Proceedings of the International Congress of Mathematicians
Berkeley, California, USA, 1986

Representations of Reductive Lie Groups

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1. Introduction. Representation theory is built around two of the fundamental ideas of mathematics: symmetry and linearization. Unfortunately, only one of these is an idea which we would be proud to bring home and introduce to our parents. The development of the subject has often paralleled these dual origins, with insights of breathtaking power and beauty hidden within a maze of technical preliminaries. In this paper, I hope to sketch the development of one of these insights (the Kirillov-Kostant philosophy of coadjoint orbits) as it applies to representations of reductive Lie groups. The technical preliminaries, along with their own more difficult beauties, I will largely neglect.

To begin, suppose G is a group. To regard G as a group of symmetries means to consider a set X on which G acts. That is, for each element g of G there is a permutation $x \to g \cdot x$ of X; and the permutation associated to the product of two group elements is the product of the permutations associated to the elements. Formally,

DEFINITION 1.1. An *action* or *permutation representation* of a group G is a pair (σ, X), with X a set, and σ a homomorphism from G to the group of permutations of X. When no confusion results, we write $g \cdot x$ instead of $\sigma(g)(x)$. We sometimes say that X is a *G-space.*

Here is another formulation of the same idea.

DEFINITION $(1.1)'$. An action of a group G on a set X is a map $G \times X \to X$, $(g, x) \to g \cdot x$, satisfying the following conditions:

(a) for all g and h in G, and x in X, $g \cdot (h \cdot x) = (gh) \cdot x$; and

(b) if e is the identity element of G, then $e \cdot x = x$.

The most important example is this: if H is any subgroup of G, then G acts on the set G/H of left cosets of H in G, by $g \cdot (xH) = (gx)H$.

There are a few interesting things to say about actions even in this generality.

DEFINITION 1.2. Suppose the group G acts on the set X. Fix an element x of X. The *orbit* of x under the action is the subset

$$G \cdot x = \{g \cdot x | g \in G\} \subset X.$$

The author is an Alfred P. Sloan Foundation Fellow. This work is supported, in part, by National Science Foundation Grant DMS-8504029.

The *isotropy group* of the action at x is the subgroup

$$G(x) = \{g \in G | g \cdot x - x\} \subset G.$$

The action is said to be *transitive* if X consists of exactly one orbit. In that case, we say that X is a *homogeneous space* for G.

Any group acts in a unique way on the empty set. Since the number of orbits is zero, the action is not transitive.

LEMMA 1.3. *Suppose G acts on X. Then X is the disjoint union of all the orbits of X on G. This is the unique decomposition of X into a union of homogeneous spaces for G.*

This lemma shows that transitive actions have a particular importance. To describe all of them, we need to know what an equivalence between two actions is.

DEFINITION 1.4. Suppose G acts on two sets X and Y. A map f from X to Y is called *G-equivariant* if it respects the action of G: $f(g \cdot x) = g \cdot f(x)$. The actions are called *equivalent* if there is a G-equivariant bijection from X to Y.

LEMMA 1.5. *Suppose G acts on X, and $x \in X$. Then there is a well-defined map from the coset space $G/G(x)$ to the orbit $G \cdot x$ (Definition 1.2), given by $gG(x) \to g \cdot x$. This map is an equivariant bijection from $G/G(x)$ onto the orbit $G \cdot x$.*

Suppose G acts transitively on X and Y; fix points x and y in these sets. Then X is equivalent to Y if and only if the isotropy groups $G(x)$ and $G(y)$ are conjugate as subgroups of G.

In one sense, these results describe group actions quite completely. It is misleading to take them too seriously, however. To know no more of a sphere than that it is equivalent to $\mathrm{SO}(3)/\mathrm{SO}(2)$ is not to know enough.

Almost any kind of additional structure can be imposed on Definition $(1.1)'$. For example, suppose G is a Lie group, and X is a manifold. Then an action of G of X is called *smooth* if the map from $G \times X$ to X is a smooth map. The questions one wants to ask about a group action usually depend on some such additional structure. For example, if G and X are finite, one can ask for the cardinality of X. If they are topological, one can examine the topological type of X. If they are algebraic varieties, then the singularities of X may be of interest.

All of these questions—and the theory of group actions generally—are nonlinear, in the simple-minded sense that they do not involve vector spaces. Therefore they are quite hard. Some hint of this can be seen even in Lemma 1.5: finding all the subgroups of a group is an enormously difficult task, even for such apparently innocuous examples as the symmetric group on n letters. The idea of (linear) representation theory is that it is sometimes helpful to replace a group action on a set by a related action on a vector space.

DEFINITION 1.6. Suppose G acts on the set X. Write $\mathbf{C}X$ for the complex vector space of functions on X with values in $\mathbf{C}$. Define an action of G on $\mathbf{C}X$

by

$$(g \cdot f)(x) = f(g^{-1} \cdot x) \qquad (g \in G,\ f \in \mathbf{C}X,\ x \in X).$$

We call this action the *regular representation* of G on $\mathbf{C}X$. We sometimes write the operators of this action as $\lambda(g)$: $\lambda(g)f = g \cdot f$.

The regular representation on functions on a G-space has the properties abstracted in the following definition.

DEFINITION 1.7. A (*linear*) *representation* of a group G is a vector space V endowed with an action of G by linear transformations. That is, it is a pair (π, V), with V a vector space, and π a homomorphism from G into the group of automorphisms of V. We sometimes say that V *carries a representation of* G.

We will be interested almost exclusively in complex representations; that is, in the case when V is a vector space over $\mathbf{C}$. Some real representations will appear as auxiliary objects, however.

A reformulation along the lines of Definition (1.1)′ is possible, and often helpful.

DEFINITION (1.7)′. A representation of a group G on a vector space V is a map $G \times V \to V$, $(g, v) \to g \cdot v$, satisfying the following conditions:

(a) for all g and h in G, and v in V, $g \cdot (h \cdot v) = (gh) \cdot v$;

(b) if e is the identity element of G, then $e \cdot v = v$; and

(c) if v and w are in V, and a and b are scalars, then $g \cdot (av + bw) = a(g \cdot v) + b(g \cdot w)$.

Just as in the case of general actions, it is this definition on which additional structure is almost easily imposed. For example, if G is a topological group, and V is a topological vector space, we say that the representation is *continuous* if the map from $G \times V$ to V is continuous.

The analogue for representations of transitive actions (Definition 1.2) is a little subtle.

DEFINITION 1.8. Suppose (π, V) is a representation of the group G. An *invariant subspace* of V is a linear subspace W of V, which is preserved by the action of G:

$$\pi(g \cdot w) \in W \quad (\text{all } g \in G,\ w \in W).$$

The representation is said to be *irreducible* if there are exactly two invariant subspaces.

A continuous representation is said to be *irreducible* if there are exactly two closed invariant subspaces. (This is not quite consistent with the nontopological definition.)

The subspaces $\{0\}$ and V are always invariant. To say that the representation is irreducible means that V is not zero, and that there are no other invariant subspaces.

The most obvious way to construct invariant subspaces is to start with a nonzero vector v, and look at the subspace $\langle G \cdot v \rangle$ generated by the orbit of v. These subspaces do not behave as well as orbits for actions, however. The difficulty is that a nonzero vector w in $\langle G \cdot v \rangle$ may in turn generate a strictly smaller subspace. If V is infinite-dimensional, this process can go on forever; there may be no minimal invariant subspaces. Another aspect of the same problem is that even an irreducible representation is not characterized by the isotropy group of a single nonzero vector. There are no easy analogues of Lemmas 1.3 and 1.5 for linear representations. Nevertheless, our goal is to find some sort of analogues. A little more precisely, we have

ABSTRACT ABSTRACT HARMONIC ANALYSIS PROBLEM 1.9. I. For a reasonable representation (π, V) of a reasonable group G, show that (π, V) is (in some reasonable sense) a "direct sum" of irreducible representations of G.

II. For a reasonable group G, find all the reasonable irreducible representations of G.

Putting everything (symmetry and linearization, that is) together, we arrive at

CONCRETE ABSTRACT HARMONIC ANALYSIS PROBLEM 1.10. Suppose we have a mathematical problem P.

I. Translate P into a problem GXP about a space X on which a group G acts.

II. Translate GXP into a problem GVP about a vector space V of (something like) functions on X.

III. Decompose V as a representation of G, into irreducible representations V_i.

IV. Solve the problem for each irreducible representation V_i.

V. Combine these solutions into a solution of GVP.

VI. Translate back from GVP to P.

A shining example of a problem amenable to this process is the problem of finding the eigenvalues of the Laplace operator on the $(n-1)$-dimensional sphere. There we can take X to be the sphere S^{n-1}; G to be the orthogonal group $\mathrm{O}(n)$; and V to be $L^2(S^{n-1})$. The result is the theory of spherical harmonics. We will give two more examples: one quite foolish but fairly simple; and the other quite technical, but illustrative of the success of the process in the context of reductive groups.

The first problem is that of finding the number N of p-element subsets of an n-element set S. We take X to be the set of such subsets, and V to be the space of all functions on X. The symmetric group G of permutations of S acts on V. We have $N = \dim V$. The irreducible representations of G are completely understood; they are parametrized by the partitions of n. Suppose that p is at most $[n/2]$; the other case is similar. Then it turns out that V is a direct sum of exactly $p+1$ irreducible representations V_i $(i = 0, \ldots, p)$. Here V_i is the irreducible representation corresponding to the partition $(n-i, i)$. The dimensions of the irreducible representations are known; and

$$\dim V_i = [n!(n-2i+1)]/[i!(n-i)!(n-i+1)].$$

By induction on p, it is easy to show that

$$(\dim V_0) + \cdots + (\dim V_p) = n!/[p!(n-p)!].$$

This is then the dimension of V, and hence the cardinality of X.

The next example is Matsushima's formula; details about it may be found in [**3**] and [**14**]. Suppose M is a connected, compact, locally symmetric Riemannian manifold. ("Locally symmetric" means that the map sending v to $-v$ on the tangent space at each point p exponentiates to an isometry θ_p on some neighborhood of p. A Riemann surface always has this property, for example.) Consider the problem of finding the deRham cohomology of M. There is no group immediately evident in this problem; M may admit no isometries. But let $M^\sim$ be the universal cover of M. The local isometries θ_p extend to all of $M^\sim$; and so there is a large group G of isometries of $M^\sim$. The fundamental group Γ of M, regarded as the group of deck transformations of $M^\sim$, is a discrete subgroup of G. Set $X = G/\Gamma$, a compact homogeneous space for G. As a representation, we take $V = L^2(X)$. It turns out that V is a Hilbert space direct sum of a countable number of irreducible representations V_i $(i = 1, 2, \ldots)$.

To any nice continuous representation of G on W, it is possible to attach a collection of vector spaces $H^j(G, W)$, indexed by nonnegative integers j. These are called the *continuous cohomology of G with coefficients in W*. We have

$$H^0(G, W) = \{w \in W | g \cdot w = w, \text{ all } g \in G\};$$

and the higher H^j are defined as derived functors. It turns out that

$$H^j(M, \mathbf{C}) \cong H^j(G, V) \cong \bigoplus H^j(G, V_i);$$

this is Matsushima's formula. Finally, the groups $H^j(G, W)$ can be explicitly determined for any irreducible representation W occurring among the V_i.

The only part of this program which is not effective is the explicit determination of the V_i. Nevertheless, our explicit knowledge of the continuous cohomology of all possible V_i places rather strong *a priori* restrictions on what the cohomology of M can be like.

With applications such as this in mind, this paper will consider the following question: what can a reasonable representation of a reductive Lie group look like? (This question has by no means been answered completely.) §2 discusses what "reasonable" means, and indicates the rough shape of the answer suggested by the Kirillov-Kostant method of coadjoint orbits.

§3 shows how the Jordan decomposition of matrices separates the orbit method into three parts: hyperbolic, elliptic, and nilpotent. A little more precisely, it suggests that there ought to be three fundamental constructions of reasonable representations, and that all reasonable representations are obtained by applying them in succession.

The next three sections examine these three constructions. The hyperbolic step (giving continuous families of representations) is implemented by parabolic induction, developed in the 1950's by Gelfand-Naimark and Harish-Chandra. What is involved is fairly easy real analysis; the representation spaces are L^2 function spaces.

The elliptic step (giving discrete families of representations) was first seriously developed by Harish-Chandra in the 1960's, in his theory of square-integrable representations. It has been greatly extended in the past ten years, by Zuckerman and others. The methods are (in spirit, at least) complex-analytic; representations appear on (something like) Dolbeault cohomology spaces.

The nilpotent step is expected to give only a finite number of representations for each group. It has not yet been systematically developed beyond a very primitive form, and there is no general construction of the representations; but already it is possible to guess something about the appearance of the representations it will eventually produce.

I would like to thank Bert Kostant for teaching me about coadjoint orbits (or at least permitting me to drink from his firehose). Michel Duflo has explained a number of technical points.

2. Unitary representations and coadjoint orbits. Here is the class of "reasonable" representations we will consider in connection with Problem 1.9.

DEFINITION 2.1. Suppose G is a topological group. A representation (π, V) of G is called *unitary* if

(a) π is continuous (defined after Definition (1.7)′),

(b) V is a complex Hilbert space, and

(c) for each element g of G, the operator $\pi(g)$ is unitary.

The set of equivalence classes of irreducible unitary representations of G is called the *unitary dual* of G, and written $\hat{G}$.

Even unitary representations need not decompose as (Hilbert space) direct sums of irreducible representations. An example is the regular representation of $\mathbf{R}$ on $L^2(\mathbf{R})$ (Definition 1.6). In this example, the Fourier transform decomposes the space as a kind of continuous direct sum of irreducible representations. The "summands" are the spaces V_t = multiples of the function χ_t, where $\chi_t(x) = e^{itx}$. (V_t is not a subspace of V. It is a representation of $\mathbf{R}$, however; it is a space of functions, and it is preserved by the regular representation.) This is like a direct sum in the sense that every vector v in $L^2(\mathbf{R})$ is written as a continuous combination $v = \int a_t \chi_t \, dt$. We summarize the basic L^2 properties of the Fourier transform by writing $V = \int_{\oplus} V_t \, dt$ and saying that V is the *direct integral* of the representations V_t.

We leave to the reader's imagination the general definition of the direct integral (of a family (π_x, V_x) of unitary representations, parametrized by the points of a measure space X). When it is properly formulated, we get

THEOREM 2.2 (SEE [**10**]). *Suppose π is a unitary representation of a group G on a separable Hilbert space V. Then (π, V) is equivalent to a direct integral of irreducible unitary representations. If G is a type* I *separable locally compact group, then this decomposition is unique.*

Type I is another term we prefer to leave undefined. Discrete groups are type I only if they are nearly abelian. All of your favorite Lie groups (for example, the reductive ones as defined in §3 below) are type I. For part I of Problem 1.9, we have therefore found a reasonable class of groups to go with our reasonable class of representations.

The method of coadjoint orbits seeks to describe irreducible unitary representations of Lie groups in more or less geometric terms. A complete introduction to it may be found in [**7**, **9**], and [**5**]. We will give an outline, concentrating on the goals of the method rather than on the means by which they are achieved. This requires some notation.

Suppose G is a Lie group. We will always assume that G has finitely many connected components. Put

$$\begin{aligned} G_0 &= \text{identity component of } G,\\ \mathfrak{g} &= \text{Lie algebra of } G, \\ \mathfrak{g}^* &= \text{vector space of real} - \text{valued linear functionals on } \mathfrak{g}. \end{aligned} \tag{2.3}$$

We regard $\mathfrak{g}$ as the tangent space of G at the identity element; then $\mathfrak{g}^*$ is the cotangent space at the identity.

Suppose G acts smoothly on a manifold X, and x is in X. Recall that $G(x)$ is the isotropy group at x (Definition 1.2); write $\mathfrak{g}(x)$ for its Lie algebra. The action mapping restricts to a smooth map

$$G \times \{x\} \to X, \qquad g \to g \cdot x. \tag{2.4a}$$

The differential of this map at the identity is a map from $\mathfrak{g}$ to $T_x(X)$ (the tangent space at X); its kernel is precisely $\mathfrak{g}(x)$. We get

$$\mathfrak{g}/\mathfrak{g}(x) \hookrightarrow T_x(X). \tag{2.4b}$$

If X is a homogeneous space, this is an isomorphism.

Fix now g in G, and consider the other restriction

$$\lambda_g \colon \{g\} \times X \to X, \qquad x \to g \cdot x. \tag{2.5a}$$

The differential of this map is an isomorphism from $T_x(X)$ to $T_{g \cdot x}(X)$. In particular, we get the *isotropy representation*

$$(\tau_x, T_x(X)) \tag{2.5b}$$

of $G(x)$, defined by

$$\tau_x(g) = d\lambda_g(x) \qquad (g \in G(x)). \tag{2.5c}$$

This is a finite-dimensional real representation.

G acts on itself by conjugation:

$$g \cdot x = gxg^{-1}. \tag{2.6a}$$

The isotropy group of the point e in G (the identity) is all of G. The isotropy representation (of G on the tangent space $\mathfrak{g}$ of G at (e) is called the *adjoint representation*, and written Ad:

$$\mathrm{Ad}(g) \colon \mathfrak{g} \to \mathfrak{g}. \tag{2.6b}$$

From any representation π on a vector space V, we can construct the *contragredient representation* π^* on the space V^* of linear functionals on V. We simply regard π as a group action, associate to it the regular representation of G on functions on V, and restrict to the linear functionals. Explicitly,

$$(\pi^*(g)\xi)(v) = \xi(\pi(g^{-1})v) \tag{2.7}$$

(for ξ in V^*, g in G, and v in V). The contragredient of the adjoint representation is called the *coadjoint representation*, and denoted

$$(\mathrm{Ad}^*, \mathfrak{g}^*). \tag{2.8}$$

We will be interested in the coadjoint representation more as a group action than as a representation. That is, we will consider not invariant subspaces of $\mathfrak{g}^*$, but orbits. We will therefore speak of the *coadjoint action* and *coadjoint orbits.*

Here at last is the beginning of the Kirillov-Kostant orbit method.

PHILOSOPHY OF COADJOINT ORBITS (FIRST APPROXIMATION) 2.9. If G is a Lie group, the set $\hat{G}$ of equivalence classes of irreducible unitary representations of G is related to the set of orbits of G on $\mathfrak{g}^*$.

This is a startling idea on its face; the two sets in question appear to be completely unrelated. Mathematical experience might lead one to suspect that there is a trick here—that, properly understood, irreducible unitary representations and coadjoint orbits will turn out to be the same thing almost by definition. There may be such a trick, but it has eluded group representers for more than twenty years now. I prefer to believe that there is real magic.

To understand this philosophy better, we need to make it more precise. The case of abelian groups is an example which helps. Notice first that a unitary operator on a one-dimensional complex Hilbert space is just multiplication by a scalar of absolute value 1. Write $\mathbf{T}$ for the circle group. Then we have just seen that one-dimensional unitary representations are the same as homomorphisms into $\mathbf{T}$.

LEMMA 2.10. *Suppose H is an abelian group. Then any irreducible unitary representation has dimension one. If H is a connected abelian Lie group, then any such homomorphism χ is determined by its differential $d\chi$. If we identify* $\mathrm{Lie}(\mathbf{T})$ *with* $i\mathbf{R}$, *then $d\chi$ corresponds in turn to a linear functional if on $\mathfrak{g}$. This gives an inclusion*

(a) $\hat{G} \subset \mathfrak{g}^*$, $\chi \to f$, *defined by*

(b) $\chi(\exp X) =$ *multiplication by* $e^{if(X)}$.

Conversely, suppose f is any linear functional on H. Then f is identified with a character χ by (b), *if and only if f maps the kernel of the exponential map into* $2\pi\mathbf{Z}$:

(c) $f(X) \in 2\pi\mathbf{Z}$, *all* $X \in \ker(\exp)$.

When H is abelian, an element $f \in \mathfrak{h}^*$ satisfying condition (c) is said to be *integral.* If H is simply connected, the condition is trivially satisfied for all f. If H is a torus, the kernel of the exponential map is a lattice in $\mathfrak{h}$; the integral weights are just 2π times the dual lattice in $\mathfrak{h}^*$.

Lemma 2.10 suggests that the philosophy of coadjoint orbits ought to involve only orbits satisfying some kind of integrality condition. Here is a general formulation.

DEFINITION 2.11. Suppose G is a Lie group, and $f \in \mathfrak{g}^*$ (notation (2.3)). Recall that the isotropy group of the coadjoint action at f is written $G(f)$ (Definition 1.2), and that its identity component is $G(f)_0$. We say that f is *integral* if either of the following equivalent conditions is satisfied.

(a) There is a unitary character $\pi_0(f)$ of $G_0(f)$, with differential if.

(b) There is a finite-dimensional irreducible unitary representation $\pi(f)$ of $G(f)$, with differential $[d\pi(f)](X) = if(X) \cdot \mathrm{Id}$ $(X \in \mathfrak{g})$.

We say that the pair $(f, \pi(f))$ is an *integral datum* for G. We can now formulate a better version of the orbit method.

PHILOSOPHY OF COADJOINT ORBITS (SECOND APPROXIMATION) 2.12. If G is a Lie group, then $\hat{G}$ is in one-to-one correspondence with the set of integral data for G (Definition 2.11). In particular, $\hat{G}$ is in finite-to-one correspondence with the set of integral coadjoint orbits for G. Lemma 2.10 says that this is true if G is connected and abelian. It is rather easy to deduce that it is also true when the identity component of G is abelian. In [**6**], it is proved to be true for G simply connected nilpotent; it follows in the case when the identity component of G is nilpotent.

The next obvious class of groups to consider is solvable groups. There are solvable Lie groups which are not type I (cf. Theorem 2.2). They do not satisfy Philosophy 2.12. This is almost good, however, because irreducible unitary representations are not the most important tools for harmonic analysis in the non-type I case. Auslander and Kostant showed in [**2**] that Philosophy 2.12 applies to simply connected type I solvable groups. As in the nilpotent case, we would like to drop the hypothesis that G be simply connected. There are formal problems with this, however. Although the integrality condition in Definition 2.11 is very pretty, it now turns out not to be quite right. The adjustments needed are rather delicate, but they involve some important ideas; so we will outline them.

Suppose V is a finite-dimensional real vector space. Suppose ω is a nondegenerate symplectic form on V; that is, ω is a skew-symmetric bilinear form on V, with radical zero. Alternatively, one can think of ω as a 2-form on V. (Nondegeneracy in this interpretation means that for any nonzero v in V, there is a w such that $\omega(v \wedge w)$ is nonzero.) The *symplectic group of* V is

$$\mathrm{Sp}(V) = \{g \in \mathrm{GL}(V) | \omega(gv, gw) = \omega(v, w), \text{ all } v, w\}. \tag{2.13a}$$

(We may also write $\mathrm{Sp}(\omega)$, or $\mathrm{Sp}(V, \omega)$.) Its Lie algebra is

$$\mathfrak{sp}(V) = \{X \in \mathfrak{gl}(V) | \omega(Xv, w) + \omega(v, Xw) = 0\}. \tag{2.13b}$$

We want to define a two-fold cover $\mathrm{Mp}(V)$ of $\mathrm{Sp}(V)$. To do so properly would take a little too long. It is not hard to characterize the cover, however. If V is zero, then $\mathrm{Sp}(V)$ is trivial; we define

$$\mathrm{Mp}(0) = \mathbf{Z}/2\mathbf{Z}. \tag{2.14}$$

If V is nonzero, we define $\mathrm{Mp}(V)$ to be the unique (connected) two-fold cover of $\mathrm{Sp}(V)$. In all cases, we get a short exact sequence

$$\{1, \varepsilon\} \to \mathrm{Mp}(V) \to \mathrm{Sp}(V). \tag{2.15}$$

$\mathrm{Mp}(V)$ is called the *metaplectic group.*

Suppose M is a manifold. A *symplectic structure* on M is a closed 2-form ω on M, which is nondegenerate as a bilinear form on each tangent space $T_m(M)$. If we are given such a structure, we say that M is a *symplectic manifold.*

THEOREM 2.16. *Suppose G is a Lie group, and F is a coadjoint orbit. Then F has a natural, G-invariant symplectic structure ω_F, defined as follows. Fix $f \in F$, and write ω_f for the symplectic form on the tangent space $T_f(F)$. Identify this tangent space with $\mathfrak{g}/\mathfrak{g}(f)$ (cf.* (2.4b)*). Given tangent vectors x and y, choose representatives X and Y in $\mathfrak{g}$. Then $\omega_f(x,y) = f([X,Y])$. The isotropy representation τ_f of $G(f)$ (cf.* (2.5)*) preserves ω_f.*

That the formula given defines a symplectic form on $T_f(F)$ is almost trivial. That ω_F is a closed 2-form on F is only slightly harder; it comes down to the Jacobi identity in $\mathfrak{g}$.

DEFINITION 2.17. Suppose G is a Lie group, and $f \in \mathfrak{g}^*$. By Lemma 2.16, there is a homomorphism $\tau_f : G(f) \to \mathrm{Sp}(\omega_f)$. The *metaplectic cover of $G(f)$* is the pullback via τ of the metaplectic cover of $\mathrm{Sp}(\omega_f)$ (cf. (2.15)). It is denoted $G(f)^{\mathrm{mp}}$. We have

$$1 \to \{1, \varepsilon\} \to G(f)^{\mathrm{mp}} \to G(f) \to 1.$$

A representation $\pi(f)^{\mathrm{mp}}$ of $G(f)^{\mathrm{mp}}$ is called *genuine* if

(1) The nontrivial element ε of the kernel of the covering map acts by -1 in $\pi(f)^{\mathrm{mp}}$.

It is called *admissible* if it is genuine, and its differential satisfies

(2) $d\pi(f)^{\mathrm{mp}}(X) = if(X) \cdot \mathrm{Id}$.

In this case, the pair $(f, \pi(f)^{\mathrm{mp}})$ is called an *admissible datum* for G. The element f (or the orbit $G \cdot f$) is called *admissible* if there is an admissible representation of $G(f)^{\mathrm{mp}}$.

With this definition, Duflo has shown that all questions of covering groups fit together properly; irreducible unitary representations of type I solvable Lie groups are parametrized by G-conjugacy classes of admissible data. This suggests another approximation to the orbit method, which the reader can easily formulate.

When this philosophy is applied to semisimple groups (to be defined in §3 below), several things go wrong. The first problem appears when G is $\mathrm{SO}(3)$, the rotation group in three dimensions. In this case, the trivial representation of G is attached both to the orbit $\{0\}$, and to the smallest nonzero admissible orbit. We can no longer expect to have a bijection between representations and admissible data.

A more fundamental problem appears when G is $\mathrm{SL}(2,\mathbf{R})$. There is a family of irreducible unitary representations called the *complementary series*, parametrized by the open unit interval $(0,1)$. Except for the one parametrized by $\frac{1}{2}$, these representations appear not to be associated to any coadjoint orbit. The correspondence from orbits to representations is therefore not surjective.

Again for $\mathrm{SL}(2,\mathbf{R})$, there is a representation which appears to be attached to a union of two orbits, and not to either one individually. (It is the spherical principal series with normalized parameter zero.) The orbits in question are not closed, and their closures have a nonempty intersection.

Other minor problems along these general lines appear for more complicated semisimple groups: the representations attached to admissible data can be reducible, or zero, or they can even fail to exist. But philosophies are not as susceptible to counterexample as theorems, or even conjectures; and something like this at least survives.

PHILOSOPHY OF COADJOINT ORBITS (THIRD APPROXIMATION) 2.18. Suppose G is a type I Lie group. Attached to a finite set of admissible data for G (Definition 2.17), and some boundary conditions along the closures of the corresponding orbits, there is a unitary representation of G. All nice irreducible unitary representations arise in this way.

For the rest of this paper, we will be concerned with implementing the first part of this philosophy: attaching Hilbert spaces and operators to manifolds and group actions. We ignore completely two fascinating problems associated with the second part: giving an *a priori* definition of "nice," and attaching orbits to (nice) representations.

3. Coadjoint orbits for reductive groups. The first reductive group to understand (even before reductive is defined) is $\mathrm{GL}(n,\mathbf{R})$, the group of n by n invertible real matrices. Its Lie algebra is

$$\mathfrak{gl}(n,\mathbf{R}) = \text{all } n \text{ by } n \text{ real matrices.} \tag{3.1a}$$

The adjoint action is by conjugation of matrices:

$$\mathrm{Ad}(g)(X) = gXg^{-1}. \tag{3.1b}$$

The Lie algebra carries a nondegenerate symmetric bilinear form, called the *trace form*, and denoted $\langle\ ,\ \rangle$:

$$\langle X, Y\rangle = \operatorname{tr} XY. \tag{3.1c}$$

It is preserved by the adjoint action. The trace form defines an identification of $(\mathfrak{gl}(n,\mathbf{R}))^*$ with all n by n matrices. The linear functional f corresponds to the matrix $X(f)$ defined by

$$f(Y) = \langle X(f), Y\rangle. \tag{3.1d}$$

Because of the invariance of $\langle\ ,\ \rangle$ under Ad, this identification sends coadjoint orbits to adjoint orbits. That is,

coadjoint orbits for $\mathrm{GL}(n,\mathbf{R})$ are in one-to-one correspondence with conjugacy classes of n by n real matrices. (3.2)

We recall now some elementary facts about conjugacy classes of matrices.

DEFINITION 3.3. Suppose X is an n by n real matrix. We say that X is *nilpotent* if X^k is zero for some k; or, equivalently, if all the (complex) eigenvalues of X are zero. We say X is *semisimple* if X is diagonalizable over $\mathbf{C}$. It is *elliptic* if it is semisimple, and all the eigenvalues are purely imaginary. It is *hyperbolic* if it is semisimple and all the eigenvalues are real; or, equivalently, if it is diagonalizable over $\mathbf{R}$.

PROPOSITION 3.4 (JORDAN DECOMPOSITION). *Suppose X is a real n by n matrix. Then there are unique matrices X_h, X_e, and X_n, with the following properties:*

(a) $X = X_h + X_e + X_n$;

(b) *X_h is hyperbolic, X_e is elliptic, and X_n is nilpotent*; *and*

(c) *X_h, X_e, and X_n all commute with each other.*

They have in addition the following property:

(d) *any matrix commuting with X commutes also with X_h, X_e, and X_n.*

The usual Jordan decomposition, into semisimple and nilpotent parts, has semisimple part

$$X_s = X_h + X_e. \tag{3.5}$$

The *Cartan involution* for $\mathrm{GL}(n,\mathbf{R})$ is the automorphism θ defined by

$$\theta g = {}^t g^{-1}, \tag{3.6a}$$

for g in $\mathrm{GL}(n,\mathbf{R})$. Its differential, also denoted by θ, is the automorphism

$$\theta X = -{}^t X \tag{3.6b}$$

of $\mathfrak{gl}(n,\mathbf{R})$. (Since both group and algebra consist of matrices, the notation is inconsistent.) On the Lie algebra, the $+1$ eigenspace of θ is the Lie algebra of skew-symmetric matrices. It consists of elliptic elements, and the trace form is negative definite there. The -1 eigenspace consists of symmetric matrices, all of which are hyperbolic; the trace form is positive definite there.

At this point, it is convenient to introduce general reductive groups. The definition used here is borrowed from [**8**].

DEFINITION 3.7. A Lie group G (having finitely many components) is called *reductive* if there is a homomorphism $\eta\colon G \to \mathrm{GL}(n,\mathbf{R})$ with the following properties:

(1) the kernel of η is finite;

(2) the image of η is θ-stable.

G is called *semisimple* if it is reductive, and the center of G_0 is finite.

We write θ for the unique lifting of θ to G which is trivial on the kernel of η, and call it the *Cartan involution of* G. The differential of η identifies $\mathfrak{g}$ with a Lie algebra of matrices. Elements of $\mathfrak{g}$ are called *semisimple, nilpotent, hyperbolic*, or *elliptic* when the corresponding matrices are. Put

$$K = \text{fixed points of } \theta \text{ on } G,$$

$$\mathfrak{k} = \mathrm{Lie}(K) = \text{fixed points of } \theta \text{ on } \mathfrak{g},$$

$$\mathfrak{s} = -1 \text{ eigenspace of } \theta \text{ on } \mathfrak{g}.$$

We get the *Cartan decompostion* $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$. By the remarks after (3.6), the trace form $\langle\ ,\ \rangle$ is positive definite on $\mathfrak{s}$ and negative definite on $\mathfrak{k}$. By the Cartan decomposition, $\langle\ ,\ \rangle$ is therefore nondegenerate. We use it as in (3.1d) to identify $\mathfrak{g}^*$ with $\mathfrak{g}$: the linear functional f on $\mathfrak{g}$ corresponds to the element $X(f)$ satisfying $f(Y) = \langle X(f), Y\rangle$, for all Y in $\mathfrak{g}$.

Here are some important structural facts. For $\mathrm{GL}(n, \mathbf{R})$, they amount to Proposition 3.4, and the fact that every real elliptic matrix is conjugate to a skew-symmetric one.

PROPOSITION 3.8. *Suppose G is a real reductive group, and X is in* $\mathfrak{g}$.

(a) *The components X_h, X_e, and X_n of the Jordan decomposition of X all lie in* $\mathfrak{g}$.

(b) *If X is hyperbolic, it is conjugate under* $\mathrm{Ad}(G)$ *to an element of* $\mathfrak{s}$.

(c) *If X is elliptic, it is conjugate under* $\mathrm{Ad}(G)$ *to an element of* $\mathfrak{k}$.

DEFINITION 3.9. Suppose G is a reductive group, and $f \in \mathfrak{g}^*$. Write $X(f)$ for the corresponding element of $\mathfrak{g}$ (Definition 3.7), and

$$X(f) = X(f)_h + X(f)_e + X(f)_n$$

for its Jordan decomposition (Proposition 3.4). The corresponding linear functionals on $\mathfrak{g}$ are written f_h, f_e, and f_n; and $f = f_h + f_e + f_n$ is the *Jordan decomposition* of f. We call f *hyperbolic, nilpotent*, etc. if $X(f)$ is. The *semisimple part* of f is $f_s = f_h + f_e$.

Here is how the Jordan decomposition is to be used to organize the problem of associating representations to orbits. Suppose we are given f. We will use constantly the fact that

$$G(f) = \text{centralizer of } X(f) \text{ in } G, \tag{3.10a}$$

$$\mathfrak{g}(f) = \{Y \in \mathfrak{g} \mid [X(f), Y] = 0\}. \tag{3.10b}$$

Proposition 3.8 allows us to replace f by a conjugate, and get

$$\theta f_h = -f_h, \qquad \theta f_e = f_e. \tag{3.11a}$$

It follows that the isotropy groups

$$G(f_h), \quad G(f_e), \quad \text{and} \quad G(f_s) = G(f_h) \cap G(f_e) \tag{3.11b}$$

are all preserved by θ; they are therefore reductive (via the restrictions of the map η used for G itself). The elements X_e and X_n commute with X_h, and so belong to $\mathfrak{g}(f_h)$. We can therefore identify f_e and f_n (by restriction) with elements of $\mathfrak{g}(f_h)^*$. We get a chain

$$G(f_h) \supset [G(f_h)](f_e) \supset [[G(f_h)](f_e)](f_n); \tag{3.11c}$$

these are the same groups as

$$G(f_h) \supset G(f_s) \supset G(f). \tag{3.11c$'$}$$

The datum we are given is (more or less) a representation of $G(f)$. From this, we propose to get a representation of G in three steps. First, we will get a representation of $G(f_s)$ (the nilpotent step); then a representation of $G(f_h)$ (the elliptic step); then a representation of G (the hyperbolic step). The terminology arises because (for example) the subgroup $G(f_s)$ (from which we begin in the elliptic step) is the isotropy group for the elliptic element f_e in $G(f_h)$ (to which we are going).

As was indicated in the introduction, the order of these steps exactly reverses the historical one; and in fact the general treatment of the first step is still in the future. Undaunted by a suspicion that nothing comes from nothing, however (and having utterly misspent our youths and childhoods), we will treat the last two steps in the next two sections. The last section will discuss prospects for the first step.

4. Parabolic induction and the hyperbolic step. All that we know so far of coadjoint orbits is that they are symplectic homogeneous spaces. (For semisimple groups, that is all there is to know: Kirillov, Kostant, and Souriau have shown independently that any such space is a finite covering of an orbit.) The orbit method asks us to build a representation out of an orbit $X = G \cdot f$. There is only one obvious representation in sight, namely the regular representation on X (Definition 1.6), or something closely related to it. Experimental evidence shows that this is too large: it is almost never irreducible, for example.

A more careful analysis would suggest building a bundle on X out of the admissible datum $(f, \pi(f)^{\mathrm{mp}})$. The difficulty is that $\pi(f)^{\mathrm{mp}}$ must first be untwisted into a representation of $G(f)$ (and not just of $G(f)^{\mathrm{mp}}$). This seems to require something like tensoring it with the metaplectic representation of $\mathrm{Mp}(\omega_f)$. We are left with an infinite-dimensional bundle on a space which was too large to begin with. This looks bad enough to be promising, but no progress has been made in this direction.

If the symplectic structure does not suffice to produce a representation, it is reasonable to ask what additional structure would help. A useful way to phrase that question is this: what more complicated objects happen to be symplectic manifolds in a natural way?

The first answer is cotangent bundles. Suppose Y is any manifold. Then T^*Y carries a natural symplectic structure. If G acts on Y, then it acts on T^*Y, preserving this structure. To the symplectic manifold T^*Y, one can associate the unitary representation of G on $L^2(Y)$. More precisely, we should consider the representation on square-integrable sections of the half-density bundle on Y. This makes sense even if Y has no invariant measure.

More generally, suppose $\mathcal{L}$ is a Hermitian line bundle on Y. Then R. Urwin has shown that there is a *connection bundle* $C_{\mathcal{L}}$ over Y, such that a selfadjoint connection on $\mathcal{L}$ is precisely a section of $C_{\mathcal{L}}$. The space $C_{\mathcal{L}}$ has a symplectic structure; and if $\mathcal{L}$ is a homogeneous line bundle, then G acts on $C_{\mathcal{L}}$. To the symplectic G-space $C_{\mathcal{L}}$, we associate the representation of G on square-integrable sections of $\mathcal{L}$ (twisted by half densities).

The first technique for attaching a representation to an orbit X is therefore to try to realize X as (roughly speaking) the cotangent bundle of some homogeneous space Y; or (more precisely) as the connection bundle for a homogeneous line bundle on Y.

In the case of reductive groups, we can apply this technique to hyperbolic elements. Fix f_h in $\mathfrak{g}^*$ hyperbolic, and write X_h for the corresponding Lie algebra element (Definition 3.7). We have an eigenspace decomposition

$$\mathfrak{g} = \sum_{r \in \mathbf{R}} \mathfrak{g}^r. \tag{4.1a}$$

Here

$$\mathfrak{g}^r = \{Y \in \mathfrak{g} | [X_h, Y] = rY\}. \tag{4.1b}$$

By (3.10), $G(f_h)$ preserves this decomposition, and

$$\mathfrak{g}^0 = \mathfrak{g}(f_h). \tag{4.1c}$$

The Jacobi identity shows that

$$[\mathfrak{g}^r, \mathfrak{g}^s] \subset \mathfrak{g}^{r+s}. \tag{4.1d}$$

The $\mathrm{ad}(X_h)$-invariance of the trace form $\langle\ ,\ \rangle$ shows that

$$\langle \mathfrak{g}^r, \mathfrak{g}^s \rangle = 0 \quad \text{if } r + s \neq 0. \tag{4.1e}$$

Define

$$\mathfrak{n}_h = \sum_{r>0} \mathfrak{g}^r. \tag{4.2a}$$

By (4.1), $\mathfrak{n}_h$ is a nilpotent subalgebra of $\mathfrak{g}$, normalized by $G(f_h)$. Define

$$N_h = \exp(\mathfrak{n}_h), \tag{4.2b}$$

$$P_h = G(f_h)N_h. \tag{4.2c}$$

The group P_h is what is called a *parabolic subgroup* of G. We are trying to make the space $G/G(f_h)$ look like a certain kind of bundle over a smaller homogeneous space. The smaller space will be G/P_h.

Here is an outline of the hyperbolic step of the orbit method. Recall that the preceding steps (yet to be discussed) are to have given us (roughly) a unitary representation π_h of $G(f_h)$. Extend this representation to all of P_h by making it trivial on N_h. Form the induced Hermitian bundle $\mathcal{V}_h$ on G/P_h. The representation of G that we want is that on the space of square-integrable global sections of $\mathcal{V}_h$. More precisely, we want sections of $\mathcal{V}_n$ twisted by the half-density bundle on G/P_h.

The experts will note that this can be elaborated into an orbit-theoretic partial calssification of unitary representations, in the spirit of Duflo's theorem in [4] for general algebraic Lie groups. A datum for this partial classification is a pair (f, π^{mp}) (up to conjugacy by G). Here f is hyperbolic; π^{mp} is an irreducible unitary genuine representation of $G(f)^{\mathrm{mp}}$ (Definition 2.17); and the imaginary part of the infinitesimal character of π^{mp} is $i \cdot f$.

5. Cohomological induction and the elliptic step. Having done what we can with cotangent bundles, we ask again: what other objects are also symplectic manifolds? The next answer we consider is: Kähler manifolds. These are manifolds which are both complex and symplectic, in a compatible way.

More precisely, suppose V is a real vector space. Recall that a *complex structure* on V is a map

$$J: V \to V \tag{5.1a}$$

such that

$$J^2 = -\mathrm{Id}. \tag{5.1b}$$

Here is another formulation. Write

$$V_{\mathbf{C}} = V \otimes_{\mathbf{R}} \mathbf{C} = \{v + iw | v, w \in V\}, \tag{5.2a}$$

the *complexification on* V. $V_{\mathbf{C}}$ is a complex vector space. *Complex conjugation* on V is the conjugate linear automorphism σ defined by

$$\sigma(v + iw) = v - iw. \tag{5.2b}$$

Giving a complex structure on V is equivalent to giving a complex subspace

$$V^{0,1} \subset V_{\mathbf{C}}, \tag{5.3a}$$

with the property that

$$V_{\mathbf{C}} = \sigma(V^{0,1}) \oplus V^{0,1} = V^{1,0} \oplus V^{0,1}. \tag{5.3b}$$

We call $V^{0,1}$ the *anti-holomorphic subspace.* (The equivalence sends J to the $-i$ eigenspace of J on $V_{\mathbf{C}}$.)

DEFINITION 5.4. A *Kähler structure* on a real vector space V consists of

(1) a complex structure J, and

(2) a symplectic structure ω

(cf. (5.1) and (2.13)). These are required to satisfy

(a) $\omega(Jv, w) = -\omega(v, Jw)$.

An equivalent condition is

(a)′ $\omega(Jv, Jw) = \omega(v, w)$.

If the complex structure is considered to be defined by the subspace $V^{0,1}$, then the requirement is equivalent to

(a)″ $\omega_{\mathbf{C}}(v, w) = 0$, all $v, w \in V^{0,1}$.

Here $\omega_{\mathbf{C}}$ denotes the complex-linear extension of ω to $V_{\mathbf{C}}$.

An equivalent formulation is this: a Kähler structure on V consists of

(1) a complex structure J, and

(2) a nondegenerate Hermitian form on the (complex) vector space V.

(Recall that a Hermitian form is a complex-valued sesquilinear form on V, satisfying $\langle v, w\rangle = \overline{\langle w, v\rangle}$.) The two definitions are related by

$$\omega(v,w) = \operatorname{Im}\langle v, w\rangle, \qquad \langle v, w\rangle = \omega(Jv, w) + i\omega(v, w).$$

We call $\langle\ ,\ \rangle$ the *Kähler form* on V. Its signature (p,q) is called the *signature* of the Kähler structure.

Finally, we can think of a Kähler structure as consisting of

(1) a nondegenerate symmetric bilinear form B on V, and

(2) a nondegenerate symplectic form W on V.

These are subject to the following condition: suppose the automorphism J of V is defined by

(a) $B(v,w) = \omega(Jv, w)$.

Then $J^2 = -\mathrm{Id}$.

DEFINITION 5.5. Suppose M is a manifold. A *Kähler structure* on M is a complex structure and a symplectic structure, which give a Kähler structure on each tangent space $T_m M$.

Thus a Kähler manifold is simultaneously complex, symplectic, and (possibly indefinite) Riemannian; and any two of these structures determine the third.

From the point of view of the orbit method, recall that what we seek is a smaller version of the space of functions (or sections of a line bundle) on M. A natural choice is the holomorphic functions. Just as the functions on a space Y have (morally) half as many degrees of freedom as those on T^*Y, so also the holomorphic functions on a complex manifold have half the freedom of the smooth ones.

We tread on thin analytic ice here, however. The absence of "bump functions" subjects holomorphic functions to global constraints which have no parallel in real analysis. To understand the problem, let us consider an example.

EXAMPLE 5.6. Take G to be $\mathrm{GL}(2n, \mathbf{R})$. Recall that $\mathfrak{g}^*$ consists of $2n$ by $2n$ matrices, that is, of linear transformations of $\mathbf{R}^{2n}$. Fix an identification of $\mathbf{R}^{2n}$ with $\mathbf{C}^n$, and let

(a) $f =$ matrix of multiplication by i.

If the identification is made properly,

(b) $f = \left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right)$

This element is elliptic. A moment's thought shows that its centralizer looks like

(c) $G(f) \cong \mathrm{GL}(n, \mathbf{C})$.

The orbit $F = G \cdot f$ consists of all matrices f' with square -1, that is, of all complex structures on $\mathbf{R}^{2n}$.

We claim that F carries a Kähler structure, or, what amounts to the same thing, that it has a G-invariant complex structure giving (with ω_f) a Kähler structure on $\mathfrak{g}/\mathfrak{g}(f)$. To see that, we use the second interpretation of complex structures, as subspaces of $(\mathbf{R}^{2n})_{\mathbf{C}} = \mathbf{C}^{2n}$. The space F is now identified with a subset of the (compact complex) Grassman manifold $\mathrm{Gr}(n, 2n)$, of n-dimensional subspaces in $\mathbf{C}^{2n}$. The condition on the subspace imposed by (5.3b) is open, so F is open in $\mathrm{Gr}(n, 2n)$. This gives an invariant complex structure; we omit the verification that F is Kähler. The complex dimension of F is n^2; the signature of the Kähler form is $((n^2 - n)/2, (n^2 + n)/2)$.

The homogeneous Hermitian line bundles on F are parametrized by unitary characters of the isotropy group $GL(n, \mathbf{C})$. They in turn are parametrized by $\mathbf{Z}$, by

$$\chi_m(g) = (\det(g)/|\det(g)|)^m.$$

Write $\mathcal{L}_m$ for the bundle corresponding to χ_m. It turns out that the space of holomorphic sections $\mathcal{L}_m$ is always finite-dimensional; the dimension is positive exactly when m is at most zero. The representations of G which arise on the holomorphic sections are the Cartan powers of the fundamental representation attached to the middle simple root; they are never unitary, except for the trivial representation when m is zero.

Part of the difficulty is that the complex manifold F has a large compact subvariety. Write B for the standard inner product on $\mathbf{R}^{2n}$, extended by complex linearity to $\mathbf{C}^{2n}$. Consider the following subspace of $\mathrm{Gr}(n, 2n)$:

(d) $F_K = \{V | B(V, V) = 0\}$.

It is evidently an algebraic subvariety. The fact that B is definite on $\mathbf{R}^{2n}$ implies that F_K is contained in F. In fact it is the orbit of F under the orthogonal group:

(e) $F_K = O(2n) \cdot f \cong O(2n)/U(n)$.

F_K is therefore a compact complex subvariety of F, of complex dimension $(n^2 - n)/2$.

A number of clues here suggest that instead of using holomorphic functions, one ought to consider the higher Dolbeault cohomology of F with coefficients in $\mathcal{L}_m$,

(f) $V^p(m) = H^{0,p}(F, \mathcal{L}_m)$.

(This suggestion was made in a slighly different context by Kostant and Langlands, about twenty years ago.) These spaces at least carry representations of G.

One good clue as to the nature of the representations is this. Put

(g) $(V^p(m))_K = H^{0,p}(F_K, \mathcal{L}_m)$.

This finite-dimensional space carries a representation of K, which is explicitly computable by the Bott-Borel-Weil theorem. We find

(h) $(V^p(m))_K \neq 0$ iff either $p = 0$ and $m \leq 0$, or $p = (n^2 - n)/2$ and $m \geq (n - 1)$.

The restriction map for cohomology gives a map from $V^p(m)$ to $(V^p(m))_K$. This suggests that $V^p(m)$ is most interesting under the conditions in (h) above.

We have already discarded the case of nonpositive m as uninteresting; so we are forced to consider the other case. The determinant of the action of $G(f)$ on the holomorphic cotangent space at zero is χ_{2n}, so one can imagine that there is a "half-density" shift by n involved somewhere.

What finally emerges (after some more serious labor) is this. Suppose m is greater than or equal to n, and p is $(n^2 - n)/2$. There there is a G-invariant dense subspace

(i) $\mathcal{H}(m-n) \subset V^p(m)$,

which has a G-invariant inner product making it into a Hilbert space. For m greater than n, the resulting unitary representation is the one attached to the orbit $[(m-n)/2]F$.

This example is very helpful in answering the question of approximately how representations ought to be constructed from elliptic orbits. It is much less clear in detail: we brought an inner product into the picture at the end of the example, entirely out of nowhere. The alert reader may have noticed some difficulty earlier as well: the Dolbeault cohomology space arises as (for example) the cohomology of a complex of $(0,p)$-forms. There is no easy reason for the differential to have closed range, so it is difficult even to define a topology on $V^p(m)$.

On compact manifolds, both problems are cured by Hodge theory: one finds harmonic representatives of everything, and does analysis with those. There are three difficulties with that idea in the present case. First, the Kähler form is indefinite, so the Laplacian is not elliptic. The harmonic forms are therefore not quite so nice. Second, the manifold is noncompact; so there are convergence questions to answer before one can integrate forms to get an inner product on cohomology. Third, the indefiniteness of the Kähler form makes the local inner product on $(0,p)$-forms indefinite; so it is not clear that the inner product on cohomology is positive.

For the moment, these problems appear insuperable. (Schmid has solved them for the discrete series. Some inroads are made on the general case in [**11**].) For the purposes of representation theory, however, they may be regarded as completely resolved by Zuckerman's theory of cohomological parabolic induction (see [**12**]). That theory builds representations using a formal imitation of complex analysis on appropriate homogeneous spaces. It is Zuckerman's method which allows us to complete the elliptic step of the orbit method. To simplify the description, however, we will stay in the complex analysis setting.

Suppose now that G is reductive, and that f_e is an elliptic element of $\mathfrak{g}^*$. Write X_e for the corresponding Lie algebra element. It turns out that $\mathrm{ad}(X_e)$ has purely imaginary eigenvalues, so we have an eigenspace decomposition

$$\mathfrak{g}_{\mathbf{C}} = \sum_{r \in \mathbf{R}} (\mathfrak{g}_{\mathbf{C}})^r. \tag{5.7a}$$

Here

$$(\mathfrak{g}_{\mathbf{C}})^r = \{Y \in \mathfrak{g} \mid [iX_e, Y] = rY\}. \tag{5.7b}$$

This is preserved by $G(f_e)$, and the zero weight space is $\mathfrak{g}(f_e)_{\mathbf{C}}$. The analogues of (4.1d) and (e) hold as well.

Define

$$\mathfrak{u}_e = \sum_{r>0} (\mathfrak{g}_{\mathbf{C}})^r \tag{5.8a}$$

and

$$\mathfrak{q}_e = \mathfrak{g}(f_e)_{\mathbf{C}} + \mathfrak{u}_e. \tag{5.8b}$$

The subspace

$$\mathfrak{q}_e/\mathfrak{g}(f_e)_{\mathbf{C}} \subset [\mathfrak{g}/\mathfrak{g}(f_e)]_{\mathbf{C}} \tag{5.8c}$$

turns out to be the anti-holomorphic tangent space at f_e for a Kähler structure on the orbit

$$F_e = G \cdot f_e. \tag{5.9a}$$

Write

$$(s, r) = \text{signature of Kähler form on } F_e. \tag{5.9b}$$

Here is an outline of the elliptic step of the orbit method. (When this is part of the program described at (3.11), G is replaced by $G(f_h)$.) The unipotent step is supposed to have given us a genuine unitary representation π^{mp} of $G(f_e)^{\mathrm{mp}}$. After a twist by the square root of the determinant of the action of $G(f_e)$ on the holomorphic cotangent space at f_e, we get precisely a unitary representation (π_e, V_e) of $G(f_e)$. This induces a holomorphic Hilbert bundle $\mathcal{V}_e$ on F_e. The representation we want is on an appropriate dense subspace of the Dolbeault cohomology

$$H^{0,s}(F_e, \mathcal{V}_e). \tag{5.10}$$

As stated earlier, there is an existence theorem for these unitary representations.

"THEOREM" 5.11 [**12**]. *Suppose G is a reductive group, and f_e is an elliptic element in $\mathfrak{g}^*$. Use the notation of* (5.7) – (5.9). *Recall from Definition* 2.17 *the metaplectic cover $G(f_e)^{\mathrm{mp}}$.*

(a) *There is a genuine character ρ_e of $G(f_e)^{\mathrm{mp}}$, such that $[(\rho_e)(g)]^2$ =determinant of* $\mathrm{Ad}(g)$ *on μ_e, for g in $G(f_e)^{\mathrm{mp}}$.*

Fix an irreducible unitary genuine representation π^{mp} of $G(f_e)^{\mathrm{mp}}$. Assume that (1) *the restriction of π^{mp} to the commutator subgroup is weakly unipotent* [**12**, *Definition* 8.16]; *and* (2) *π^{mp} has differential* $(if_e)\mathrm{Id}$ *on the center of $\mathfrak{g}(f_e)$. Set $\pi_e = \pi^{\mathrm{mp}} \otimes \rho_e$, and let $\mathcal{V}_e$ be the induced holomorphic Hilbert bundle on F_e.*

(b) *The Dolbeault cohomology of F_e with coefficients in $\mathcal{V}_e$ vanishes except in degree s.*

(c) *There is a dense G-invariant subspace $V(f_e, \pi^{\mathrm{mp}}) \subset H^s(F_e, \mathcal{V}_e)$, which carries a unitary representation of G.*

What the quotation marks mean is this. The result is probably true as stated, but current techniques prove only an algebraic analogue. (An honest unitary representation is given by the algebra; all that is lacking is the geometric realization.) The first of the two hypotheses can be regarded as a desired property of the orbit method for nilpotent orbits. The second is tied to the first hypothesis in Definition 2.17. One way that both hypotheses can be satisfied is if π^{mp} is a unitary character with differential f_e.

The representation V given by the theorem may be reducible or zero; but neither of these possibilities can arise if f_e is large enough.

6. The nilpotent step. What remains is to understand what representations of our reductive group G are associated to nilpotent coadjoint orbits. This is still largely mysterious. The nilpotent orbits are typically not cotangent bundles; except for the case of the point zero, they never admit invariant Kähler structures; and group representers have thought of no other answers to the question posed at the beginning of §4.

Nevertheless, the methods of the hyperbolic and elliptic steps have something to offer here. First, the element zero of $\mathfrak{g}^*$ is both hyperbolic and elliptic (as well as nilpotent); either §4 or §5 says that the representations associated to it must be those trivial on G_0. Here is a more refined version of the same idea.

LEMMA 6.1. *In the setting of equations* (4.1) *and* (4.2), *there are open orbits* $(E_1, \ldots, E_t)$ *of* G *on* $T^*(G/P_h)$. *Each* E_i *is a finite cover of a nilpotent coadjoint orbit* F_i *(as a symplectic homogeneous space).*

This lemma suggests the following requirement.

REQUIREMENT 6.2. In the setting of Lemma 6.1, suppose $\mathcal{V}$ is a homogeneous Hermitian vector bundle on G/P_h, with an invariant flat connection. Then the unitary representation of G on L^2 sections of $\mathcal{V}$ (twisted by half-densities) must be one of those associated to the nilpotent coadjoint orbits F_i.

A requirement along the same lines can be made using the elliptic step.

REQUIREMENT 6.3. Suppose G is a reductive group, and f_e is an elliptic element in $\mathfrak{g}^*$. Use the notation (5.7)–(5.9). Suppose π^{mp} is a genuine representation of $G(f_e)^{\mathrm{mp}}$, trivial on the identity component. Set

$$\pi_e = \pi^{\mathrm{mp}} \otimes \rho_e$$

(cf. Theorem 5.11), and let $\mathcal{V}_e$ be the induced holomorphic Hilbert bundle on F_e. Then the unitary representation of G which is dense in $H^s(F_e, \mathcal{V}_e)$ must be one of those attached to nilpotent coadjoint orbits.

The nilpotent orbits to which this representation ought to be attached are those in the "associated cone"

$$\lim_{t \to 0^+} tF_e. \tag{6.4}$$

It can be shown that the representation is nonzero if and only if this limit cone has the same dimension as F_e. (The orbits in Requirement 6.2 may be obtained as the associated cone for F_h. They always have the same dimension as F_h.)

By arguments like this, one can collect evidence for proposed descriptions of the orbit method for nilpotent orbits. Additional help has come from primitive ideal theory and the theory of automorphic forms. One would like to apply knowledge about representations to these subjects. However, each of them has its own internal intuition, and it is possible to predict what the representation theory ought to say about them. These predictions can then be read as conjectures about representation theory—with luck, as descriptions of the representations attached to nilpotent orbits. The paper [**1**] is a landmark in this direction; an account of further developments will appear in [**13**].

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