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Algebraic K-Theory of Fields

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The construction of higher algebraic K-theory was achieved by the fundamental work of Quillen [**24**]. After that the main efforts were concentrated in the field of computations and applications of K-theory to concrete algebraic problems. The most intriguing are the conjectures relating algebraic K-theory to etale cohomology. Such conjectures in certain particular cases were made by Quillen and Lichtenbaum [**14**, **9**, **23**]. Nowadays all conjectures of this type are usually called the Quillen-Lichtenbaum conjectures. One of the important properties of algebraic K-theory is the exact localization sequence: if $Y \subset X$ is a closed subscheme, then there is a long exact sequence

$$\cdots \to K_i'(Y) \to K_i'(X) \to K_i'(X-Y) \xrightarrow{\partial} K_{i-1}'(Y) \to \cdots$$

($K' = K$ for regular schemes) and the resulting spectral sequence

$$E_1^{pq} = \coprod_{\operatorname{codim} x = p} K_{-p-q}(k(x)) \Rightarrow K'_{-p-q}(X).$$

Another important property is Gersten's conjecture, proved by Quillen, which makes it possible to identify the second term of this spectral sequence: $E_2^{pq} = H^p(X, K_{-q})$. These properties often reduce general problems of algebraic K-theory to the particular case of fields in which case these problems are especially explicit and intriguing.

Higher K-theory of a field F (as well as of any ring) may be defined in terms of Quillen's plus construction: $K_i(F) = \pi_i(\mathrm{BGL}(F)^+)$, where $\mathrm{BGL}(F)^+$ is the H-space having the same homology as $\mathrm{BGL}(F)$, i.e., the same as homology of the discrete group $\mathrm{GL}(F)$. Thus K-theory is closely related to the homology theory of $\mathrm{GL}(F)$.

This paper concerns some of the recent achievements in the K-theory of fields and in related areas. To a pity, I have only mentioned very briefly such an important field as etale K-theory of Dwyer-Friedlander; the ideas and methods used in this theory are very far from those discussed in the main part of this paper.

1. Norm-residue homomorphism. In view of the Moore-Matsumoto theorem, the group $K_2(F)$ may be described as a group with generators $\{a, b\}$ ($a, b \in F^*$) and relations $\{a_1a_2, b\} = \{a_1, b\} + \{a_2, b\}$, $\{a, b_1b_2\} = \{a, b_1\} + \{a, b_2\}$, $\{a, 1-a\} = 0$ ($a \neq 1$). Suppose that n is an integer prime to $\operatorname{char} F$; then we have the Kummer isomorphism $\chi: F^*/F^{*n} \xrightarrow{\sim} H^1(F, \mu_n)$. It is easy to verify that $\chi(a) \cup \chi(1-a) = 0 \in H^2(F, \mu_n^{\otimes 2})$ and hence we get a well-defined homomorphism

$$R_n = R_{n,F}: K_2(F)/n \to H^2(F, \mu_n^{\otimes 2}): \{a, b\} \mapsto \chi(a) \cup \chi(b),$$

which is called the norm-residue homomorphism. In case $F \supset \mu_n$ the choice of the primitive nth root of unity ξ makes it possible to identify G_F-modules μ_n and $\mu_n^{\otimes 2}$ and hence to identify $H^2(F, \mu_n^{\otimes 2})$ with $H^2(F, \mu_n) = {}_n\operatorname{Br}(F)$. After this identification R_n turns into a cyclic algebra homomorphism: $\{a, b\} \mapsto [A_\xi(a, b)]$ (cf. [**19**]). Thus in this case the question about surjectivity of R_n is equivalent to the classical problem of Albert whether every algebra of exponent n is similar to a product of cyclic algebras.

THEOREM 1.1 [**17, 30**]. *For any field F and any n prime to* $\operatorname{char} F$, *R_n: $K_2(F)/n \to H^2(F, \mu_n^{\otimes 2})$ is an isomorphism.*

The general case of the theorem may be easily reduced to the case (which we will consider below) when $n = p$ is prime and $F \supset \mu_p$. There are two different, but closely related, approaches to the proof of (1.1). Both approaches use essentially the computation of certain K-cohomology groups of Severi-Brauer varieties.

The first method, the original method of Merkurjev [**16**], works mostly for $p = 2$. Set provisionally $k_2 = K_2/2$. Suppose that $E = F(\sqrt{a})$ is a quadratic extension of a field F and denote by $\chi(a) \in H^1(F, \mu_2)$ the cohomology class corresponding to a under the Kummer isomorphism. The exact cohomology sequence

$$H^1(F, \mu_2) \xrightarrow{\chi(a)} H^2(F, \mu_2) \to H^2(E, \mu_2) \xrightarrow{N_{E/F}} H^2(F, \mu_2)$$

shows that the validity of (1.1) implies the exactness of the sequence

$$F^*/F^{*2} \xrightarrow{a} k_2(F) \to k_2(E) \xrightarrow{N_{E/F}} k_2(F). \tag{1.1.1}$$

Vice versa, if (1.1.1) is exact for any quadratic extension, then an easy inductive argument proves (1.1). Moreover, it is shown in [**16**] that even the exactness of

$$k_2(F) \to k_2(E) \to k_2(F)$$

for any E/F is sufficient to finish the proof. Every element of $k_2(E)$ may be written in the form $\sum_{i=1}^n \{x_i + \sqrt{a}y_i, z_i\}$ with $x_i, y_i, z_i \in F$. The norm of this element in $k_2(F)$ is equal to $\sum_{i=1}^n \{x_i^2 - ay_i^2, z_i\}$. It is not difficult to write down explicitly when the last element is equal to zero: this is equivalent (after certain cosmetic changes, including possible enlargement of n) to the existence of certain elements u_S, v_S for every nonempty $S \subset \{1, \dots, n\}$ such that, setting $z_S = \prod_{i \in S} z_i$ we will have formulae

$$x_i^2 - y_i^2 a = \prod_{S \ni i} (u_S^2 - z_S v_S^2). \tag{1.1.2}$$

Denote by F_0 the prime subfield of F and set $F_1 = F_0(a)$. Equations (1.1.2) define an affine variety T over F_1, and elements x_i, y_i, z_i, u_S, v_S define an F-valued point of this variety. Denoting the corresponding coordinate functions by $X_i, Y_i, Z_i, U_S, V_S \in F_1(T)$, we get in $k_2(F_1(T)(\sqrt{a}))$ the "universal" element with trivial norm $\sum_{i=1}^n \{X_i + \sqrt{a}Y_i, Z_i\}$. It is sufficient to show that this universal element lies in $k_2(F_1(T))$—the specialization argument finishes the proof. To prove the last statement it is sufficient to show that R_2 is an isomorphism for $F_1(T)$ and $F_1(T)(\sqrt{a})$. This is trivial for the second field since this field is purely transcendent over $F_1(\sqrt{a})$ (both kernel and cokernel of R_n do not change under purely transcendental extensions [**4**]). The field $F_2 = F_1(Z_i, U_S, V_S)$ is purely transcendental over F_1, and $F_1(T)$ is obtained from F_2 pasing several times to the function field on a conic, given by an equation of the form $X^2 - Y^2 a = *$. So the theorem follows from

PROPOSITION 1.2 [**27**]. *Suppose that* $\operatorname{char} k \neq 2$, $a, b \in k^*$, *and denote by* F *the function field on the conic, given by equation* $X^2 - aY^2 = b$. *If* $R_{2,k}$ *and* $R_{2,k(\sqrt{a})}$ *are isomorphisms, then* $R_{2,F}$ *is also an isomorphism.*

The proof of (1.2) is based on the computation of certain K-cohomology groups of the conic; it also uses extensively the theory of quadratic forms, which does not allow the use of this method for $p \neq 2$.

The second approach to the proof of (1.1), developed in [**17, 28, 30**], is in a certain sense opposite to the one discussed above. The main technical result in this approach is

PROPOSITION 1.3. *Suppose that* $p \neq \operatorname{char} F$ *and* F *contains a primitive* p*th root of unity* ξ. *Let* $a, b \in F^*$ *and denote by* X *the Severi-Brauer variety, corresponding to the cyclic algebra* $D = A_\xi(a, b)$. *The natural homomorphisms* $\ker R_{p,F} \to \ker R_{p,F(X)}$ *and* $\operatorname{coker} R_{p,F} \to \operatorname{coker} R_{p,F(X)}$ *are injective.*

Assuming (1.3), one can finish the proof of (1.1) as follows. It is well known that the maps $\ker R_{p,F} \to \ker R_{p,E}$, $\operatorname{coker} R_{p,F} \to \operatorname{coker} R_{p,E}$ are injective if E is algebraic over F of degree prime to p, so, using (1.3), one can construct an extension $\tilde{F}/F$ such that

(a) all cyclic p-algebras over $\tilde{F}$ are trivial,

(b) $\tilde{F}$ has no extensions of degree prime to p, and

(c) $\ker R_{p,F} \hookrightarrow \ker R_{p,\tilde{F}}$, $\operatorname{coker} R_{p,F} \hookrightarrow \operatorname{coker} R_{p,\tilde{F}}$.

A classical result of Milnor [**19**] shows that (a) is equivalent to the equality $K_2(\tilde{F})/p = 0$. Hence $\operatorname{Ker} R_{p,\tilde{F}} = 0$. Moreover, it is easy to see that (a) and (b) imply that $\operatorname{Br}(\tilde{F}) = 0$ and hence $\operatorname{coker} R_{p,\tilde{F}} = 0$. Now property (c) shows that $\ker R_{p,F} = \operatorname{coker} R_{p,F} = 0$.

The proof of (1.3), as well as the proof of (1.2), is based on the computation of K-cohomology groups of Severi-Brauer varieties.

PROPOSITION 1.4. *In conditions of* (1.3), $H^1(X, K_2) = N = \mathrm{Nrd}\, D^* \subset F^*$, *the natural map* $K_2(F) \to H^0(X, K_2)$ *is surjective.*

To prove this, one has to consider the spectral sequence $E_2^{ij} = H^i(X, K_{-j}) \Rightarrow K_{-i-j}(X)$. The theory of Chern classes and the Riemann-Roch theorem makes it possible to show that all differentials in this spectral sequence starting at or coming to $E^{i,j}$ terms with $i + j = 0, -1$ are killed by $(\dim X)!$ In our case $\dim X = p - 1$ and we know also that all differentials in this spectral sequence are killed by p (since D has splitting fields of degree p over F). This shows that there are no differentials starting at or coming to $E^{i,j}$ with $i + j = 0, -1$ and hence $H^1(X, K_2) = E_2^{1,-2} = E_\infty^{1,-2} = K_1(X)^{1/2}$. K-theory of Severi-Brauer varieties was computed by Quillen [**24**]:

$$K_i(X) = K_i(F) \oplus K_i(D) \oplus \cdots \oplus K_i(D^{\otimes(p-1)}).$$

Thus to finish the proof of the first statement it is sufficient to compute the topological filtration on $K_1(X) = F \oplus N \oplus \cdots \oplus N$, which is not difficult to do. Vanishing of all differentials starting at $E^{0,-2}$ imply that the edge homomorphism

$$K_2(F) \oplus K_2(D) \oplus \cdots \oplus K_2(D^{\otimes(p-1)}) = K_2(X) \to H^0(X, K_2) = E_2^{0,-2}$$

is surjective. To finish the proof of the second statement we have to show that the image of $K_2(D^{\otimes i})$ in $H^0(X, K_2) \subset K_2(F(X))$ is contained in the image of $K_2(F)$. This requires additional information about K_2 for algebras of prime index; see (3.1) below.

Proposition 1.4 is not yet sufficient for the proof of (1.3); one needs a more precise statement that $K_2(F) = H^0(X, K_2)$. This requires information about torsion in $K_2(F)$. The basic result in this direction is Hilbert's Theorem 90 for K_2.

THEOREM 1.5. *Let E/F be a cyclic extension of prime degree p and let σ be a generator of* $\mathrm{Gal}(E/F)$. *The following sequence is exact:*

$$K_2(E) \xrightarrow{1-\sigma} K_2(E) \xrightarrow{N_{E/F}} K_2(F). \tag{1.5.1}$$

The exactness of (1.5.1) is easily proved provided the norm map $N\colon E^* \to F^*$ is surjective—in this case, one constructs explicitly the homomorphism $K_2(F) \to K_2(E)/(1-\sigma)K_2(E)$ inverse to $N_{E/F}$ by means of the formula $\{a, b\} \mapsto \{\alpha, b\}$ $\mathrm{mod}(1-\sigma)K_2(E)$, where $N(\alpha) = a$. Using the same trick as above we see now that we will be done if we are able to prove that if X is a Severi-Brauer variety, corresponding to a cyclic algebra $(E/F, \sigma, a)$, then the map

$$\ker N_{E/F}/(1-\sigma)K_2(E) \to \ker N_{E(X)/F(X)}/(1-\sigma)K_2(E(X))$$

is injective. This problem is simplified by the fact that the algebra under consideration splits over E and hence $X_E = \mathbf{P}_E^{p-1}$. The proof uses, in fact, only the computation of $H^1(X, K_2)$ fulfilled above.

Applying (1.5) to the universal Kummer extension $F(\sqrt[n]{T})/F(T)$ (T is transcendental over F) or to the universal Artin-Schreier extension we get the following result, which was conjectured by Tate [**37**].

COROLLARY 1.6. *If F contains a primitive nth root of unity ξ, then ${}_nK_2(F) = \{\xi, F^*\}$, $K_2(F)$ does not have p-torsion with $p = \operatorname{char} F$.*

To finish the description of torsion in K_2, one needs the description of those elements $x \in F^*$ for which $\{\xi, x\} = 0$. This question is settled by

THEOREM 1.7 [**28, 30**]. *Suppose that F contains a primitive nth root of unity ξ and denote by F_0 the subfield of constants in F (i.e., the algebraic closure of the prime subfield). For $x \in F^*$ the following conditions are equivalent:*
(a) $\{\xi, x\} = 0 \in K_2(F)$;
(b) $x = x_0 y^n$, *where* $y \in F^*$, $x_0 \in F_0^*$, *and* $\{\xi, x_0\} = 0 \in K_2(F_0)$.

COROLLARY 1.8. *If E/F is an extension such that F is algebraically closed in E, then $K_2(F) \hookrightarrow K_2(E)$.*

The last corollary shows that in conditions of (1.4), $K_2(F) \xrightarrow{\sim} H^0(X, K_2)$. Using this and (1.4), one easily finishes the proof of (1.3); see [**17, 30**].

The proof of (1.7) is based on the study of certain l-adic cohomology groups. The crucial role plays the following fact, related to Weil's theorem about eigenvalues of Frobenius substitution on a Tate module of an abelian variety.

PROPOSITION 1.9. *Suppose that F is finitely generated and $l \neq \operatorname{char} F$. Then $H^1(F_0, Z_l(2)) \xrightarrow{\sim} H^1(F, Z_l(2))$ and $H^2(F_0, Z_l(2)) \hookrightarrow H^2(F, Z_l(2))$.*

The above results have many important applications in algebra and algebraic geometry, some of which may be found in [**30, 39, 40, 41**]. We will only mention the following, for further use.

PROPOSITION 1.10 [**28, 30**]. *If X/F is a complete rational variety, then $H^0(X, K_2) = K_2(F)$.*

2. Algebraically closed and local fields. Since the etale cohomology groups of an algebraically closed field are trivial, it is reasonable to expect that K-groups of such a field will also have a sufficiently simple structure. The following is one of the Quillen-Lichtenbaum conjectures (see [**9, 23**]).

(2.1) If F is an algebraically closed field, then $K_i(F)$ is divisible for $i \geq 1$, the torsion subgroup in $K_i(F)$ being zero if i is even, and isomorphic to $\coprod_{l \neq \operatorname{char} F} Q_l/Z_l(n)$ if $i = 2n - 1$.

This conjecture is clearly true for $i = 1$ and may be easily proved for $i = 2$ [**2**]. Apart from these trivial cases, the conjecture was known to be true in the case where F is the algebraic closure of a finite field [**22**]. For fields of positive characteristics, this conjecture was proved in [**31**]. The basic result of [**31**] is

THEOREM 2.2. *If F/F_0 is an extension of algebraically closed fields, then for any integer n the induced maps $K_i(F_0)/n \to K_i(F)/n$, ${}_nK_i(F_0) \to {}_nK_i(F)$, $K_i(F_0, Z/n) \to K_i(F, Z/n)$ are bijective.*

This theorem is a particular case of a certain simple general principle. Let V be a contravariant functor on an appropriate category of schemes with values in the category of torsion abelian groups. Suppose further that for any finite flat morphism $X \to Y$ we are given a transfer homomorphism $N_{X/Y}: V(X) \to V(Y)$, satisfying the usual properties. Suppose finally that V is homotopy invariant, i.e., $V(X \times A^1) = V(X)$ for any X.

PROPOSITION 2.3 (RIGIDITY THEOREM). *Let X/F be a connected variety over an algebraically closed field. Then for any two points $x, y: \operatorname{Spec}(F) \to X$, the induced maps $V(X) \rightrightarrows V(\operatorname{Spec} F) = V(F)$ coincide.*

It is clearly sufficient to treat the case of a smooth affine curve. Consider the bilinear pairing $\operatorname{Div}(X) \times V(X) \to V(F)$ given by $x \times u \mapsto x^*(u)$. We have to show that its restriction on $\operatorname{Div}^0(X) \times V(X)$ is trivial. Denote by $\overline{X}$ the smooth projective model of X and set $X_\infty = \overline{X} - X$. If f is a rational function on $\overline{X}$, defined and equal to one on X_∞, then the principal divisor (f) lies in the kernel of our pairing: f defines a covering $X_0 \to \mathbf{A}^1_F = \mathbf{P}^1_F - 1$, where X_0 is obtained from X by deleting points where f is equal to one. The usual properties of transfer imply that the image of $(f) \times u$ in $V(F)$ coincides with the image of $(0 - \infty) \times N_{X_0/\mathbf{A}^1_F}(u|_{X_0})$, which is zero in view of homotopy invariance. Thus our pairing factors through $\operatorname{Pic}^0(\overline{X}, X_\infty) \otimes V(X)$. The group $\operatorname{Pic}^0(\overline{X}, X_\infty)$ coincides with the group of F-points of the corresponding Rosenlicht jacobian of $\overline{X}$ (see [**26**]) and hence is divisible. Since $V(X)$ is torsion we deduce that $\operatorname{Pic}^0(\overline{X}, X_\infty) \otimes V(X) = 0$.

COROLLARY 2.3.1. *Let F/F_0 be an extension of algebraically closed fields and let X_0/F_0 be a connected variety. If $x, y: \operatorname{Spec} F \to X_0$ are any two F_0-points, then the induced maps $V(X_0) \rightrightarrows V(F)$ coincide.*

COROLLARY 2.3.2. *In conditions of* (2.3.1) *for any F_0-point $x: \operatorname{Spec} F \to X_0$, the image of the corresponding homomorphism $V(X_0) \to V(F)$ is contained in the image of $V(F_0)$.*

Choose a rational point $\operatorname{Spec} F_0 \to X_0$ and apply (2.3.1) to x and $y: \operatorname{Spec} F \to \operatorname{Spec} F_0 \to X_0$.

COROLLARY 2.3.3. *Suppose, in addition, that V commutes with limits:*

$$V(\operatorname{Spec} \varinjlim A_i) = \varinjlim V(\operatorname{Spec} A_i).$$

Then $V(F) = V(F_0)$ for any extension F/F_0 of algebraically closed fields.

F may be written as $\varinjlim A$ where A runs through all finitely generated F_0-subalgebras of F. Our conditions imply that $V(F) = \varinjlim V(\operatorname{Spec} A)$. The homomorphism $V(\operatorname{Spec} A) \to V(F)$ is induced by a F_0-point $\operatorname{Spec} F \to \operatorname{Spec} A$, corresponding to the imbedding $A \hookrightarrow F$. In view of (2.3.2) the image of this homomorphism is contained in the image of $V(F_0)$. Since this is true for any A we deduce that $V(F_0) \to V(F)$ is surjective. The injectivity of this map is trivial.

The present proof of (2.3), which is a slight modification of the original proof of the author [**31**], is due to Gabber, Gillet, and Thomason. The use of relative Picard groups instead of absolute ones allowed these authors to prove the following important generalization of (2.3).

PROPOSITION 2.4. *Let O be a henselian ring with field of fractions F and residue field k and let $X/\operatorname{Spec} O$ be a smooth affine curve. Further, let x, y: $\operatorname{Spec} O \to X$ be two sections that coincide in the closed point of $\operatorname{Spec} O$. Suppose, in addition, that*

(a) $nV(X) = 0$, *where* $(n, \operatorname{char} k) \doteq 1$,
(b) $V(O) \hookrightarrow V(F)$.

Then the induced maps $x^*, y^*: V(X) \to V(O)$ coincide.

Choose a projective closure $\overline{X}$ of X and set $X_\infty = \overline{X} - X$. The sections x, y define relative divisors D_x, D_y on $\overline{X}$ (relative to X_∞). Their difference is divisible by n in $\operatorname{Pic}(\overline{X}, X_\infty)$ since $\operatorname{Pic}(\overline{X}, X_\infty)/n \hookrightarrow H^2_{\text{et}}(X, j_!(\mu_n))$ (where $j: X \hookrightarrow \overline{X}$) and $H^2_{\text{et}}(X, j_!(\mu_n)) = H^2_{\text{et}}(X_0, (j_0)_!(\mu_n))$, where X_0 is the closed fiber of X, in view of the proper base change theorem in etale cohomology. In view of condition (b) it is sufficient to prove the coincidence of maps $V(X_F) \rightrightarrows V(F)$, where X_F is the generic fiber of X. The last fact follows in the same manner as in the proof of (2.3), since the difference of the corresponding points is divisible by n in the relative Picard group of X_F.

REMARK 2.5. Condition (b) of Proposition 2.4 is often satisfied in algebraic K-theory in view of Quillen's theorem [**24**]. Moreover, this condition may be avoided in many cases of interest.

Using induction and tricks similar to those used in the proof of (2.3.3) we deduce from (2.4) the following important

THEOREM 2.6 (GABBER (UNPUBLISHED), GILLET AND THOMASON [**10**]). *Let V/F be a smooth variety and let $v \in V$ be a rational point. Denote by O_v^h the henselization of a local ring O_v. For any m prime to $\operatorname{char} F$, the natural homomorphism $K_*(O_v^h, Z/m) \to K_*(F, Z/m)$ is bijective.*

Denote by I_v^h the maximal ideal of the local ring O_v^h. It is not difficult to deduce from (2.6) that $H_i(\operatorname{GL}(O_v^h, I_v^h), Z/m) = 0$ $(i \geq 1)$; see [**32**]. Consider now the simplicial scheme BGL_n/F and denote by $X_{n,i}^h$ the henselization of $(\operatorname{BGL}_n)_i = (\operatorname{GL}_n)^i$ in unity; denote further by $O_{n,i}^h$ the coordinate ring of $X_{n,i}^h$ and by $I_{n,i}^h$ its maximal ideal. Since face and degeneracy maps of BGL_n respect unity, we see that $X^h_{n,i}$ also form a simplicial scheme. The evident maps $X^h_{n,i} \to \mathrm{GL}^i_n \xrightarrow{\mathrm{pr}_k} \mathrm{GL}_n$ define matrices $a_k \in \mathrm{GL}_n(O^h_{n,i}, I^h_{n,i})$. We will denote by $u_{n,i}$ the chain $[a_1, \ldots, a_i] \in C_i(\mathrm{GL}_n(O^h_{n,i}, I^h_{n,i}), Z/m)$. Now one constructs, using induction on i and the fact that $H_i(\mathrm{GL}(O^h_{n,i}, I^h_{n,i}), Z/m) = 0$, chains $c_{n,i} \in C_{i+1}(\mathrm{GL}(O^h_{n,i}, I^h_{n,i}), Z/m)$ such that

$$d(c_{n,i}) = u_{n,i} - \sum_{j=0}^{i} (-1)^j (d_j)^* (c_{n,i-1}).$$

These considerations enable one to generalize (2.6):

COROLLARY 2.7. *Let (R, I) be a henselian pair, where R is an F-algebra. Then $\tilde{H}_*(\mathrm{GL}(R, I), Z/m) = 0$ and $K_*(R, Z/m) \xrightarrow{\sim} K_*(R/I, Z/m)$.*

The second statement follows from the first one. For the proof of the first statement it is sufficient to show that the imbedding $\tilde{C}_*(\mathrm{GL}_n(R, I), Z/m) \hookrightarrow \tilde{C}_*(\mathrm{GL}(R, I), Z/m)$ is nul-homotopic. Consider matrices $b_1, \ldots, b_i \in \mathrm{GL}_n(R, I)$. These matrices define a morphism $\operatorname{Spec} R \to \mathrm{GL}^i_n$, taking $\operatorname{Spec} R/I$ to unity. Since (R, I) is a henselian pair, this morphism factors uniquely through a morphism $f_b \colon \operatorname{Spec} R \to X^h_{n,i}$. The desired nul-homotopy may be defined now by a formula $s([b_1, \ldots, b_i]) = (f_b)^*(c_{n,i})$.

The same method of evaluation of "universal homotopy operators" $c_{n,i}$ may be applied also in many other situations. The following results are proved in [**32**].

THEOREM 2.8. *Let R be a henselian discrete valuation ring with maximal ideal I, fraction field F, and residue field k. For any m prime to* $\operatorname{char} F$ *we have canonical isomorphisms of pro-groups*:

$$H_*(\mathrm{GL}(R), Z/m) \to \{H_*(\mathrm{GL}(R/I^n), Z/m)\}_n,$$
$$K_*(R, Z/m) \to \{K_*(R/I^n), Z/m)\}_n.$$

To deduce the second statement from the first one it is necessary to use a version of Hurewitz's theorem for pro-spaces, proved by Panin [**43**].

COROLLARY 2.8.1. *In conditions of Theorem* 2.8,

$$K_*(R, Z/m) \xrightarrow{\sim} K_*(k, Z/m)$$

provided that $(m, \operatorname{char} k) = 1$.

COROLLARY 2.8.2. *Let k be an algebraically closed field of positive characteristics p and let F be the algebraic closure of the fraction field of the ring of Witt vectors over k. Then for any m prime to p there are canonical isomorphisms $K_*(k, Z/m) = K_*(F, Z/m)$.*

This corollary together with Theorem 2.2 shows that the groups $K_i(F, Z/m)$ do not depend on the algebraically closed field F (provided that m is prime to $\operatorname{char} F$); this enables us to finish the proof of the Quillen-Lichtenbaum conjecture for fields of zero characteristics. It is more natural, however, to apply the method of universal homotopy operators to the proof of the following theorem.

THEOREM 2.9 [**32**]. *Let F denote either the field* $\mathbf{R}$ *of real numbers or the field* $\mathbf{C}$ *of complex numbers. The natural morphism* $\mathrm{BGL}(F)^+ \to \mathrm{BGL}(F)^{\mathrm{top}}$ *induces isomorphisms on homology and homotopy groups with finite coefficients.*

Using, in addition, the Stability Theorem 4.6 (see below) we get the following result, confirming partially the isomorphism conjecture of Friedlander-Milnor [**21**].

COROLLARY 2.9.1. $\mathrm{BGL}_n(F) \to \mathrm{BGL}_n(F)^{\mathrm{top}}$ *induce isomorphisms on* $H_i(-, Z/m)$ *with* $i \leq n$.

COROLLARY 2.9.2. *Modulo uniquely divisible groups, the K-theory of the fields* $\mathbf{R}$ *and* $\mathbf{C}$ *is as displayed in the following table.*

$i \bmod 8$	0	1	2	3	4	5	6	7
$K_i(\mathbf{R})$	0	$Z/2$	$Z/2$	Q/Z	0	0	0	Q/Z
$\downarrow$	0	incl.	0	2	0	0	0	iso
$K_i(\mathbf{C})$	0	Q/Z	0	Q/Z	0	Q/Z	0	Q/Z

REMARK 2.10. A different and more algebraic approach to the proof of the Quillen-Lichtenbaum conjecture was proposed later by Jardine [**42**]. His method is also based on the use of (2.6).

3. The K-theory of division algebras. The K_2-theory of division algebras was already used above in the proof of (1.4). The result that was necessary there looks as follows. If D/F is a central simple algebra and E/F is its splitting field of finite degree, then we can consider the canonical homomorphism $g_E\colon K_2(E) = K_2(D_E) \xrightarrow{N_{E/F}} K_2(D)$.

THEOREM 3.1 [**17**]. *If index of algebra D is squarefree, then $K_2(D)$ is generated by images of $K_2(E)$ over all finite splitting fields E/F.*

REMARK 3.1.1. It seems possible that the restriction on index is not really necessary for the validity of (3.1). This is a rather interesting problem. The same question may be asked for higher K-groups.

Let X/F be the Severi-Brauer variety corresponding to D. In view of (1.10) $H^0(X, K_2) = K_2(F)$ and we get a canonical homomorphism $\mathrm{Nrd}\colon K_2(D) \to K_2(X) \to H^0(X, K_2) = K_2(F)$.

THEOREM 3.2 [**17**]. *Let D be an algebra of squarefree degree over a field F.*
(a) *If c.d.$F = 2$, then* $\mathrm{Nrd}\colon K_2(D) \xrightarrow{\sim} K_2(F)$.

(b) *If F is a global field, then there is an exact sequence* $0 \to K_2(D) \xrightarrow{\mathrm{Nrd}} K_2(F) \to \coprod_v Z/2 \to 0$ *where v runs through real points of F in which D is nontrivial.*

Theorem 3.2 follows easily from (3.1) and Hilbert's Theorem 90 for K_2. I am sure that the assumption about index is superfluous for its validity. The following important result of Merkurjev [**18**] is much deeper.

THEOREM 3.3. *For any quaternion algebra D/F, the reduced norm* Nrd: $K_2(D) \to K_2(F)$ *is injective.*[1]

For the proof of (3.3) Merkurjev uses his method of universal problems (cf. §1). In this case the field of definition of the universal problem is the function field on the product of three-dimensional quadrics. To prove the analog of (1.2) it is necessary to compute certain D-operator K-cohomology groups of these quadrics. The first step in this direction is provided by the theorem of Swan [**36**], which computes the operator K-theory of an arbitrary quadric. Further, one has to study the differentials in the BGQ-spectral sequence. The theory of Chern classes does not help this time, but Merkurjev has invented a direct method to prove the vanishing of the necessary differentials.

REMARK 3.4. (a) Merkurjev has given also the description of the image of reduced norm.

(b) It is reasonable to expect that injectivity of reduced norm holds for any algebra of squarefree degree, but at present I do not see how to attack this problem.

In the local case, to compute the K-theory of a division algebra one can use a version of methods of the previous section.

THEOREM 3.4 [**35**]. *Let R be a henselian discrete valuation ring with fraction field F and let D be a division algebra over F. Denote by A the maximal order in D and by I its maximal ideal. For any m prime to* char F *there is a canonical isomorphism of pro-groups $K_i(A, Z/m) \to \{K_i(A/I^n, Z/m)\}_n$.*

COROLLARY 3.4.1. *If m is prime to* char $\overline{R}$, *then*

$$K_i(A, Z/m) \xrightarrow{\sim} K_i(A/I, Z/m).$$

COROLLARY 3.4.2. *Let F be a usual local field (i.e., a finite extension of the field of p-adic numbers) and D/F a division algebra of degree prime to p. For all $i \geq 1$ there are canonical isomorphisms* Nrd: $K_i(D) \xrightarrow{\sim} K_i(F)$.

COROLLARY 3.4.3 [**11**]. *In conditions of* (3.4.2), Nrd: $K_2(D) \xrightarrow{\sim} K_2(F)$ *for any division algebra D.*

Acting as in the proof of (2.9) we get, moreover,

PROPOSITION 3.5. *Denote by H the classical quaternion algebra over* $\mathbf{R}$. *The natural map* $\mathrm{BGL}(H)^+ \to \mathrm{BGL}(H)^{\mathrm{top}}$ *induce isomorphisms on homology and homotopy groups with finite coefficients.*

4. Milnor K-theory. For any field F, its Milnor ring $K_*^M(F)$ is defined as a quotient ring of the tensor algebra $T(F^*)$ by a homogeneous ideal, generated by tensors $a \otimes (1-a) \in T_2(F^*) = F^* \otimes F^*$ [**2**]. The image of $a_1 \otimes \cdots \otimes a_n$ in $K_n^M(F)$ will be denoted $\{a_1, \ldots, a_n\}$. There is a canonical ring homomorphism $K_*^M(F) \to K_*(F)$, which is isomorphic in degrees ≤ 2. The example of finite

[1] *Remark added in proof*: Theorem 3.3 was proved independently by M. Rost [**47**].

fields shows that in degrees $n \geq 3$ the map $K_n^M(F) \to K_n(F)$ is not in general surjective; however, I know of no examples where this map is not injective (cf. (4.7.1) below).[2] The following conjecture is principal in the understanding of the structure of $K_*(F)$.

CONJECTURE 4.1. *Denote by F_0 the subfield of constants in F. The ring $K_*(F)$ is generated by $K_1(F) = F^*$ and $K_*(F_0)$.*[3]

In positive characteristics, (4.1) would imply that the homomorphism $K_*^M(F) \to K_*(F)$ is an isomorphism modulo torsion.

The following conjecture is a particular case of the general conjectures of Beilinson (see §7).

CONJECTURE 4.2. *For any field F and any n prime to* $\operatorname{char} F$, *the norm-residue homomorphism $K_*^M(F)/n \to \coprod_{i\geq 0} H^i(F, \mu_n^{\otimes i})\colon \{a_1, \ldots, a_i\} \mapsto \chi_n(a_1) \cup \cdots \cup \chi_n(a_i)$ is an isomorphism of rings.*

Another interesting conjecture concerning $K_*^M(F)$ is Milnor's conjecture about quadratic forms [20]. Suppose that $\operatorname{char} F \neq 2$ and denote by $W(F)$ the Witt ring of nondegenerate quadratic forms over F. Let $I(F)$ denote the maximal ideal of $W(F)$, consisting of even-dimensional forms. For any $a \in F^*$ set $\langle\langle a\rangle\rangle = 1 \perp -a \in I(F)$ and $\langle\langle a_1, \ldots, a_n\rangle\rangle = \langle\langle a_1\rangle\rangle \cdot \cdots \cdot \langle\langle a_n\rangle\rangle$. $I(F)$ is additively generated by $\langle\langle a\rangle\rangle$ and hence $I^n(F)$ is additively generated by n-fold Pfister forms $\langle\langle a_1, \ldots, a_n\rangle\rangle$. It is easy to see that the discriminant defines an isomorphism $I(F)/I^2(F) \xrightarrow{\sim} F^*/F^{*2}$. Since the 2-fold Pfister form $\langle\langle a, 1-a\rangle\rangle$ is trivial we have a well-defined ring homomorphism

$$K_*^M(F)/2 \to \coprod_{n\geq 0} I^n(F)/I^{n+1}(F)\colon \{a_1, \ldots, a_n\} \mapsto \langle\langle a_1, \ldots, a_n\rangle\rangle \bmod I^{n+1}(F),$$

which is surjective by the remarks above.

CONJECTURE 4.3 (MILNOR [20]). $K_n^M(F)/2 \xrightarrow{\sim} I^n(F)/I^{n+1}(F)$.

For $n \leq 2$ this conjecture was verified in [20].

Among the general properties of Milnor K-groups we will note the following:

(4.4) Let O be a discrete valuation ring with valuation v, fraction field F, and residue field k. Then there are canonical homomorphisms $\partial\colon K_n^M(F) \to K_{n-1}^M(k)$, which are completely characterized by the formula

$$\partial(\{x_1, \ldots, x_n\}) = v(x_1) \cdot \{\bar{x}_2, \ldots, \bar{x}_n\} \qquad (x_2, \ldots, x_n \in O^*)$$

[2].

REMARK 4.4.1. The following diagram commutes:

$$\begin{array}{ccc} K_n^M(F) & \to & K_n(F) \\ \partial \downarrow & & \downarrow \partial \\ K_{n-1}^M(k) & \to & K_{n-1}(k) \end{array}$$

[2] *Remark added in proof*: Ch. Wiebel pointed out to me that the map $K_n^M(Q) \to K_n(Q)$ is not injective for $n \geq 4$.

[3] *Remark added in proof*: The formulation of this conjecture should be modified; in the present form it is easily seen to be false even in degree 4.

(4.5) *Transfer* (Bass and Tate [**2**], Kato [**12**]). If E/F is a finite extension, then it is possible to define canonical homomorphisms $N_{E/F}: K_n^M(E) \to K_n^M(F)$, which satisfy the usual properties of transfer and are completely characterized by the following reciprocity formula:

(4.5.1) If C/F is a complete regular curve, then for any $u \in K_{n+1}^M(C)$ one has $\sum_{x\in C} N_{F(x)/F}(\partial_x(u)) = 0$.

REMARK 4.5.2. The following diagram commutes:

$$\begin{array}{ccc} K_n^M(E) & \xrightarrow{N_{E/F}} & K_n^M(F) \\ \downarrow & & \downarrow \\ K_n(E) & \xrightarrow{N_{E/F}} & K_n(F) \end{array}$$

Milnor K-theory is closely related to homology properties of GL_n.

THEOREM 4.6 [**29**]. *Let F be an infinite field. Then the homomorphisms $H_n(\mathrm{GL}_n(F), Z) \to H_n(\mathrm{GL}_{n+1}(F), Z) \to \cdots \to H_n(\mathrm{GL}(F), Z)$ are isomorphisms. Moreover, the homology product*

$$F^* \otimes \cdots \otimes F^* = H_1(\mathrm{GL}_1(F)) \otimes \cdots \otimes H_1(\mathrm{GL}_1(F)) \to H_n(\mathrm{GL}_n(F))$$

defines an isomorphism

$$K_n^M(F) \to H_n(\mathrm{GL}_n(F))/H_n(\mathrm{GL}_{n-1}(F)) = H_n(\mathrm{GL}(F))/H_n(\mathrm{GL}_{n-1}(F)).$$

The last theorem provides a homomorphism

$$\begin{aligned} f: K_n(F) = \pi_n(\mathrm{BGL}(F)^+) &\to H_n(\mathrm{BGL}(F)^+) \\ &= H_n(\mathrm{GL}(F)) \to H_n(\mathrm{GL}(F))/H_n(\mathrm{GL}_{n-1}(F)) = K_n^M(F). \end{aligned}$$

PROPOSITION 4.5 [**29**]. (a) *The composition $K_n^M(F) \to K_n(F) \to K_n^M(F)$ coincides with multiplication by $(-1)^{n-1}(n-1)!$*

(b) *The composition $K_n(F) \to K_n^M(F) \to K_n(F)$ coincides with Chern class $c_{n,n}$.*

COROLLARY 4.7.1. *The kernel of the homomorphism $K_n^M \to K_n(F)$ is annihilated by $(n-1)!$.*

COROLLARY 4.7.2. *Suppose that O is a discrete valuation ring with fraction field F and residue field k. Then the diagram*

$$\begin{array}{ccc} K_3(F) & \xrightarrow{f} & K_3^M(F) \\ \partial\downarrow & & \downarrow\partial \\ K_2(F) & \xrightarrow{2} & K_2(F) \end{array}$$

commutes.

In connection with Milnor's conjecture (4.3) we will mention also Proposition 4.8.

PROPOSITION 4.8 [29]. *The image of the homomorphism* $f\colon K_3(F) \to K_3^M(F)$ *coincides with the kernel of Milnor's homomorphism*

$$K_3^M(F) \to I^3(F)/I^4(F).$$

5. K_3 and Bloch's group. For any field F denote by $D(F)$ the free abelian group with basis $[x]$ ($x \in F^* - 1$) and by $r\colon D(F) \to F^* \otimes F^*$ the homomorphism $[x] \mapsto x \otimes (1-x)$. There is an involution s on $F^* \otimes F^*$, given by $s(a \otimes b) = -(b \otimes a)$. It is easy to verify that the induced homomorphism $D(F) \to (F^* \otimes F^*)_s$ is trivial on elements of the form

$$[x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \left[\frac{1-x}{1-y}\right] \qquad (x \neq y \in F^* - 1).$$

We will denote by $T(F)$ the factor-group of $D(F)$ by the subgroup, generated by the above elements. The kernel of the induced homomorphism $T(F) \to (F^* \otimes F^*)_s$ is denoted $B(F)$ and is called the Bloch's group of F. Thus we have an exact sequence $0 \to B(F) \to T(F) \to (F^* \otimes F^*)_s \to K_2(F) \to 0$. For $x \neq 1$ put $\langle x \rangle = [x] + [x^{-1}]$; put also $\langle 1 \rangle = 0$.

LEMMA 5.1. (a) $x \mapsto \langle x \rangle$ *defines a homomorphism* $F^* \to {}_2T(F)$; *in particular,* $\langle x^2 \rangle = 0$.

(b) *The element* $c = [x] + [1-x] \in T(F)$ *does not depend on the choice of* $x \in F^* - 1$.

(c) $3c = \langle -1 \rangle$.

(d) *If equation* $x^2 + 1 = 0$ *has solutions in* F, *then* $3c = 0$; *if equation* $x^2 - x + 1 = 0$ *has solutions in* F, *then* $2c = 0$.

The group $B(F)$ has the following relation to $K_3(F)$. Denote by $\mathrm{GM}(F)$ the subgroup of $\mathrm{GL}(F)$, consisting of monomial matrices. This group is quasiperfect, so one can apply to $\mathrm{BGM}(F)$ Quillen's plus-construction. The homotopy groups of $\mathrm{BGM}(F)^+$ coincide in view of the Barrat-Priddy-Quillen theorem with stable homotopy groups of BF^* and hence are more or less understandable. The imbedding $\mathrm{GM}(F) \hookrightarrow \mathrm{GL}(F)$ induces a map $\mathrm{BGM}(F)^+ \to \mathrm{BGL}(F)^+$ and hence homomorphisms $\pi_i^s(BF^*) = \pi_i(\mathrm{BGM}(F)^+) \to K_i(F)$. These homomorphisms are surjective in dimensions ≤ 2.

THEOREM 5.2 [33]. *If the field* F *is infinite, then*

$$\operatorname{coker}(\pi_3(\mathrm{BGM}(F)^+) \to K_3(F)) = B(F)/2c.$$

The proof is done by means of homological methods. One proves first of all that

$$\operatorname{coker}(\pi_3(\mathrm{BGM}(F)^+) \to K_3(F)) = \operatorname{coker}(H_3(\mathrm{GM}(F)) \to H_3(\mathrm{GL}(F))).$$

Next one computes $H_3(\mathrm{GL}_2(F))/H_3(\mathrm{GM}_2(F))$. This step is very close to the proof of Bloch's theorem [6]. Consider the complex $C_*(F)$ with $C_i(F)$ equal to the free abelian group, generated by $(i+1)$-tuples $(x_0, \ldots, x_i)$ of distinct points of $\mathbf{P}^1(F)$. It is easy to see that all homology groups of this complex are zero, except for H_0, which is equal to Z. The natural action of $\mathrm{GL}_2(F)$ on $C_*(F)$ gives rise to a spectral sequence $H_p(\mathrm{GL}_2(F), C_q(F)) \Rightarrow H_{p+q}(\mathrm{GL}_2(F), Z)$. The action of $\mathrm{GL}_2(F)$ on the basis of $C_i(F)$ is transitive for $i = 0, 1, 2$ and the stabilizers of (0), $(0, \infty)$, $(0, \infty, 1)$ are correspondingly equal to $B_2 = \left(\begin{smallmatrix} F^* & * \\ 0 & F^* \end{smallmatrix}\right)$, $T_2 = \left(\begin{smallmatrix} F^* & 0 \\ 0 & F^* \end{smallmatrix}\right)$, and F^*. Thus the term E^1 of the spectral sequence looks as follows:

$$H_*(B_2) \quad H_*(T_2) \quad H_*(F^*) \quad \coprod^{*}_{x \in \mathbf{P}^1(F) - \{0, \infty, 1\}} Z[x] \quad \coprod^{*}_{x \neq y \in \mathbf{P}^1(F) - \{0, \infty, 1\}} Z[x, y]$$

where $[x]$ (resp. $[x, y]$) is the orbit of $(0, \infty, 1, x)$ (resp. $(0, \infty, 1, x, y)$). Using the fact that $H_*(B_2) = H_*(T_2)$, one computes easily the differential d_1. The interesting E_2-terms look as follows (s is the involution induced in $H_*(T_2)$ by the permutation of factors):

$$\begin{array}{llll}
H_3(T_2)_s & & & \\
H_2(T_2)_s = H_2(F^*) \oplus (F^* \otimes F^*)_s & (F^* \otimes F^*)^s & 0 & \\
F^* & 0 & 0 & \\
Z & 0 & 0 & T(F)
\end{array}$$

The only nontrivial differential starting at $T(F)$ is the differential $d_3 \colon T(F) \to H_2(F^*) \oplus (F^* \otimes F^*)_s$, which is given by the formula $d_3([x]) = x \wedge (1 - x) - x \otimes (1 - x) \in \Lambda^2(F^*) + (F^* \otimes F^*)_s$. It is clear that $E^\infty_{0,3} = \ker d_3 = B(F)$. Our spectral sequence defines a filtration on $H_3(\mathrm{GL}_2(F))$. We have also a filtration on $H_3(\mathrm{GM}_2(F))$ arising from the Hochschild-Serre spectral sequence, corresponding to the extension $1 \to T_2 \to \mathrm{GM}_2(F) \to S_2 \to 1$. In both cases the zero's term of filtration coincides with the image of $H_3(T_2)$. Thus the homomorphism $H_3(\mathrm{GM}_2(F)) \to H_3(\mathrm{GL}_2(F))$ takes $H_3(\mathrm{GM}_2(F))^0$ onto $H_3(\mathrm{GL}_2(F))^0$. Comparing the $E^2_{2,1}$-terms of the spectral sequences under consideration, one shows easily that $H_3(\mathrm{GM}_2(F))^1$ is mapped onto $H_3(\mathrm{GL}(F))^1$. Finally $H_3(\mathrm{GM}_2(F)) = H_3(\mathrm{GM}_2(F))^1 + H_3(S_2)$ and the image of $H_3(S_2)$ in $H_3(\mathrm{GL}_2(F))$ is clearly contained in $H_3(T_2)$. Thus

$$H_3(\mathrm{GL}_2(F))/H_3(\mathrm{GM}_2(F)) = H_3(\mathrm{GL}_2(F))/H_3(\mathrm{GL}_2(F))^1 = E^\infty_{0,3} = B(F).$$

Next one checks that the kernel of $H_3(\mathrm{GL}_2(F)) \to H_3(\mathrm{GL}_3(F))$ is contained in the image of $H_3(\mathrm{GM}_2)$. After that one has only to compute the intersection of $H_3(\mathrm{GL}_2(F))$ and $H_3(\mathrm{GM}(F))$ in $H_3(\mathrm{GL}(F))$. In view of the isomorphism $H_3(\mathrm{GL}(F))/H_3(\mathrm{GL}_2(F)) = K_3^M(F)$, this is equivalent to the computation of the kernel of $H_3(\mathrm{GM}(F)) \to K_3^M(F)$. The answer is as follows: consider on $H_3(\mathrm{GM}(F))$ the filtration, arising from the Hochschild-Serre spectral sequence; then $H_3(\mathrm{GM}(F))^2 \cap H_3(\mathrm{GL}_2(F)) = H_3(\mathrm{GM}_2(F))$. Since $H_3(\mathrm{GM}(F)) = H_3(\mathrm{GM}(F))^2 + H_3(S)$ we deduce that

$$H_3(\mathrm{GL}(F))/H_3(\mathrm{GM}(F)) = B(F)/\operatorname{Im}(H_3(S))$$

and it is sufficient to check now that $\operatorname{Im}(H_3(S)) = 2c$.

To apply Theorem 5.2 it is necessary to know the group $\pi_3(B\mathrm{GM}(F)^+)$ and its image in $K_3(F)$. Using the spectral sequence $H_i(F^*, \pi_j^s(\mathrm{pt})) \Rightarrow \pi_{i+j}^s(BF^*)$ one easily proves Proposition 5.3.

PROPOSITION 5.3. *Denote by* $\overline{K}_3^M(F)$ *the image of* $K_3^M(F)$ *in* $K_3(F)$ *and by* G (*resp.* $G\mu$) *the subgroup of* GM, *consisting of monomial matrices with entries* ± 1 (*resp. with entries from the group* μ *of roots of unity*).

(a) $\mathrm{Im}(\pi_3(\mathrm{BGM}(F)^+) \to K_3(F)) = \overline{K}_3^M(F) + \mathrm{Im}((\pi_3(BG\mu)^+) \to K_3(F))$.

(b) *There is a canonical surjective homomorphism* $\mathrm{Tor}(\mu,\mu) = \mathrm{Tor}(F^*,F^*) \to \mathrm{Im}(\pi_3(\mathrm{BGM}(F)^+))/K_3^M(F) + \mathrm{Im}(\pi_3(BG^+))$.[4]

COROLLARY 5.3.1. *The group* $B(F)$ *does not change under purely transcendental extensions.*

COROLLARY 5.3.2. *Denote by* F_0 *the subfield of constants in* F. *Then* $K_3(F)/K_3(F_0) + K_3^M(F) = B(F)/B(F_0)$.

In the case of K_3, Conjecture 4.1 may be specified as follows:

CONJECTURE 5.4. $B(F) = B(F_0)$.

We will need below two slightly different descriptions of $B(F)$.

(5.5) Denote by $\overline{F^* \otimes F^*}$ the factor group of $F^* \otimes F^*$ by the subgroup generated by elements $a \otimes (-a)$, and by $T'(F)$ the factorgroup of $T(F)$ by the subgroup generated by elements $\langle a\rangle$. Since $r(\langle a\rangle) = a \otimes (-a)$ we get an induced homomorphism $T'(F) \to \overline{F^* \otimes F^*}$, whose kernel we will denote by $B'(F)$.

LEMMA 5.5.1. $B'(F) = B(F)/\langle -1\rangle$.

(5.6) One can describe $B(F)$ equally in terms of relations on $a\otimes(1-a)$ directly in $F^* \otimes F^*$. It is easy to verify that

$$r\left([x]-[y]+\left[\frac{y}{x}\right]-\left[\frac{1-x^{-1}}{1-y^{-1}}\right]+\left[\frac{1-x}{1-y}\right]\right)$$
$$= x \otimes \frac{1-x}{1-y} + \frac{1-x}{1-y} \otimes x = r\left(\left\langle x\frac{1-x}{1-y}\right\rangle - \langle x\rangle - \left\langle \frac{1-x}{1-y}\right\rangle\right).$$

Thus the kernel of r contains elements

$$[x]-[y]+\left[\frac{y}{x}\right]-\left[\frac{1-x^{-1}}{1-y^{-1}}\right]+\left[\frac{1-x}{1-y}\right]$$
$$-\left\langle x\frac{1-x}{1-y}\right\rangle + \langle x\rangle + \left\langle\frac{1-x}{1-y}\right\rangle_{(x\neq y\neq F^*-1)},$$
$$\langle xyz\rangle - \langle xy\rangle - \langle xz\rangle - \langle yz\rangle + \langle x\rangle + \langle y\rangle + \langle z\rangle,$$
$$\langle x^2\rangle - 4\langle x\rangle$$

(where, as always, $\langle 1\rangle = 0$). Denote by $T''(F)$ the factor group of $D(F)$ by the above elements, and by $B''(F)$ the kernel of the homomorphism $T''(F) \to F^* \otimes F^*$.

[4] *Remark added in proof*: More precisely, the relation between $K_3(F)$ and $B(F)$ is given by the exact sequence $0 \to \mathrm{Tor}(F^*,F^*)^\sim \to K_3(F)_{\mathrm{ind}} \to B(F) \to 0$ where $K_3(F)_{\mathrm{ind}} = K_3(F)/K_3^M(F)$ and $\mathrm{Tor}(F^*,F^*)^\sim$ is the unique nontrivial extension of $Z/2$ by means of $\mathrm{Tor}(F^*,F^*)$.

LEMMA 5.6.1. $B''(F) = B(F)$.

6. Divisibility in Bloch's group.[5] All fields considered in this section are supposed to contain an algebraically closed subfield.

Suppose that F is a discretely valuated field with valuation ring O and residue field k. Choose a local parameter π. This choice defines a homomorphism $s_\pi: F^* \to k^*: x \mapsto \overline{x/\pi^{v(x)}}$ and induced homomorphisms $F^* \otimes F^* \to k^* \otimes k^*$, $\overline{F^* \otimes F^*} \to \overline{k^* \otimes k^*}$. We will take $\overline{x} = \infty$ for $x \notin O$ and we will define elements $[0], [\infty], [1] \in T'(k)$ as zero.

LEMMA 6.2. *$[x] \mapsto [\overline{x}]$ defines a homomorphism $T'(F) \xrightarrow{s} T'(k)$. Moreover, the following diagram commutes*:

$$\begin{array}{ccc} T'(F) & \to & \overline{F^* \otimes F^*} \\ {\scriptstyle s}\downarrow & & {\scriptstyle s_\pi}\downarrow \\ T'(k) & \to & \overline{k^* \otimes k^*} \end{array}$$

Hence s takes $B'(F)$ to $B'(k)$.

If E/F is a finite extension, then $N_{E/F}: K_3(E) \to K_3(F)$ defines in view of (5.3) and (4.5) the transfer $N_{E/F}: B(E) \to B(F)$. A slight modification of the proof of (2.3) now gives

PROPOSITION 6.2. *Let C be a smooth connected curve over an algebraically closed field F. For any two points $x, y \in C$ the specialization homomorphisms $s_x, s_y: B(F(C)) \to B(F)$ coincide on B/n and ${}_nB$.*

THEOREM 6.3. *If F is algebraically closed, then $B(F)$ is uniquely divisible.*

Since $\overline{F^* \otimes F^*}$ and $K_2(F)$ are uniquely divisible, it is sufficient to prove the unique divisibility of $T'(F) = T(F)$. The divisibility of $T(F)$ was proved in [**6**]. It follows from the formulae $[x^p] = p(\sum_{\xi \in \mu_p} [\xi x])$ $(p \neq \operatorname{char} F)$, $[x^p] = p^2[x]$ $(p = \operatorname{char} F)$, which are valid for any field F. To prove these formulae consider the element $[t^p] - p(\sum_{\xi \in \mu_p} [\xi t]) \in B(F(t))$. Since $B(F(t)) = B(F)$ this element coincides with any of its specializations. But specializing at zero we get zero. Now, to prove the unique divisibility, we will define a homomorphism $T'(F) \to T'(F)$ inverse to multiplication by p by means of the formula $[x] \to \sum_{y^p = x} [y]$. We have to check that the defining relation on $[x]$ goes to zero, i.e., to check the formula (where $u, v, w \notin \mu_p \cup 0$ and $w^p = (1 - u^p)/(1 - v^p)$):

$$\sum_{\xi \in \mu_p} [\xi u] - \sum_{\xi \in \mu_p} [\xi v] + \sum_{\xi \in \mu_p} [\xi \cdot v/u] - \sum_{\xi \in \mu_p} [\xi \cdot wv/u] + \sum_{\xi \in \mu_p} [\xi \cdot w] = 0.$$

Note that this element lies in ${}_pB(F)$. Now fix w and consider the curve C, given by equation $(1 - V^p)w^p = 1 - U^p$. We can consider the universal element in

[5] *Remark added in proof*: A different and much more powerful approach to the study of divisibility in $K_3(F)_{\text{ind}}$ (and hence in $B(F)$) is developed in [**46**].

${}_pB(F(C))$ for which the element under consideration is a specialization. Specializing this element in the point $U = 1$, $V = 1$ we will get zero and it is sufficient to apply Proposition 6.2.

Suppose from now on that $\operatorname{char} F \neq 2$. Let E/F be a quadratic extension: $E = F(\alpha)$, $\alpha^2 = a \in F^*$. Denote by A the image of $E^* \otimes F^* \to E^* \otimes E^*$. One sees easily that $N_{E/F} \otimes \mathrm{id}: E^* \otimes F^* \to F^* \otimes F^*$ induce a homomorphism $N: A \to F^* \otimes F^*$. Let $T''(E/F)$ denote the inverse image of A in $T''(E)$. It is not difficult to find generators and relations for this group. The important point is that relations are of "rational character" (i.e., may be parametrized by means of F-rational varieties).

PROPOSITION 6.4. *There exists a canonical homomorphism (given by rational formulae) $L_{E/F}: T''(E/F) \to T''(F)$, making the following diagram commutative*

$$\begin{array}{ccc} T''(E/F) & \to & A \\ {\scriptstyle L_{E/F}} \downarrow & & {\scriptstyle N} \downarrow \\ T''(F) & \to & F^* \otimes F^* \end{array}$$

and hence inducing the homomorphism $L_{E/F}: B(E) \to B(F)$.

$L_{E/F}$ is given explicitly on generators. To check that relations go to zero one remarks that the image of any relation is a rationally parametrized element of $B(F)$ and hence should be zero.

THEOREM 6.5. *Denote by t the generator of* $\mathrm{Gal}(E/F)$. *The following sequence is exact:* $B(E) \xrightarrow{1-t} B(E) \xrightarrow{L_{E/F}} B(F)$.

As in the proof of (1.5) we reduce, first of all, the general case to the case where $N_{E/F}: E^* \to F^*$ is surjective. In the present situation this is trivial: to make $b \in F^*$ a norm it is sufficient to pass to the function field on a conic C with equation $X^2 - aY^2 = b$. The field $E(C)$ is purely transcendental over E and hence $B(E(C)) = B(E)$. Thus

$$\ker L_{E/F}/(1-t)B(E) \hookrightarrow \ker L_{E(C)/F(C)}/(1-t)B(E(C)).$$

Supposing now that $N_{E/F}: E^* \to F^*$ is surjective we define a map $f: T''(F) \to T''(E/F)_t$ by means of the formula

$$f([x]) = [-z] + [(1+z^t)z/(1+z)] - [-(1+z^t)/(1+z)],$$

where $z \in E^*$ is such that $N_{E/F}(z) = x$ and $\mathrm{Tr}(z) \neq -2$. One verifies then that f and $L_{E/F}$ are mutually inverse. Set $M = \mathrm{Im}(T''(E/F) \to A)$. We get two short exact sequences of t-modules:

$$0 \to B(E) \to T''(E/F) \to M \to 0, \qquad 0 \to M \to A \to K_2(E) \to 0.$$

One computes easily the homology groups of $G = \mathrm{Gal}(E/F)$ with coefficients in A and $K_2(F)$ (in the second case using essentially the results of §1). This makes it possible to compute homology with coefficients in $M: H_i(G, M) = Z/2$. Now it is easy to check that $H_1(G, T''(E/F)) \to H_1(G, M)$ is surjective and hence $(1-t)T''(E/F) \cap B(E) = (1-t)B(E)$.

Applying (6.5) to the universal Kummer Extension we get

COROLLARY 6.6. *For any F (containing an algebraically closed subfield) of characteristics $\neq 2$ the group $B(F)$ does not have 2-torsion.*

THEOREM 6.7. *In conditions of* (6.6) *the group $B(F)$ is uniquely* 2*-divisible.*[6]

(6.7.1) Let C be a conic over F. If $B(F)$ is 2-divisible then $B(F(C))$ is also 2-divisible.

Let E be a quadratic extension of F splitting C. If $u \in B(F(C))$ then $2u = N_{E(C)/F(C)}(u_{E(C)}) = N_{E/F}(u_{E(C)})_{F(C)}$ (since $B(E(C)) = B(E)$). We can write $N_{E/F}(u_{E(C)}) = 4v$ and hence $u = 2v_{F(C)}$.

COROLLARY 6.7.2. *If F is a function field on a product of conics defined over an algebraically closed field, then $B(F)$ is* 2*-divisible.*

Using the description of the 2-torsion in K_2 one can easily prove the exactness of the sequence $0 \to B(F)/2 \to T'(F)/2 \to \Lambda^2(F^*/F^{*2})$. This makes it possible to write down the universal elements of $B/2$. Fields of definition of these universal elements are function fields on products of conics, so we deduce from (6.7.2) that these universal elements are zero. The specialization argument finishes the proof.

COROLLARY 6.8. *Let F be as above and let F_0 denote its subfield of constants. Then $K_3(F) = K_3(F_0) + K_3^M(F) + 2K_3(F)$.*

COROLLARY 6.9. *For F as above, $K_3^M(F)/2 \xrightarrow{\sim} I^3(F)/I^4(F)$.*

In fact, $\ker(K_3^M(F) \to I^3/I^4) = \operatorname{Im}(K_3(F) \to K_3^M(F))$ but the image of all three terms, which appear in (6.8), is clearly contained in $2K_3^M(F)$.

COROLLARY 6.10. *Let F be as above and let X/F be a smooth variety. Then the images of $(K_3(F(X)) \to \coprod_{\operatorname{codim} x=1} K_2(F(x)))$ and $(K_3^M(F(X)) \to \coprod_{\operatorname{codim} x=1} K_2(F(x)))$ coincide.*[7]

REMARK 6.11. It seems that (6.10) together with Merkurjev's Theorem 3.3 are sufficient to prove Hilbert's Theorem 90 for K_3^M (for quadratic extensions) and, in particular, to prove that $K_3^M(F)/2 = H^3(F, \mu_2)$; but we have not yet checked all the details.[8]

7. Higher Chow groups. To visualize the relations between K-theory and etale cohomology Beilinson [3] conjectured the existence of a certain "universal" cohomology theory on the category of schemes, which is directly related both to K-theory and to etale cohomology (this theory should be analogous to the

[6] *Remark added in proof*: The group $B(F)$ is uniquely divisible for any F, containing an algebraically closed subfield [46].

[7] *Remark added in proof*: Statements 6.9 and 6.10 are true for any field F [46].

[8] *Remark added in proof*: Hilbert's Theorem 90 for $K_3^M(F)/2 = H^3(F, \mu_2)$ are proved in [45, 48].

integral singular cohomology theory in topology). A closely related list of conjectures was proposed by Lichtenbaum [**15**]. According to Beilinson there should exist complexes of sheaves $\Gamma(i)$ on the big Zariski site, satisfying (among others) the following properties:

(a) $\Gamma(i) = 0$ for $i < 0$, $\Gamma(0) = Z$, $\Gamma(1) = O^*[-1]$.

(b) For $i \geq 1$ the complex $\Gamma(i)$ is acyclic outside $1, \dots, i$; for a smooth X, $H^i(\Gamma(i))$ coincides with the sheaf of Milnor groups K_i^M.

(c) For any n invertible on a "good" smooth X one has $\Gamma(i) \otimes^L Z/n = \tau_{\leq i} R\pi_* Z/n[i]$, where $\pi: X_{\text{et}} \to X_{\text{Zar}}$ is the canonical morphism.

(d) There exists a spectral sequence $H^i(X, \Gamma(j)) \Rightarrow K'_{2j-i}(X)$, which is split up to standard factorials by means of Chern classes. The resulting filtration on K'-theory coincides with γ-filtration.

Recently Bloch has constructed a theory which satisfies properties (a) and (d). There is no doubt that this is the expected theory, but it is very difficult to attack the remaining properties. We will work in the category of quasiprojective varieties over a field. Define the standard simplex Δ^n as a hyperplane in A^{n+1}, defined by the equation $t_0 + \cdots + t_n = 1$. Δ^n form a cosimplicial variety. For any variety X define $z^i(X, n)$ to be a subgroup in the group $Z^i(X \times \Delta^n)$, consisting of those cycles which properly intersect $X \times \Delta^m$ for any face $\Delta^m \subset \Delta^n$. $z^i(-,-)$, is clearly a complex (of degree -1) of sheaves in any reasonable topology; the expected complex $\Gamma(i)$ is obtained from $z^i(-,-)$ by reindexing. Bloch set $\mathrm{CH}^i(X, n)$ equal to the nth homology group of $z^i(X, -)$ (the group $\mathrm{CH}^i(X, 0)$ coincides evidently with Chow groups of cycles of codimension i modulo rational equivalence). The groups $\mathrm{CH}^i(X, n)$ are contravariant functors of X with respect to flat maps; one can define the inverse image on $\mathrm{CH}^i(X, n)$ with respect to arbitrary maps if one restricts to the subcategory of smooth varieties. The groups $\mathrm{CH}^i(X, n)$ are covariant functors of X with respect to proper maps. Bloch has also proved the following properties of $\mathrm{CH}^i(X, n)$:

(7.1) $\mathrm{CH}^i(X, n)$ are homotopy invariant: $\mathrm{CH}^i(X, n) = \mathrm{CH}^i(X \times \mathbf{A}^1, n)$.

(7.2) *Localization.* If $Y \subset X$ is a closed subvariety of pure codimension d, then there is an exact sequence $\mathrm{CH}^i(X - Y, n+1) \to \mathrm{CH}^{i-d}(Y, n) \to \mathrm{CH}^i(X, n) \to \mathrm{CH}^i(X - Y, n) \to \cdots \to \mathrm{CH}^i(X - Y, 0) \to 0$.

(7.3) *Products.* For any X, Y there are canonical pairings $\mathrm{CH}^i(X, n) \otimes \mathrm{CH}^j(Y, n) \to \mathrm{CH}^{i+j}(X \times Y, n + m)$. Combining these products with inverse image along the diagonal one gets on $\mathrm{CH}^*(X, *)$ (for X smooth) a structure of bigraded ring.

(7.4)

$$\mathrm{CH}^1(X, q) = \begin{cases} \operatorname{Pic} X, & q = 0, \\ \Gamma(X, \sigma_X^*), & q = 1, \\ 0, & q = 2. \end{cases}$$

(7.4) *Relations to K-theory.* Bloch shows that $\mathrm{CH}^*(X, *)$ satisfy the Gillet axioms [**44**] and hence there is a theory of Chern classes with values in $\mathrm{CH}^*(X, *)$.

He proves that these Chern classes define isomorphisms

$$\mathrm{CH}^i(X, n) \otimes Q = \mathrm{gr}^i K'_n(X) \otimes Q.$$

(7.5) Gersten conjecture is true for $\mathrm{CH}^i(X, n)$.

The spectral sequence relating higher Chow groups to K-theory was constructed earlier by Landsburg [**13**].

Consider the case of a field. It is clear that $\mathrm{CH}^i(\operatorname{Spec} F, n) = 0$ when $n < i$. One can easily verify also that $\mathrm{CH}^n(\operatorname{Spec} F, n) = K_n^M(F)$. These facts correspond exactly to the properties conjectured by Beilinson in (b) above. However, the remaining point of (b) means that $\mathrm{CH}^i(\operatorname{Spec} F, n) = 0$ when $n \geq 2i$. This seems to be an extremely difficult question for $i > 1$. Finally, we mention that the group $\mathrm{CH}^2(\operatorname{Spec} F, 3)$ coincides with $K_3(F)/K_3^M(F)$ and thus is very close to Bloch's group $B(F)$.

8. Etale K-theory. For any simplicial scheme X one can construct a certain pro-space X_{et}—its etale topological type [**1, 8**]. $X \mapsto X_{\mathrm{et}}$ is a functor from schemes to pro-spaces. The main property of X_{et} is that its fundamental group coincides with the fundamental group of X as defined by Grothendieck, and its cohomology groups with finite coefficients coincide with etale cohomology groups of X.

For a variety over C its etale K-theory may be defined as complex K-theory of the pro-space X_{et}. In the general case, one can proceed as follows [**7**]. Fix a prime integer l and denote $Z[l^{-1}]$ by R. We will consider only schemes over R. For any X one has morphisms of pro-spaces

$$X_{\mathrm{et}} \to (\operatorname{Spec} R)_{\mathrm{et}} \leftarrow (\mathrm{BGL}_n)_{\mathrm{et}}.$$

Consider now the space of relative l-adic functions [**7, 8**]

$$\mathrm{Hom}_l(X_{\mathrm{et}}, (\mathrm{BGL}_n)_{\mathrm{et}})_{R_{\mathrm{et}}}$$

and set

$$K_i^{\mathrm{et}}(X) = \varinjlim_n \pi_i(\mathrm{Hom}_l(X_{\mathrm{et}}, \mathrm{BGL}_n)_{\mathrm{et}})_{R_{\mathrm{et}}})$$

and

$$K_i^{\mathrm{et}}(X, Z/l^\nu) = \varinjlim_n \pi_i(\mathrm{Hom}(X_{\mathrm{et}}, (\mathrm{BGL}_n)_{\mathrm{et}})_{R_{\mathrm{et}}}, Z/l^\nu).$$

Etale K-theory is easy to compute in view of spectral sequences relating it to etale cohomology (which are strongly convergent if X has finite l-cohomological dimension)

$$E_2^{p,q} = H^p_{\mathrm{cont}}(X_{\mathrm{et}}, Z_l(q/2)) \Rightarrow K_{q-p}^{\mathrm{et}}(X),$$
$$E_2^{p,q} = H^p(X_{et}, Z/l^\nu(q/2)) \Rightarrow K_{q-p}^{\mathrm{et}}(X, Z/l^\nu),$$
$$(E_2^{p,q} \text{ is zero if } q \text{ is odd}).$$

For X quasiprojective over a noetherian R-algebra there are natural maps $K_i(X) \to K_i^{\mathrm{et}}(X)$, $K_i(X, Z/l^\nu) \to K_i^{\mathrm{et}}(X, Z/l^\nu)$.

THEOREM 8.1 [7]. *Let A be the ring of integers in an algebraic number field F and let l be a prime integer. If $l = 2$ assume, in addition, that $F \ni \sqrt{-1}$. Then the natural map*

$$K_i(A) \otimes Z_l \to K_i^{\mathrm{et}}(A[1/l])$$

is surjective.

It should be mentioned that Quillen-Lichtenbaum conjectures for number fields are equivalent to the fact that the homomorphism considered in (8.1) is bijective.

Let A be an R-algebra, containing a primitive l^νth root of unity ξ. The group $\pi_2(BA^*, Z/l^\nu)$ coincides with the group ${}_{l^\nu}A$ of l^νth roots of unity in A. The image of ξ under a canonical homomorphism $\pi_2(BA^*, Z/l^\nu) \to K_2(A, Z/l^\nu)$ (induced by the evident morphism $BA^* \to \mathrm{BGL}(A)^+$) is denoted by β and is called the Bott element. Let X be a scheme of finite l-cohomological dimension over A. It is easy to see that etale K-theory of X, $K^{\mathrm{et}}(X, Z/l^\nu)$, is 2-periodical and this periodicity is given by multiplication by β. Thus we get an induced map $K_*(X, Z/l^\nu)[\beta^{-1}] \to K_*^{\mathrm{et}}(X, Z/l^\nu)$ (one has to be more careful when $l = 2$ or 3 since in these cases there are difficulties with ring structure on $K_*(X, Z/l^\nu)$). The fundamental result, relating algebraic and etale K-theory is the following theorem of Thomason [38].

THEOREM 8.2. *Under mild additional hypotheses (see* [38] *for the exact formulation) the induced map*

$$K_*(X, Z/l^\nu)[\beta^{-1}] \to K_*^{\mathrm{et}}(X, Z/l^\nu)$$

is an isomorphism.

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