (1981), 231–270.

[**K3**] T. Kuusalo, *Quasiconformal mappings without boundary extensions*, Ann. Acad. Sci. Fenn. Ser. A I Math. **10** (1985), 331–338.

[**L1**] M. A. Lavrentieff, *Sur une classe de représentations continues*, Mat. Sb. **42** (1935), 407–423.

[**L2**] ——, *Sur un critère différentiel des transformations homéomorphes des domaines à trois dimensions*, Dokl. Akad. Nauk. **20** (1938), 241–242.

[**L3**] O. Lehto and K. I. Virtanen, *Quasiconformal mappings in the plane*, Springer-Verlag, 1973.

[**L4**] J. Lelong-Ferrand, *Transformations conformes et quasi-conformes des variétés riemanniennes compactes (Démonstration de la conjoncture de A. Lichnerowicz)*, Acad. Roy. Belg. Cl. Sci. Mém. Collect. **39** (1971), 1–44.

[**L5**] ——, *Étude d'une classe d'applications liées à des homomorphismes d'algèbres de fonctions, et généralisant les quasi conformes*, Duke Math. J. **40** (1973), 163–186.

[**L6**] L. G. Lewis, *Quasiconformal mappings and Royden algebras in space*, Trans. Amer. Math. Soc. **158** (1971), 481–492.

[**L7**] C. Loewner, *On the conformal capacity in space*, J. Math. Mech. **8** (1959), 411–414.

[**M1**] O. Martio, S. Rickman, and J. Väisälä, *Topological and metric properties of quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A I Math. **488** (1971), 1–31.

[**M2**] B. Mazur, *On embeddings of spheres*, Bull. Amer. Math. Soc. **65** (1959), 59–65.

[**M3**] D. Menchoff, *Sur une généralisation d'un théorème de M. H. Bohr*, Mat. Sb. **44** (1937), 339–354.

[**M4**] C. B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43** (1938), 126–166.

[**M5**] G. D. Mostow, *Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms*, Inst. Hautes Études Sci. Publ. Math. **34** (1968), 53–104.

[**N1**] Z. Nehari, *The Schwarzian derivative and schlicht functions*, Bull. Amer. Math. Soc. **55** (1949), 545–551.

[**N2**] R. Nirenberg, *On quasi-pseudoconformality in several complex variables*, Trans. Amer. Math. Soc. **127** (1967), 233–240.

[**R1**] H. M. Reimann, *Functions of bounded mean oscillation and quasiconformal mappings*, Comment. Math. Helv. **49** (1974), 260–276.

[**R2**] Yu. G. Reshetnyak, *Space mappings with bounded distortion*, Sibirsk. Mat. Zh. **8** (1967), 629–658. (Russian)

[**R3**] ——, *Liouville's theorem on conformal mappings under minimal regularity assumptions*, Sibirsk. Mat. Zh. **8** (1967), 835–840. (Russian)

[**R4**] ——, *Stability estimates in Liouville's theorem and the L^p-integrability of the derivatives of quasiconformal mappings*, Sibirsk. Mat. Zh. **17** (1976), 868–896. (Russian)

[**R5**] S. Rickman, *Extension over quasiconformally equivalent curves*, Ann. Acad. Sci. Fenn Ser. A I Math. **436** (1969), 1–12.

[**R6**] ——, *On the number of omitted values of entire quasiregular mappings*, J. Analyse Math. **37** (1980), 100–117.

[**S1**] D. Sullivan, *Hyperbolic geometry and homeomorphisms*, Geometric Topology, Academic Press, New York, 1979, pp. 543–555.

[**S2**] ——, *On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions*, Riemann Surfaces and Related Topics: Proceedings of the 1978 Stony Brook Conference, Ann. of Math. Studies, No. 97, Princeton Univ. Press, Princeton, N.J., 1981, pp. 465–496.

[**S3**] ——, *Quasiconformal homeomorphisms and dynamics* I. *Solution of the Fatou-Julia problem on wandering domains*, Ann. of Math. **122** (1985), 401–418.

[**S4**] ——, *Quasiconformal homeomorphisms and dynamics* II: *Structural stability implies hyperbolicity for Kleinian groups*, Acta Math. **155** (1985), 243–260.

[**T1**] O. Teichmüller, *Extremale quasikonforme Abbildungen und quadratische Differentiale*, Abh. Preuss. Akad. Wiss. Mat.-Nat. Kl. **22** (1940), 1–197.

[**T2**] P. Tukia, *On two-dimensional quasiconformal groups*, Ann. Acad. Sci. Fenn. Ser. A I Math. **5** (1980), 73–78.

[**T3**] ——, *A quasiconformal group not isomorphic to a Möbius group*, Ann. Acad. Sci. Fenn. Ser. A I Math. **6** (1981), 149–160.

[**T4**] ——, *Quasiconformal extension of quasisymmetric mappings compatible with a Möbius group*, Acta Math. **154** (1985), 153–193.

[**T5**] P. Tukia and J. Väisälä, *Quasiconformal extension from dimension n to $n+1$*, Ann. of Math. **115** (1982), 331–348.

[**T6**] ——, *Bilipschitz extensions of maps having quasiconformal extensions*, Math. Ann. **269** (1984), 561–572.

[**V1**] J. Väisälä, *On quasiconformal mappings in space*, Ann. Acad. Sci. Fenn. Ser. A I Math. **298** (1961), 1–36.

[**V2**] ——, *On quasiconformal mappings of a ball*, Ann. Acad. Sci. Fenn. Ser. A I Math. **304** (1961), 1–7.

[**V3**] ——, *Lectures on n-dimensional quasiconformal mappings*, Lectures Notes in Math., Vol. 229, Springer-Verlag, 1971.

[**V4**] ——, *A survey of quasiregular maps in* $\mathbf{R}^n$, Proc. Internat. Congr. Math. (Helsinki, 1978), Acad. Sci. Fennica, Helsinki, 1980, pp. 685–691.

[**V5**] ——, *Quasimöbius maps*, J. Analyse Math. **44** (1984/85), 218–234.

MATHEMATICAL SCIENCES RESEARCH INSTITUTE, BERKELEY, CALIFORNIA 94720, USA

UNIVERSITY OF MICHIGAN, ANN ARBOR, MICHIGAN 48109, USA