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Singularities of Ray Systems

The simplest example of a ray system is the system of all normals to a given surface in Euclidean space. Hamilton (1824) turned the theory of ray system into a part of symplectic geometry; since Maslov's thesis (1965) ray systems are called Lagrangian submanifolds.

The normals to a surface foliate some neighbourhood of that surface; but away from that neighbourhood various normals start intersecting one another (Fig. 1). The resulting complicated and beautiful geometry

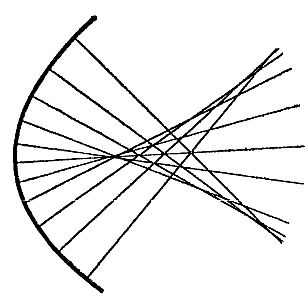


Fig. 1

was hidden up to 1972, when the relation between singularities of ray systems and Euclidean reflection groups was discovered.

This relation, for which there is no a priori reason, turned out to be a powerful method for the analysis of singularities. By 1978 it became clear that the Euclidean reflection groups also govern the singularities of Huygens evolvents.

Huygens (1654) discovered that the evolvent of a plane curve has a cusp singularity at each point of contact with the curve (Fig. 2). Plane curve evolvents and their higher-dimensional generalizations are the wave fronts on manifolds with boundary. The singularities of wave fronts, as well as those of ray systems, are classified by reflection groups.

While the ray and front systems on manifolds without boundary are related to the A, D and E series of the Weyl groups, the singularities of evolvents are described by the B, C, F series (those having Dynkin diagrams with double connections).

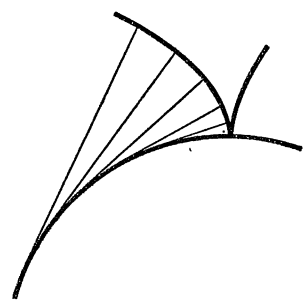


Fig. 2

The relation of the remaining reflection groups $(I_2(p), H_3, H_4)$ to singularity theory was unknown until recently. This situation has changed since the fall of 1982 when it was discovered that the group H_3 (the group of symmetries of the icosahedron) governs the singularities of evolvent systems at the inflection points of plane curves.

The icosahedron appears at an inflection point as mystically as it does in Kepler's law of planetary distances. I believe, however, that in our problem the appearance of the icosahedron is more relevant than in Kepler's case; I hope that the remaining group H_4 will appear naturally in the analysis of the more complicated singularities of ray systems and wave fronts.

The main theme of this paper is the application of the relation between singularities of ray systems and reflection groups. The results I shall discuss are now included in symplectic and contact geometries under the names of Lagrangian and Legendrian singularity theories. But one may consider them as part of the calculus of variations, or of control theory, of PDE theory, or of classical mechanics, of optics, or of wave theory, of algebraic geometry, or of general singularity theory. Some of these results deal with objects so basic, that it seems strange that the classics have missed them. For example, the local classification of projections of surfaces in general position in the usual 3-space was discovered only in 1981.

The number of nonequivalent projection germs is 14: the point neighbourhoods on generic surfaces generate exactly 14 different patterns when the surfaces are seen from different points of 3-space.

The reason is perhaps the difficulty of the proofs: they depend on the relations (sometimes unexpected) to invariant theory, Lie algebras, reflexion groups, algebraic geometry, and Deligne mixed Hodge structures. Some of the results were stimulated by applications of singularity theory to perturbation analysis of Hamiltonian dynamical systems, and even to number theory, but most new concepts came from the problem of bypassing an obstacle in Euclidean 3-space.

In order to describe these new results I must recall some well-known notions.

1. Symplectic geometry

A *symplectic structure* on an even-dimensional smooth manifold is a closed nondegenerate differential 2-form on it.

Examples: 1. The oriented area element defines a symplectic structure on the plane. 2. The direct product of symplectic manifolds has a natural symplectic structure. 3. The phase space of classical mechanics (the total space of the cotangent bundle of a smooth manifold) has a natural "$dp \wedge dq$" symplectic structure. 4. One may equip the manifold of oriented lines in Euclidean space with the symplectic structure of the total space of the cotangent bundle of the sphere, since these two manifolds are diffeomorphic. 5. The *characteristic direction* at a point of a hypersurface in a symplectic manifold is the skew-ortho-complement to the tangent plane. The *characteristics* on a hypersurface are the integral lines of its field of characteristic directions. The manifold of characteristics inherits a symplectic structure from the original manifold. 6. In particular, the manifold of extremals of general variational problem, lying at the same level manifold of the Hamiltonian function, is equipped with a natural symplectic structure. 7. Consider the space of odd-degree binary forms. There exists a unique (up to constant multiple) nondegenerate SL_2-invariant bilinear skew form on this even-dimensional linear space. This form defines a natural symplectic structure on the space of binary forms. 8. The binary forms in x and y, with coefficient in front of x^{2k+1} equal to 1, form a hyperplane in the space of all forms. The manifold of characteristics of this hyperplane can be identified with the manifold of even-degree polynomials in x of the form $x^{2k}+\ldots$ We have thus equipped this space of even-degree polynomials with a symplectic structure. 9. The one-parameter group of shifts along the x-axis preserves this symplectic structure. The Hamiltonian function of this group is a polynomial of degree 2, known already to Hilbert (1893). The manifold of characteristics of a level hypersurface of the Hamiltonian function can be identified with the manifold of polynomials $x^{2k-1}+\dots$ with sum of roots equal to 0. We thus get a natural symplectic structure on this space of polynomials.

THEOREM (G. Darboux, 1882). *All the symplectic structures on manifolds of a fixed dimension are locally diffeomorphic.*

Thus, every symplectic structure is locally reducible to the normal form $\sum dp_i \wedge dq_i$ by a suitable choice of local "*Darboux coordinates*" p_i, q_i.

Let us now consider submanifolds of a symplectic manifold. The restriction to the submanifold of the symplectic structure is a closed 2-form, but it is not necessarily nondegenerate. In Euclidean space there is not only the inner geometry of a submanifold, but also an extensive theory of exterior curvatures. In the symplectic case the situation is much simpler:

THEOREM (A. B. Givental, 1981). *The germ of a submanifold of a symplectic manifold is determined (up to a symplectic diffeomorphism) by the restriction of the symplectic form to the tangent spaces of the submanifold.*

An intermediate theorem, dealing with vectors nontangent to the submanifold, was proved by A. Weinstein (1973). Unlike the Weinstein theorem, the Givental theorem implies the classification of the germs of generic submanifolds in a symplectic space: one uses the classification of the degeneracies of symplectic structures obtained by J. Martinet (1970) and his followers.

Examples: 1. The germs of a generic 2-surface in a symplectic manifold are locally symplectomorphic (symplectically diffeomorphic) to those of the surface $p_2 = p_1^2$, $q_1 = 0$, $p_3 = q_3 = \dots = 0$ (we use the Darboux coordinates). 2. On a 4-submanifold one encounters stably the curves of elliptic and hyperbolic Martinet singularities with normal forms

$$p_2 = p_1p_3 \pm q_1q_2 + q_3^3/6, \quad p_3 = 0, \quad p_4 = q_4 = \dots = 0.$$

The ellipticity and hyperbolicity concern the character of the motions in a dynamical system related intrinsically to the submanifold. The relevant divergence-free vector field on a 3-dimensional manifold has a curve of singular points. The classification at singular curves turns out to be less pathological than that at singular points (the latter being almost as difficult as the whole of celestial mechanics).

I have thus described the first steps of the symplectic singularity theory of smooth submanifolds.

A *Lagrangian submanifold* of a symplectic manifold is a submanifold on which the restriction of the symplectic structure vanishes, and which has highest possible dimension (equal to half of the dimension of the symplectic manifold).

Examples: 1. The fibers of the cotangent bundle. 2. The manifold of lines normal to a smooth submanifold (of arbitrary dimension) in Euclidean space. 3. The set of all polynomials $x^{2m}+\dots$ divisible by x^m.

A *Lagrangian fibration* is a fibration whose fibers are Lagrangian submanifolds.

Examples: 1. The cotangent bundle. 2. The fibration sending an oriented line in Euclidean space to the corresponding unit vector at the origin.

All Lagrangian fibrations of a given dimension are locally symplectomorphic (in the neighbourhood of each point of the total space).

A *Lagrangian mapping* is a diagram $V \to E \to B$, where the first arrow is an immersion of a Lagrangian submanifold, and the second is a Lagrangian fibration (Fig. 3).

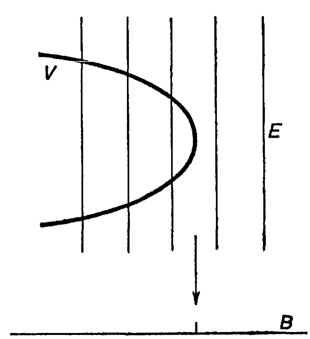


Fig. 3

Examples: 1. The gradient mapping: $q \mapsto \partial S/\partial q$. 2. The normal mapping: associate to each vector normal to a submanifold in Euclidean space its end point. 3. The Gaussian mapping: associate to each point of a transversally oriented hypersurface in Euclidean space the unit vector at the origin in the direction of the normal at that point. (The corresponding Lagrangian manifold consists of the normals to that hypersurface.)

An *equivalence* between Lagrangian mappings is a fiber-preserving symplectomorphism between the total spaces of the fibrations, mapping the first Lagrangian submanifold onto the second one.

The set of critical values of a Lagrangian mapping is called its *caustic*. The caustics of equivalent mappings are diffeomorphic.

Example. The caustic of the normal mapping of a surface is the envelope of its normals, i.e., its focal surface (the surface of the curvature centers).

Every Lagrangian mapping is locally equivalent to a gradient one (to a normal one, to a Gaussian one). The singularities of generic gradient (normal, Gaussian) mappings are equivalent to those of generic Lagrangian mappings. These singularities are classified by the Euclidean reflection groups A, D, E.

Example. Consider a medium of dust-like particles moving inertially whose velocities form a potential field. After a time interval t a particle moves from x to $x+t\,\partial S/\partial x$. We obtain a one-parameter family of smooth mappings $\mathbf{R}^3\to\mathbf{R}^3$.

These are Lagrangian mappings. Indeed, a potential field of velocities defines a Lagrangian section of the cotangent bundle. The phase flow of Newton's equation sends the initial Lagrangian manifold to new Lagrangian manifolds, which, however, need not be sections (for large t): their projections to the base space may have singularities (Fig. 4). The caustics

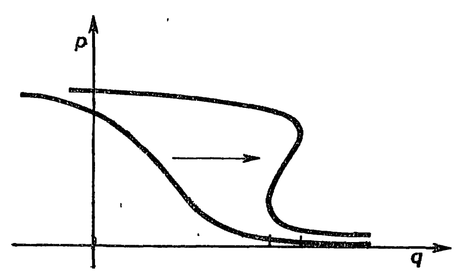


Fig. 4

of these mappings are the places where the density of particles becomes infinite. According to Ya. B. Zel'dovich (1970), a similar model (taking into account gravitation and expansion of the Universe) describes the generation of the large-scale nonuniformity of the distribution of matter in the Universe.

The theory of Lagrangian singularities implies that a new-born caustic has the shape of a saucer (at moment t after its birth the saucer's axes

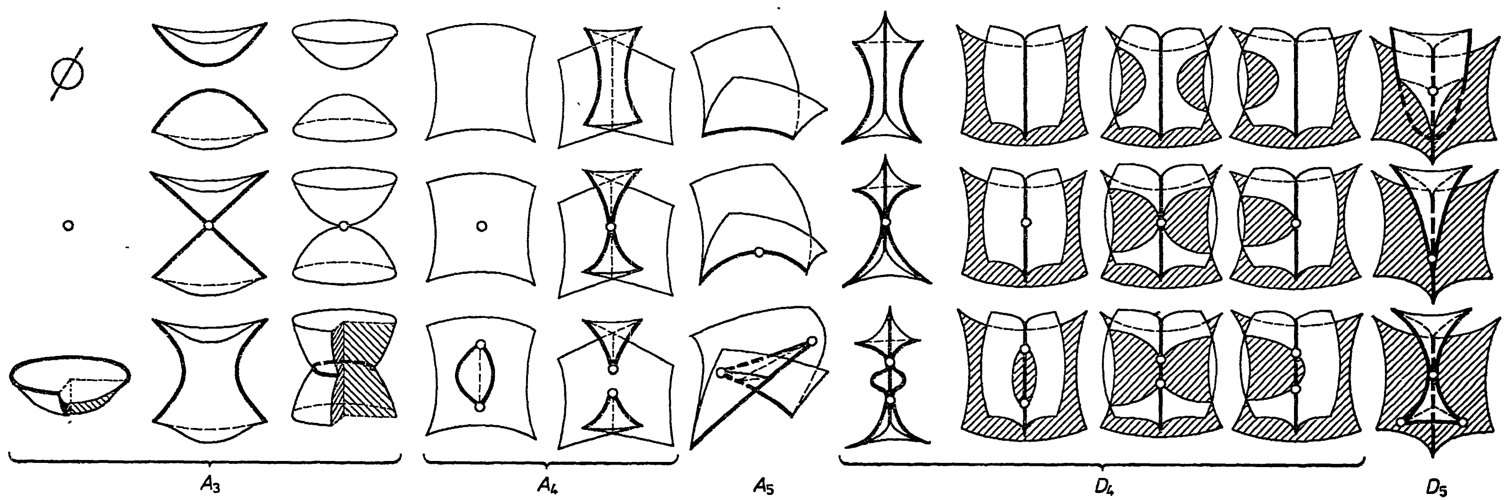


Fig. 5

are of order $t^{1/2}$, its depth of order t and thickness of order $t^{3/2}$). The saucer's birth corresponds to A_3. All metamorphoses of caustics in generic one-parameter families of Lagrangian mappings in 3-space are presented in Fig. 5 (1976).

THEOREM (1972). *The germs of generic Lagrangian mappings of manifolds of dimension* $\leqslant 5$ *are stable and simple (have no moduli) at every point. The simple stable germs of Lagrangian mappings are classified by the* A, D, E *Euclidean reflection groups, as explained below.*

2. Contact geometry

A *contact structure* on an odd-dimensional smooth manifold is a nondegenerate field of hyperplanes in the tangent spaces. The exact meaning of "nondegenerate" is irrelevant because of the "Darboux contact theorem": in the neighbourhood of a generic point, all generic fields of hyperplanes on a manifold of a fixed odd dimension are diffeomorphic.

Examples: 1. The *space of contact elements* of a smooth manifold consists of all its tangent hyperplanes. The velocity of an element belongs to the hyperplane defining the contact structure, if and only if the velocity of the contact point belongs to that element. 2. The space of 1-jets of functions $y = f(x)$ has a natural contact structure $dy = p\,dx$ ($p = \partial f/\partial x$ for the 1-jet of $y = f(x)$ at x).

The external geometry of a submanifold of a contact space is locally determined by the internal one, i.e., by the contact structure traces on the tangent spaces (the Givental contact theorem).

An integral submanifold of a contact manifold is said to be *Legendrian* if it has the highest possible dimension.

Examples: 1. The set of all contact elements tangent to a fixed submanifold (of arbitrary dimension). 2. In particular, the contact elements at a given point form a Legendrian manifold (the fibre of the bundle of contact elements). 3. The set of 1-jets of a function.

A fibration is said to be *Legendrian* if its fibers are Legendrian submanifolds.

Examples: 1. The projective cotangent fibration (a contact element is sent to its contact point). 2. The fibration of 1-jets of a function over its 0-jets (forgetting derivatives).

All Legendrian fibrations of a given dimension are locally contactomorphic (at every point of the total space).

The projection of a Legendrian submanifold to the base of a Legendrian fibration is called a *Legendrian mapping.* Its image is called a *front.*

Examples: 1. *The Legendre transformation*: A hypersurface in a projective space can be lifted to the space of its contact elements as a Legendrian submanifold. The manifold of contact elements of the projective space fibers over the dual projective space (associate to a contact element the hyperplane containing it). This fibration is Legendrian. The projection maps the lifted Legendrian manifold to the hypersurface which is projectively dual to the original hypersurface. Thus the projective dual of a smooth hypersurface is a Legendrian mapping front. 2. *The equidistant mapping*: Pick a point on every oriented normal to a hypersurface in Euclidean space, at distance t from the hypersurface (along the normal). We get a Legendrian mapping whose front is equidistant from the given hypersurface.

Legendrian equivalence, stability and simplicity are defined by analogy with the Lagrangian case.

Every Legendrian mapping is locally equivalent to a mapping defined by a Legendre transformation, and to an equidistant mapping. The local Legendrian singularity theory coincides with that of singularities of Legendre transformations (or equidistant mappings, or wave fronts).

THEOREM (1973). *The germs of generic Legendrian mappings of manifolds of dimension $\leqslant 5$ are stable and simple at every point. The simple stable germs of Legendrian mappings are classified by the A, D, E Euclidean reflection groups: the Legendrian mapping fronts are holomorphically equivalent to the varieties of singular orbits of the corresponding reflection groups.*

Example. The singularities of a generic wave-front in 3-space are (semicubical) cuspidal edges (A_2), and swallow-tails (A_3, Fig. 6: at these points the front is diffeomorphic to the surface in the space of polynomials x^4+ax^2+bx+c, consisting of the polynomials having multiple roots).

Remark. The necessity to complexify in the above theorem suggests that Euclidean reflection groups may have different real forms.

All Lagrangian singularities can be constructed from the Legendrian ones. For this, one considers Legendrian submanifolds of the space of 1-jets of functions. By forgetting the value of the functions one projects the jet space onto the phase space. The Legendrian manifolds' germs are projected isomorphically onto the Lagrangian ones. For instance, the caustic of a Lagrangian mapping is the projection of the cuspidal edge of the Legendre mapping front under a generic projection with 1-dimensional fibers.

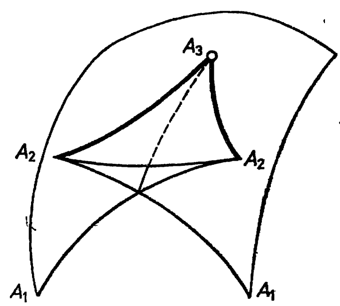


Fig. 6

THEOREM (O. V. Lyashko, 1979). *All holomorphic vector fields transversal to a front of a simple singularity can be mapped one onto another by front-preserving holomorphic diffeomorphisms germs.*

Example. A generic vector field in a neighbourhood of the "most singular point" of the swallow-tail $\{x^4+ax^2+bx+c = (x+a)^2\ldots\}$ is reducible to the normal form $\partial/\partial c$ (Fig. 7) by a swallow-tail-preserving diffeomorphism.

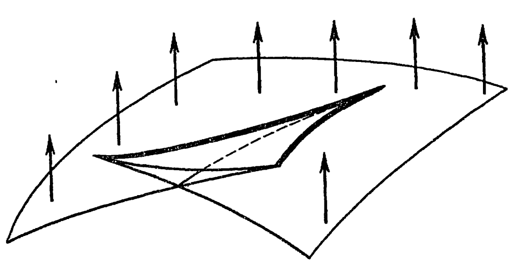


Fig. 7

The reduction to normal form of various geometric objects by wave-front or caustic-preserving diffeomorphisms is the main technical tool in the geometry of ray systems and wave fronts. For instance, the study of the metamorphoses of moving wave fronts is based on a result "dual" to the Lyashko theorem.

THEOREM (1976). *Generic holomorphic functions equal to 0 at the "most singular" point of a simple singularity front can be mapped one onto another by front-preserving holomorphic diffeomorphisms germs.*

Example. A generic function at the most singular point of a swallow-tail is reducible to the normal form a+const by a swallow-tail-preserving diffeomorphism.

The theorem above follows from the equivariant Morse lemma. We use it as follows. The momentary wave fronts form a "large front" in the space-time. Reduce the time function in the space-time to normal form by a large-front-preserving diffeomorphism. We obtain the normal form of the metamorphosis of the momentary front.

The infinitesimal diffeomorphisms preserving a front are the vector fields tangent to it. Their study leads to a "convolution operation" on the invariants of the reflection group. This operation associates to a pair of invariants (i.e., of functions on the orbit space) a new invariant — the scalar product of the gradients of the given functions (lifted from the orbit space to the Euclidean space).

The linearization of this operation is a bilinear symmetric operation on the space cotangent to the orbit space at 0.

THEOREM (1979). *The linearized convolution of the invariants is equivalent to the operation* $(p, q) \mapsto S(p \cdot q)$ *on the local algebra of the corresponding singularity, where* $S = D + (2/h)E$, h *is the Coxeter number, and* D *is the Euler quasihomogeneous derivation.*

For the exceptional groups this theorem was proved by A. B. Givental. In his joint work with A. N. Varchenko (1981) the theorem is extended to higher quasihomogeneous singularities. In this extension they substitute the Euclidean structure by the intersection form of a suitable nondegenerate period mapping. This period mapping comes from a family of holomorphic differential forms on the fibers of the Milnor fibration associated to a versal deformation of a function. A nondegenerate intersection form determines (according to the parity of the number of variables of the function) either a locally flat pseudo-Euclidean metric with a standard singularity at the Legendrian front, or a symplectic structure which is holomorphically extendable to the front.

Example. The set of odd-degree polynomials having highest coefficient equal to 1 and sum of the roots equal to 0 is thus equipped with a new symplectic structure. The variety of polynomials with maximal possible number of double roots is a Lagrangian subvariety.

3. Applications of Lagrangian and Legendrian singularities

The theory was first developed for the study of asymptotics of oscillatory integrals by the stationary phase method. I shall not discuss these (very important) applications here in detail, but shall rather mention: (1) Varchenko's (1976) proof of the formula describing the exponent of the main term of oscillatory integrals in terms of the Newton boundary of the phase function; (2) the example due to the same author of the nonsemicontinuity of this exponent; and (3) V. N. Karpushkin's (1981) proof of a uniform with respect to the parameters estimate from above of the double oscillatory integrals (for simple integrals such an estimate was obtained by I. M. Vinogradov, and for triple ones it was disproved by Varchenko's nonsemicontinuity example).

The uniform estimate also holds for all members of generic families of functions depending on a small number l of parameters (Duistermaat proved it in 1974 for $l \leqslant 6$; Colin de la Verdiere in 1977 for $l \leqslant 7$; Karpushkin in 1982 for $l \leqslant 9$); $l = 73$ is too large (the Varchenko example becomes possible).

The study of asymptotic expansions of oscillatory integrals in the complex domain has led Varchenko (1980–1981) to the construction of a mixed Hodge structure, which he calls the *asymptotic structure.* He has proved that its Hodge numbers coincide with the mixed Hodge numbers constructed algebraically by Steenbrink (1976). Among the corollaries of Varchenko's theory are: (1) the constancy of the Hodge structure invariants along the "$\mu = \text{const}$" statum, and (2) the fact that the "inner modality" of quasihomogeneous functions coincides with their true modality. In real algebraic geometry the mixed structure gives some generalizations of the Petrovskii–Oleinik inequalities.

THEOREM (1978). *The local Poincaré index of a gradient vector field in* $\boldsymbol{R}^{2n}$ *is bounded from above by the middle Hodge number* $|\mathrm{ind}| \leqslant h_1^{n,n}$.

The singularity mixed Hodge structure associates to a finite multiplicity critical point of a function a finite set of rational numbers, the critical points *spectrum.* The spectrum's left end is the smallest exponent of the oscillatory integrals with a given phase function (along complex chains). The examples show the semicontinuity of this exponent, as well as of all the other spectrum points. For instance, the spectrum obtained through a deformation reducing the multiplicity by one, divides the initial spectrum (in the same way as the axes of an ellipsoid divide the axes of the initial ellipsoid).

The spectrum semicontinuity conjecture (1978) was recently confirmed by the works on an apparently unrelated to it algebraic geometry problem: *how large can the numbers of (Morse) singular points on a hypersurface of degree d in CP^n be?*

Bruce (1981) gives an estimate from above, which asymptotic (for the surfaces in CP^3) is $d^3/2+\dots$ (the best estimates from below are of order $3d^3/8$, S. Chmutov, 1983). Comparing the first exactly known answers (0, 1, 4, 16, 31, 64) with the mixed Hodge structure, I have formulated the following

CONJECTURE. *The number of singular points does not exceed the number of integer points m of the cube $(0, d)^n$, for which $(n-2)d/2+1 < \sum m_i \leqslant nd/2$.*

For surfaces in 3-space this implies an estimate from above $23d^3/48+\dots$ Trying to prove this cojecture A. B. Givental in October of 1982 improved the lower order terms in the Bruce estimate. His proof uses some Rayleigh–Fisher–Courant type inequalities and makes transparent the relation of the problem to the spectrum semicontinuity conjecture.

A. N. Varchenko immediately applied to this problem the Steenbrink (1976) theorem on the limits of the Hodge structures. Thus he proved both the conjectured estimate of the number of singular points and the spectrum semicontinuity (the last — for quasihomogenous function deformations, generated by adding lower weight monomials). The same way he proved the semicontinuity of the left end of the spectrum for all functions in 3 variables and for functions in n variables having "far away" Newton polyhedra.

I shall also mention the applications of Lagrangian singularities to the mechanical quadrature theory, i.e., to the problem of integer points in large domains. Let V be the volume of a smooth boundary domain G in the Euclidean $\mathbf{R}^n$, and $N(\lambda)$ the number of integer points inside λG, $R(\lambda) = \lambda^n V - N(\lambda)$. The Lagrangian singularity theory implies the following results:

THEOREM (Colin de Verdiere, 1977). *For $n \leqslant 7$ generically*

$$|R(\lambda)| \leqslant C\lambda^{n-2+2/(n+1)}.$$

THEOREM (Varchenko, 1981). *The average $|R(\lambda)|^2$ over all lattices obtained from the integer point lattice by rotations and shifts, does not exceed $C\lambda^{n-1}$.*

The convex analytic case was studied by Randol (1969). The exponent $(n-1)/2$ is what one might expect according to the law of large numbers (if the λ^{n-1} cells were divided by the boundary independently). The proof of the last theorem is inspired by the Duistermaat (1974) proof of the Maslov "canonical operator" unitarity.

The statistics of Newton diagrams of singularities has led to another inequality related to integer points.

THEOREM (K. A. Sevastianov, S. V. Konyagin, 1982). *The number of vertices of a volume V convex polyhedron in $\boldsymbol{R}^n$, whose vertices are integer points, does not exceed $CV^{(n-1)/(n+1)}$ (the same estimate holds also for the number of faces of arbitrary dimension).*

The influence of the boundary inflections on the remainder term of the asymptotic of the number of integer points is a particular case of the interrelations between the integer and smooth structures of $\boldsymbol{R}^n$, which are crucial for many branches of calculus.

For instance, the order of approximation of a typical point of a submanifold by the hyperplanes defined by the equations with not too large integer coefficients is essential for the resonance phenomena in the theory of nonlinear oscillations (the flattening of the fast frequencies' manifold enhances the sticking at resonances).

In his study of evolutions of action variables in Hamiltonian systems, N. N. Nehoroshev introduced "steepness exponents" of the unperturbed Hamiltonian function. The calculation of these exponents for a generic Hamiltonian function has inspired the theory of tangential singularities.

4. Tangential singularities

These are singularities of the arrangement of a projective surface with respect to its tangents of all dimensions.

Example. The tangential classification of points on a generic surface in 3-space (Fig. 8) was found by O. A. Platonova and E. E. Landis (1979). A line (p) of parabolic points divides the surface into the domain (e) of elliptic points and that of hyperbolic ones (h) containing the curve (f) of inflection points of the asymptotic lines with the biinflection points (b), the selfintersection points (c), and the points of tangency to the parabolic line (t).

This classification is useful both for Nehoroshev's exponent estimate and for the classification of projection degenerations.

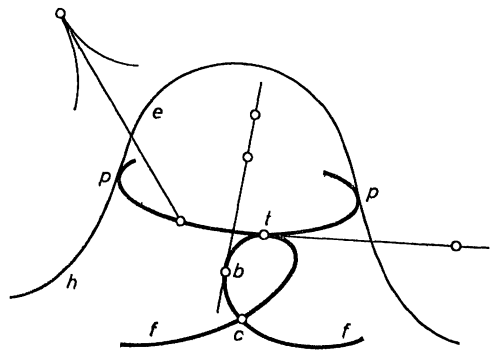


Fig. 8

THEOREM (O. A. Platonova, O. P. Shcherbak, 1981). *Project a generic surface from $\boldsymbol{R}P^3$ to a plane along the straight lines passing through a projection center (a point outside the surface).*

All the projections thus obtained are locally equivalent to the 14 *projections of surfaces $z = f(x, y)$ along the x-axis, where f is given by the list*

$$x, \quad x^2, \quad x^3+xy, \quad x^3\pm xy^2, \quad x^3+xy^3, \quad x^4+xy, \quad x^4+x^2y+xy^2,$$

$$x^5\pm x^3y+xy, \quad x^3\pm xy^4, \quad x^4+x^2y+xy^3, \quad x^5+xy.$$

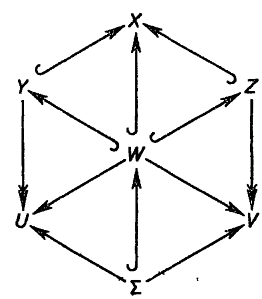
Here the projections are considered as the diagrams $V\to E\to B$ consisting of imbeddings and fibrations, and the equivalences are 3×2-diagrams, whose verticals are diffeomorphisms.

The singularities of a projection from a generic center are only Whitney folds and cusps (one sees a cusp along every asymptotic ray). Other singularities require special points of view. The finiteness of the list of normal forms of projections (and hence that of the list of visible contours) is not evident a priori, because there exists a continuum of nonequivalent singularities in generic 3-parameter families of projections of surfaces to the plane.

The hierarchy of tangencies may become more transparent in terms of the symplectic and contact geometries. Melrose (1976) remarked that the tangent ray geometry of a surface in Euclidean space depends on two hypersurfaces in the symplectic phase space: the first describes the metric and the second — the surface.

The same pair of hypersurfaces describes the hierarchy of asymptotic tangents. Thus we are able to transfer a large part of the geometry on the usual space surfaces to the general case of arbitrary hypersurface pairs in symplectic or contact spaces, using the geometrical intuition of the surface theory for the study of general variational problems with one-sided phase constraints.

Let Y and Z be two hypersurfaces in symplectic space X, intersecting transversally along a submanifold W. Projecting Y and Z onto their characteristics' manifolds U, V, we obtain a hexagonal diagram



where Σ is the (common) manifold of critical points of the projections from W to U and to V.

Example. Let $X = \{q, p\}$ be the phase space of a Euclidean free particle (q is the particle position, p — its momentum); Y — the manifold of unit vectors ($p^2 = 1$); Z — the manifold of boundary vectors (q belongs to a hypersurface Γ). Then U is the ray space, V is Γ's tangent bundle space, W — the bundle space of the boundary (not necessarily tangent) unit vectors and Σ — the spherical tangent bundle space.

Singularities of both projections $W \to U$ and $W \to V$ at a nonasymptotic tangent unit vector are Whitney folds. Each projection defines an involution on W which is the identity on Σ.

Example. We have defined two involutions σ and τ on the manifold W of boundary unit vectors of a convex plane curve (Fig. 9). The product of involutions is the Birkhoff (1927) billiards transformation.

Melrose used the involution pairs to reduce the symplectic space hypersurface pairs to a local normal form by a C^∞-symplectomorphism (in the analytic case the series obtained are generically divergent, as is the case in the Ecalle (1975) and Voronin (1981) theories of dynamical systems at resonances).

At more complicate singularities (for instance, at asymptotic unit vectors) the symplectic space hypersurface pairs have moduli. However, one can reduce the pair formed by the first hypersurface and the intersection to simple normal form (at least formally), for the first two degeneracies of the fold. Thus we can study the singularities of the mapping which associates the ray to a boundary unit vector at the generic asymptotic and biasymptotic unit vectors.

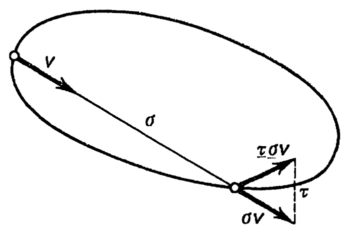


Fig. 9

The variety of critical values of this mapping is locally diffeomorphic to the product of the usual swallow-tail with a linear space. This variety lies in the symplectic space of straight lines in a standard manner:

THEOREM (1981). *All the generic symplectic structures at the point of the critical variety described above are locally reducible one into another by a critical-variety-preserving formal diffeomorphism.*

At a biasymptotic ray the variety of tangent rays is locally diffeomorphic to the product of a swallow-tail with a line. So the above theorem describes the symplectic geometry of the variety of tangent rays.

5. The obstacle problem

Consider an obstacle bounded by a smooth surface in Euclidean space. The *obstacle problem* requires a study of the singularities of the shortest path lenght from a point in the space to a fixed initial set, among paths avoiding the obstacle. This simple variational problem on a manifold with boundary is unsolved even for generic obstacles in 3-space.

The shortest path consists of segments of straight lines and of geodesics on the obstacle surfaces (Fig. 10). Hence let us consider the system of geodesics orthogonal to a fixed front. The system of all rays tangent to these geodesics is a Lagrangian variety in the symplectic space of all rays (as is every system of extremals of a variational problem). In the usual variational problems on manifolds without boundary the relevant Lagrangian variety is smooth (even in the presence of caustics). In the obstacle problem it may acquire singularities. The above theorem implies the following

COROLLARY (1981). *The Lagrangian variety in the generic obstacle problem has a semicubical cuspidal edge at the generic asymptotic rays and an "open swallow-tail" singularity at the bi-asymptotic rays.*

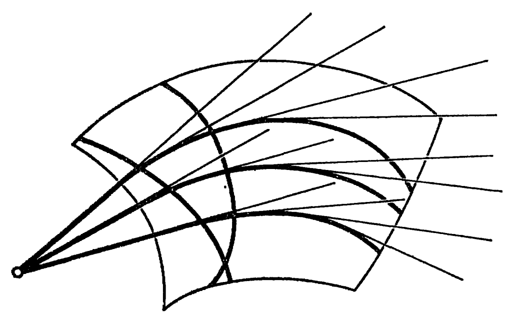


Fig. 10

The *open swallow-tail* is the surface in 4-space $\{x^5+Ax^3+Bx^2+Cx+D\}$ consisting of polynomials with at least triple roots. The differentiation of polynomials maps the open swallow-tail onto the usual one. The opening of the swallow-tail eliminates the selfinetersections but preserves the cuspidal edge (Fig. 11).

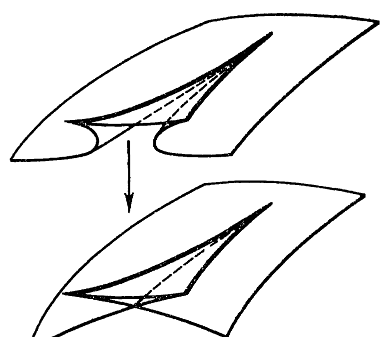


Fig. 11

THEOREM (1981). *The cuspidal edges of the wave fronts moving generically in 3-space form an open swallow-tail in the space-time (over the usual swallow-tail of the caustic).*

THEOREM (O. P. Shcherbak, 1982). *Consider a generic one-parameter family of space curves and suppose that for a given value of the parameter (of the time) the family curve has a biflatness point (of type 1, 2, 5). Then the projective dual curves form a surface in the space-time, which is locally diffeomorphic to the open swallow-tail.*

The open swallow-tail is a first representative of a large series of singularities. Consider the set of polynomials with a root of fixed comultiplicity k, $\big((x-a)^{n-k}(x^k+\dots)\big)$ in the space of polynomials $x^n+\lambda_1 x^{n-2}+\dots+\lambda_{n-1}$. The differentiation of polynomials preserves the comultiplicities of the roots.

THEOREM (A. B. Givental, 1981). *The sequence of varieties of polynomials with roots of fixed comultiplicity stabilizes as the degree n increases, starting with $n = 2k+1$ (i.e., at the moment of the dissociation of the self-intersections).*

Example. The open swallow-tail is the first stable variety over the usual swallow-tail.

The following Givental theory of triads (1982) formalizes the appearance of the open swallow-tail in the obstacle problem.

DEFINITION. A *symplectic triad* (H, L, l) consists of a smooth hypersurt face H in a symplectic manifold, and of a Lagrangian manifold L tangento H (with first order tangency) along a Lagrangian manifold hypersurface l.

The *Lagrangian variety generated by the triad* is the image of l in the manifold of characteristics of H.

Example 1. In the obstacle problem with boundary $\Gamma\subset \boldsymbol{R}^n$ let us consider the distance, along the geodesics of Γ, to the initial front as a function $s\colon \Gamma\to\boldsymbol{R}$. The manifold L of all the extensions of the 1-forms ds from Γ to $\boldsymbol{R}^n$ forms a triad together with the hypersurface $H\colon p^2 = 1$.

This triad generates precisely the variety of rays tangent to the geodesics of our system of extremals on Γ.

Example 2. Consider the symplectic space of polynomials $\mathscr{F} = x^d+\lambda_1 x^{d-1}+\dots+\lambda_d$ of an even degree $d = 2m$. The polynomials, divisible by x^m, form a Lagrangian submanifold L. Let h be the Hamiltonian function of the shifts along the x axis. (This polynomial in λ is

$$h = \sum(-1)^i\mathscr{F}^{(i)}\mathscr{F}^{(j)},\quad i+j = d,\quad \mathscr{F}^{(i)} = d^i\mathscr{F}/dx^i.)$$

The hypersurface $h = 0$ is tangent to the Lagrangian manifold L along its hypersurface l of polynomials divisible by x^{m+1} and forms with them a triad.

The variety generated by this triad is the Lagrangian open swallow-tail of dimension $m-1$ (the set of polynomials $x^{d-1}+a_1x^{d+3}+\ldots+a_{d-2}$ having a root of larger multiplicity than half the degree).

THEOREM (A. B. Givental, 1982). *The triad of Example 2 is stable. Germs of generic triads at all points are symplectically equivalent to those of Example 2.*

COROLLARY. *The variety of rays, tangent to the geodesics of the system of extremals in the generic obstacle problem is locally symplectically equivalent to the Lagrangian open swallow-tail.*

In contact geometry two sorts of Legendrian varieties are associated to the obstacle problem: the varieties of the contact elements of the fronts and the varieties of 1-jets of multi-valued time functions. The varieties of the first type ar egenerically the Legendrian open swallow-tails (they are diffeomorphically lifted Lagrangian swallow-tails). The varieties of the second type are the cylinders over the former.

Example. Consider the obstacle bounded by a plane curve with an ordinary inflection point. The fronts are the curve evolvents. They have two singularities: a usual cusp (of order 3/2) at the boundary curve of the obstacle and a 5/2-singularity at its inflectional tangent (Fig. 12). The

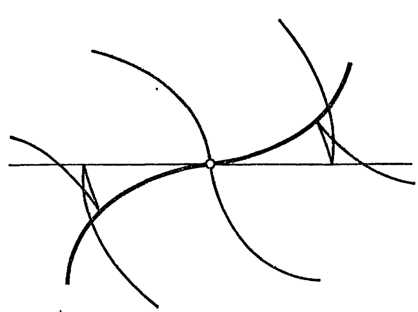


Fig. 12

Legendrian variety is nonsingular over the generic points of the obstacle curve, but over the inflectional tangent points the Legendrian variety has a cuspidal edge of order 3/2.

Let us consider the 3-space of the plane contact elements (fibered over the plane). All contact elements of all the evolvents of a generic curve form a surface in this 3-space. Let us consider the 3-space of polynomials x^3+ax^2+bx+c (fibered over the plane of their derivatives). All those polynomials having multiple roots form a surface in this 3-space.

THEOREM (1978). *The germ of the first surface at the tangent at an inflection point of a generic curve is diffeomorphic to the germ of the second surface at zero, by a fibre-preserving diffeomorphism.*

This surface (Fig. 13), together with the $c = 0$ surface representing the plane contact elements at the obstacle boundary points, forms the variety of singular orbits for the reflection group B_3. This remark has led to the boundary singularity theory (1978), of which I shall only mention the following.

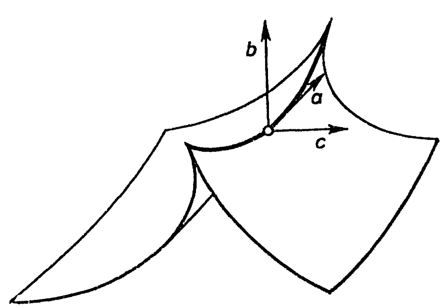


Fig. 13

Example (I. G. Shcherbak, 1982). Consider a generic curve on a generic surface in Euclidean 3-space. At some points the curve touches the surface curvature line. The boundary Lagrangian singularity theory implies that this situation is governed by the exceptional Weyl group F_4: the union of the focal sets of the surface and of the curve with all the surface normals at the points of the curve forms a variety which is locally diffeomorphic to the F_4 caustic.

The boundary Lagrangian singularity theory implies an amusing "Lagrange duality", which interchanges the singularity of a function on the ambient space with that of its restriction to the boundary: this duality is a modernized version of the "Lagrange multipliers rule" (I. G. Shcherbak, 1982).

Returning to an inflection point of a plane curve, consider the graph of the (multi-valued) time function for the obstacle problem. The level sets of this time function are the obstacle evolvents. Hence the graph has the shape drawn in Fig. 14 (1978); this surface has two cuspidal edges (of orders 3/2 and 5/2).

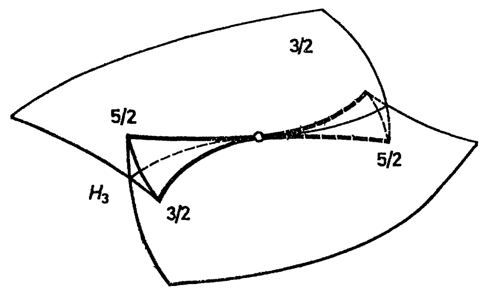


Fig. 14

When I showed this surface to A. B. Givental (1982), he recognized the singular orbit variety of the group H_3 of symmetries of the icosahedron drawn by O. V. Lyashko (1981). Givental's conjecture was rapidly confirmed:

THEOREM (O. P. Shcherbak, 1982). *The graph of the (multi-valued) time function in the generic plane obstacle problem is diffeomorphic to the variety of singular orbits at the inflection points of the obstacle boundary.*

THEOREM[1] (O. V. Lyashko, 1981). *The variety of singular orbits of H_3 is diffeomorphic of the space of polynomials $x^5+ax^4+bx^2+c$ having a multiple root.*

Lyashko's theorem describes the variety of singular orbits of H_3 as the union of the tangents to the curve (t, t^3, t^5) in 3-space, while Shcherbak deals with any curve $(t+\ldots, t^3+\ldots, t^5+\ldots)$.

A generic front in the 3-space obstacle problem must have a singularity of the same type at the point of tangency of an asymptotic ray with the obstacle surface.

In this paper I have not even mentioned many important aspects of the Lagrangian and Legendrian singularity theory, especially the global ones, such as the theory of the coexistence of singularities (the Lagrangian and Legendrian cobordism theories reduced to homotopy problems by Ya. M. Eliashberg, the Lagrangian and Legendrian characteristic

[1] A similar description of $\Sigma(H_4)$ was found by O. P. Shcherbak in 1983: it is based on an inclusion of the H_4 graded local algebra defined by the invariants convolution, into the E_8 graded local algebra: $x^3+y^5+axy^3+by^3+cx+d$.

classes of V. A. Vasiliev, which are generalizations to higher singularities of V. P. Maslov's class, and so on).

I have not even mentioned the extensive classification of the simple projections (Goryunov, 1981), the theory in which, for instance, the exceptional root system F_4 is an ancestor of a whole family of descendants F_μ. One can find details of those theories and the extensive relevant bibliography in the surveys [1and [2].

In spite of the progress of the ray system geometry during the past three centuries, from Huygens up to now, the drawing of pictures very similar to those one finds in Huygens' works is still one of the main sources of new discoveries in this difficult domain where even the 3-dimensional problem is still unsolved and where numerous useful but unexpected interrelations with other branches of mathematics (such as relation of the obstacle problem to the group H_3 of symmetries of the icosahedron) still remain mysterious.

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[3] Arnold V. I., *Catastrophes Theory*, Springer, 1983.