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Problems Solved and Unsolved Concerning Linear and Nonlinear Partial Differential Equation

Current research in partial differential equations is extensive, varied and deep. A single lecture, if it is not to be a mere catalogue, can present only a partial list of recent achievements, some comments on the modern style, i.e. the kinds of problems chosen and methods used for solution, and cautious speculations on future trends. The choice of examples is of course shaped by the personal taste of the speaker and limited by his expertise.

The first part of this lecture is such an overview; it is followed by a more detailed discussion of two topics with which the speaker has some familiarity, one concerning a linear, the other a nonlinear problem in partial differential equations.

1a. Linear problems

In the last few years a number of problems concerning linear partial differential operators on manifolds with boundaries have been solved or are nearing solution. Thanks to the researches of Melrose [30], Taylor [37], Ivrii and others we understand well the propagation of signals along reflected, glancing and gliding rays, the clue to many problems in diffraction and scattering. Microlocal analysis, the modern version of wave-ray duality, has provided the tools: pseudo-differential operators, Fourier integral operators, Hamiltonian flows and Lagrange manifolds. In his recent work Charles Fefferman [15] makes use of a sophisticated version of the uncertainty principle. Another versatile modern technique

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is the use of trace formulas to link spectral and geometric information. The view from scattering theory has also been useful.

For a thorough documentation of the successes of the modern theory of linear partial differential equations we have to await the publication of Hörmander's 3-volume treatise [19], but it is clear that the successes have been so sweeping that they have radically altered the course of research in this field. I believe that in the future we shall see more applications of the methods and results of the theory of linear partial differential equations to other fields of mathematics; examples from the past are of use of PDE methods in several complex variables and quasiconformal mappings. We are likely to see more special questions raised, from sources inside and outside mathematics, and more detailed answers given; mere preoccupation with existence and uniqueness questions is likely to diminish.

Section 2 contains a brief description of wave propagation on complete manifolds of constant negative curvature. One of the tools used is the Radon transform, less popular than its more glamorous sister, the Fourier transform, but more appropriate in some situations. See also [18].

1b. Nonlinear problems

The strides that have been made recently in the theory of nonlinear PDE's are as great as in the linear theory. Unlike the linear case, no wholesale liquidation of broad classes of problems has taken place; rather it is steady progress on old fronts and on some new ones, the complete solution of some special problems, and the discovery of some brand new phenomena. The old tools — variational methods, fixed point theorems, degree of mapping and other topological methods — have been augumented by some new ones. Preeminent for discovering new phenomena, is numerical experimentation; but it is likely that in the future numerical calculations will be part of proofs.

We shall discuss, very briefly, three topics:

(i) Viscous, incompressible flows.
(ii) Hyperbolic systems of conservation laws and shock waves.
(iii) Completely integrable systems.

(i) Viscous, incompressible flows. In spite of a claim by Kaniel, [21], laid to rest by D. Michelson, the existence for all time of strong solutions of the Navier–Stokes equation, and the uniqueness of weak solutions, in three-dimensional space are very much open questions. We have learned more about the singularities of weak solutions, in particular about the Hausdorff dimension of the singular set. Already Leray has shown that every solution is continuous if we eliminate a closed set of t with zero Hausdorff measure of dimension 1/2. B. Mandelbrot has raised the question of what the Hausdorff dimension of the possible singularities of weak solutions is in space and time. The first results on this important question were obtained by V. Scheffer [34]; the latest word is the following theorem of Caffarelli, Kohn and Nirenberg [6]:

The one-dimensional Hausdorff measure of the set of singularities of a suitable weak solution in x, t-space is zero.

Turbulence, surely one of the outstanding problems of mathematics, can be described by the long-time behavior of typical solutions of the NS equations. When viscosity is large compared to the force driving the flow there exists exactly one stationary flow to which all flows tend. As the force is increased, this stationary flow becomes unstable, i.e. any slight perturbation drives it away, perhaps to another, stable, stationary flow. When the force is increased still further, this too becomes unstable and the flow tends to yet another stationary flow or possibly to a periodic flow. As the force is increased further the flow becomes more and more chaotic. This chaotic flow is concentrated around a so-called *attractor set*, i.e. a set consisting of points of accumulation of a single flow driven by a force that is independent of t. Such sets are invariant under the Navier–Stokes flow; concerning these Foias and Temam have proved the following, see [17]:

A bounded set that is invariant under the strong Navier–Stokes flow in a bounded domain has finite Hausdorff dimension.

The dimension of such sets may go to infinity as the viscosity tends to zero.

Further results along these lines have been obtained by P. Constantin and C. Foias.

The simplest testing ground for ideas of instability and turbulence of viscous fluids are the Couette–Taylor flows, i.e. flows between two concentric cylinders, the inner one rotating with some angular velocity ω. If the cylinders have infinite length, then there is a stationary flow that is independent of the angle θ and distance z along the axis of the cylinders. For ω low enough this flow is stable; as ω increases, this flow becomes unstable, yielding stability to another, z-dependent, flow consisting of a stack of Taylor vortices, named after their discoverer. As ω increases further this flow, too, becomes unstable and gives way to a θ-dependent, periodic flow; further increase in ω leads to more and more complicated flows.

There is a wealth of experimental studies of Couette–Taylor flows, revealing a bewildering variety of steady and unsteady flows, see e.g. Benjamin, [3]. The understanding of these (which must also take into account the finiteness of the cylinders) is a profound challenge to theoretical and computational fluid dynamicists.

Flows without any driving force to maintain them decay because the viscous forces dissipate energy. Recently Foias and Saut [16] have shown that the rate of decay is exponential, the same as for the corresponding linearized Stokes flow; they have further shown that the Stokes and the Navier–Stokes flows are linked by a wave operator.

When viscosity is zero, as in the Euler equation, no imposed force is needed to maintain the flow. Existence and uniqueness is known in 2 dimensions but is doubtful when $n = 3$. Extensive calculations by S. Orszag and his colloborators on the Taylor–Green vortex problem, [31], reveal a bewilderingly complicated flow; as time goes on, smaller and smaller scale features appear until the numerical method — a spectral method keeping track of more than 100 million Fourier coefficients — is unable to resolve them. Another set of calculations by Orszag employs the Taylor series in time, up to order 88, summed in a cunning fashion to elude singularities in the complex t-plane; the features revealed in the two calculations are similar.

Another set of calculations, not nearly so machine-intensive as Orszag's has been performed by Chorin, see e.g. [8]. He considers an initial value problem, periodic in space, where vorticity is initially confined to a narrow, slightly crooked tube. The basic variable is vorticity, and the calculation takes into account that the vorticity is confined to the tube, which stretches and twists with the flow. Using a number of bold simplifications the calculation is carried out long enough to indicate that after a finite time the vortex tube will be stretched so thin that its Hausdorff dimension becomes ~ 2.5, a prediction of Mandelbrot's, [29]. Another calculation by Chorin, employing a rescaling reminiscent of the renormalization group of physicists, leads to the same conclusion.

(ii) Hyperbolic systems of conservation laws and shock waves. A system of n conservation laws in one space variable x and in t is of the form

$$u_t + f(u)_x = 0,$$

$u(x, t)$ in $\boldsymbol{R}^n$; the system is strictly hyperbolic if the matrix $\nabla f(u)$ has real eigenvalues for every u in $\boldsymbol{R}^n$.

The basic problem is the initial value problem: given $u(x, 0) = u_0(x)$, show the existence of a solution $u(x, t)$ for all t, in the class of discontinuous functions, satisfying the conservation law in the sense of distributions, and an entropy condition of the form

$$s_t + g_x \leqslant 0,$$

where $s = s(u)$ is an entropy, $g = g(u)$ entropy flux. The two satisfy

$$\nabla s \, \nabla f = \nabla g,$$

and s is required to be a convex function of u.

Numerical evidence indicates strongly that various difference schemes for solving conservation laws converge; yet until recently no proof had been given for systems with more than one state variable u. Similarly, physics strongly suggests but mathematics had been unable to prove that if $u = u_\varepsilon(x, t)$ solves the viscous equation

$$u_t + f(u)_x = \varepsilon D(u)_{xx}, \quad \varepsilon > 0.$$

$$u(x, 0) = u_0(x),$$

D an appropriate $n \times n$ viscosity matrix, then as ε tends to 0, $u_\varepsilon(x, t)$ converges to a solution of the system of conservation laws that satisfies an entropy condition.

This year Ron Di Perna [12], succeeded in proving such convergence theorems for the equation governing the isentropic flow of a gas satisfying a polytropic equation of state, with the artificial viscosity $D \equiv I$. Among the many ingredients are two beautiful general ideas of Tartar and Murat, [36]. One is a characterization of strong convergence in terms of weak convergence:

Suppose $u_j(y)$ is a sequence of mappings from $\boldsymbol{R}^k$ to $\boldsymbol{R}^m$, uniformly bounded in L^∞; then there is a subsequence such that for every continuous function g in $\boldsymbol{R}^m$ the weak limits

$$\text{w-lim}\, g\big(u_j(y)\big)$$

exists. These limits can be represented as

$$\int g(u)\, d\nu_y(u),$$

where ν_y is a probability measure in $\boldsymbol{R}^m$ parametrized by y in $\boldsymbol{R}^m$. The subsequence u_j converges strongly if the measures v_y have, for each y, a single point as support.

The second idea is compensated compactness: Let v_j and w_j be two sequences of mappings from $\boldsymbol{R}^k$ to $\boldsymbol{R}^k$; if both converge weakly in the L_2 sense to v and w respectively, and if $\operatorname{div} v_j$ and curl w_j lie in compact sets in the H^{-1}_{loc} topology, then

$$\lim \int v_j \cdot w_j \, dy = \int v \cdot w \, dy.$$

Tartar himself has used these ideas to prove the convergence of viscous solutions for scalar conservation laws; Di Perna has shown how to use them for systems with two variables.

Very little is known about existence of discontinuous solutions in more than one space variable; even short time results are of recent origin, see A. Majda's report to this Congress. Yet numerical calculations, done with care and ingenuity, see e.g. Colella and Woodward, [10], converge and give solutions consistent with experiements.

In one-space dimension we know, at least for simple systems, that shock formation and interaction severely limit the amount of information that a one-dimensional flow field can contain. Something similar must be true in higher dimension, but the mechanism causing it is not understood.

In his report to this Congress, S. Klainerman will describe some recent results on long term existence of regular solutions of the initial value problem for non-linear hyperbolic equations in several space variables.

The question of uniqueness, subject to an entropy condition, is not satisfactorily settled even in one-space dimension, not even for the equation of compressible flow, in spite of the important pioneering work of Oleinik, and more recent work of Di Perna.

There are intriguing open problems concerning stationary transonic flows around given contours, with volocity prescribed at infinity. An ingenious method of Garabedian yields large families of aero-dynamically interesting smooth flows, but a basic theorem of Morawetz shows that for a generic contour no shockless flow exists. The basic problem is to prove the existence of a flow, with shocks, and to prove its uniqueness. Recent numerical calculations of A. Jameson indicate that in the potential flow of approximation there may be many solutions.

(iii) Completely integrable systems. This chapter in mathematics, barely 15 years old, continues to fascinate analysts and algebraists, as well as physicists. The effort has been truly international, and has paid off in the discovery of new completely integrable systems, many of physical interst, some containing two space variables, see Ablowitz and Fokas, [2], and Zakharov's report to this Congress. The algebraic classification of these systems has progressed, see van Moerbeke's report to this Congress, and new connections with other branches of mathematics and physics have been found, such as the τ-function of Sato, Miwa and Jimbo, [33], see Sato's and Takhtajan's reports to this Congress. Three books have appeared recently on solitons and scattering theory [1], [7] and [41], and the work of Beals and Coifman, [4]. The speaker will restrict his remarks to a few scattered comments on the analytic side of the matter.

(a) Solutions of completely integrable partial differential equations lie on infinite-dimensional tori. Numerical experiments with such equations, see e.g. [22], furnish numerical approximations that appear to lie on tori, necessarily finite-dimensional. This indicates that some infinite-dimensional analogue of the KAM theory might be true; no such result is known.

(b) The sine-Gordon equation

$$u_{tt} - u_{xx} + \sin u = 0$$

has explicit solutions

$$u(x, t) = -4\arctan\left[\frac{m}{\sqrt{1-m^2}} \frac{\sin\sqrt{(1-m^2 t)}}{\cosh mx}\right], \quad m < 1$$

that decay exponentially in x as $|x| \to \infty$ and are periodic in time. If the function $\sin u$ is replaced by $g(u)$, Coron [11] has shown that no such solution can exist when the time period T is $< 2\pi/g'(0)$; Coron and Brezis conjecture that there are no such periodic solutions of any period, except for very special functions g.

(c) The explicit solution of the initial value problem for the KdV equation

$$u_t - 6uu_x + \varepsilon^2 u_{xxx} = 0,$$

$$u(x, 0) = u(x)$$

in terms of the scattering transform makes it possible to determine explicitely the limit of the solution $u(x, t, \varepsilon)$ as $\varepsilon \to 0$. This rather interesting limit is described in Section 3.

These altogether too brief remarks on nonlinear PDE were confined mostly to problems arising in mathematical physics; it is the richest source of such material, but not the only one: geometry is another, see S. T. Yau's report to the Helsinki Congress [39]. The speaker has neither the knowledge nor the time to report on progress in this very active area in the last five years, but he cannot resist mentioning the very recent demonstration by Wente, Steffen, Struwe and Brezis and Coron of the existence of *two* surfaces of constant mean curvature spanning a prescribed plane curve, not too large; the proof is a marvel of subtlety, see Ambrosetti's report to this Congress.

There hasn't even been enough time to mention all the subjects in mathematical physics that have been traditionally, but especially in the recent past, rich sources of problems in nonlinear PDE: elasticity theory, see Ball's report to this Congress, electromagnetic theory and, more recently, magnetohydrodynamics. Two topics which need more help from mathematicians than they are getting now are multiphase flow and combustion, see e.g. [9] and [28]. In both it is of great importance to understand the nature of turbulent regimes; but in multiphase flow, as in aero- and hydrodynamics, turbulence is detrimental; in combustion it is beneficial since it promotes the mixing of fuel and oxydizer. On the other hand shockwaves are detrimental for combustion, since they produce entropy which decreases the efficiency of conversion of heat into mechanical energy.

2. The Laplace–Beltrami operator on complete Riemannian manifolds with constant negative sectional curvature

In a series of papers, [24]–[27], R. S. Phillips and the speaker have analysed, fairly completely, the spectral properties of the Laplace–Beltrami operator on manifolds F as above in the case when F has infinite volume and is geometrically finite. This extends to all dimensions the previous work of Patterson, [32], in the case $n = 2$, and allows F to have cusps of all kinds.

The universal cover of F is hyperbolic space H_n; F itself can be identified with the quotient H_n/Γ, Γ a discrete subgroup of isometries of H_n. More concretely F can be identified with a fundamental polyhedron F of $H_n \bmod \Gamma$. Conversely, if Γ is a discrete subgroup of the group of all isometries, then $H_n/\Gamma = F$ is a complete Riemannian manifold with constant negative sectorial curvature; if Γ contains elliptic elements, F has harmless singularities along submanifolds.

We shall use the Poincaré model for $\boldsymbol{H}_n$, i.e. the upper half-space (x, y), x in $\boldsymbol{R}^{n-1}$, $y > 0$, equipped with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

The set of points at infinity, $(\beta, 0)$, ∞, is denoted by B.

The L^2 norm and Dirichlet integral, invariant under isometries of $\boldsymbol{H}_u$, will be denoted by $H(u)$ and $D(u)$:

$$H(u) = \int |u^2| \frac{dx\,dy}{y^n}, \qquad D(u) = \int (|u_x|^2 + |u_y|^2) \frac{dx\,dy}{y^{n-2}}.$$

The invariant Laplace–Beltrami operator L_0 is defined in terms of these quadratic forms:

$$H(u, L_0 v) = -D(u, v)$$

for all C_0^∞ functions u and v,

$$L_0 = y^2(\Delta + \partial y^2) - (n-2) y\, \partial y,$$

where $\Delta = \sum \partial x^2$ is the Euclidean Laplace operator. Using the Friedrichs extension, L_0 can be made into a nonpositive self-adjoint operator with respect to H.

Similarly we denote the L^2 and Dirichlet integrals over F by $H_F(u)$ and $D_F(u)$; here u is any C_0^∞ automorphic function with respect to a given discrete subgroup Γ of isometries, and F is a fundamental polyhedron for Γ. In what follows we assume that F has a finite number of sides, i.e. that Γ is *geometrically finite.*

Discrete subgroups can be classified by the geometric properties of their fundamental polyhedra F into the following classes:

(i) F is compact,

(ii) F is noncompact but has finite volume:

$$V(F) = \int_F \frac{dx\,dy}{y^n} < \infty,$$

(iii) F has infinite volume.

The spectral properties of L_0 are sharply different in these cases.

In case (i) it follows by standard elliptic theory that the spectrum of L_0 is standard discrete, i.e. pure point spectrum accumulating only at $-\infty$. The present work is concerned mainly with case (iii); our results are:

(a) L_0 has absolutely continuous spectrum of infinite multiplicity in $\left(-\infty, -\frac{1}{2}(n-1)^2\right)$.

(b) L_0 has at most a finite number of point eigenvalues, all located in the interval $\left(-\frac{1}{2}(n-1)^2, 0\right)$.

(c) L_0 has no singular spectrum; even the point spectrum may be empty. However, Beardon and Sullivan [35] have shown that if F contains a cusp of highest rank, then there is at least one point eigenvalue. We have a new proof of this result, Theorem 6.4 in [26].

Jørgensen, [20] has constructed interesting examples of groups of isometries of $\boldsymbol{H}_3$ whose fundamental polyhedron has infinitely many sides. For these, Epstein, [13], has shown that L_0 has infinite-dimensional spectrum in $(-1, 0)$.

Case (ii) is a curious mixture of (i) and (iii): L_0 has absolutely continuous spectrum in $\left(-\infty, -\frac{1}{2}(n-1)^2\right)$, but only of finite multiplicity, which is equal to the number of cusps. There is no singular spectrum but there may be point spectrum, accumulating at $-\infty$. In many special cases it is known that this point spectrum is ample. In the general case, Selberg has established a relation between the density of the point spectrum and the winding number of the determinant of the scattering matrix; see also pages 205–216 of [24]. To give an absolute estimate of the number of point eigenvalues remains a challenging open problem.

We return now to case (iii). Earlier studies of the continuous spectrum of L_0 proceeded by constructing explicitly a spectral representation of L_0; the generalized eigenfunctions of L_0 entering this spectral representation are Eisenstein series, constructed by analytic continuation.

Our approach is entirely different; it is applicable to representing operators whose continuous spectrum has uniform multiplicity on the whole line. Let A be an *anti-self-adjoint* operator whose spectrum is of uniform multiplicity on the whole imaginary axis. Then the spectral representation for A can be thought of as representing the underlying Hilbert space H by $L^2(\boldsymbol{R}, N)$, N some auxiliary Hilbert space whose dimension equals the multiplicity of the spectrum of A.

Each f in H is represented by an L^2 function $K(\sigma)$, σ in $\boldsymbol{R}$, the values of K lying in N. Since A is anti-self-adjoint, Af is represented by $i\sigma K(\sigma)$.

Denote $U(t)$ the unitary group whose generator is A: $U(t) = \exp tA$. Then

$$U(t)f \leftrightarrow e^{i\sigma t} K(\sigma).$$

The Fourier transform of this representation with respect to σ gives another representation of H by $L^2(\boldsymbol{R}, N)$, where each f in H is represented by

$$f \leftrightarrow k(s), \quad k = \tilde{K}.$$

Then

$$Af \leftrightarrow -\frac{d}{ds} k(s)$$

and

$$U(t)f \leftrightarrow k(s-t), \quad U(t) = \exp tA.$$

This is called *translation representation.*

Of course, conversely, the Fourier transform of a translation representation is a spectral representation.

We show now how to construct a translation representation for the unitary group associated with the non-Euclidean wave equation

$$u_{tt} - Lu = 0,$$

where

$$L = L_0 + \left(\frac{n-1}{2}\right)^2.$$

Note that if (a), (b), (c) hold then, apart of the finite point spectrum, L has continuous spectrum of uniform multiplicity on $(-\infty, 0)$.

The group associated with the wave equation consists of the operators mapping initial data into data at time t:

$$U(t)\colon \{u(0), u_t(0)\} \to \{u(t), u_t(t)\}.$$

The generator of U is

$$A\{u, u_t\} = \{u_t, u_{tt}\} = \{u_t, Lu\} = \{u, u_t\} \begin{Bmatrix} 0 & L \\ 1 & 0 \end{Bmatrix}.$$

Note that $A^2 = \begin{Bmatrix} L & 0 \\ 0 & L \end{Bmatrix}$, thus L having continuous spectrum of uniform multiplicity on $\boldsymbol{R}_-$ is consistent with A having continuous spectrum of uniform multiplicity on $i\boldsymbol{R}$.

Most properties of the non-Euclidean wave equation follow from standard hyperbolic theory:

(i) The initial value problem is properly posed.

(ii) Signals propagate with speed $\leqslant 1$.

(iii) If the initial data are automorphic, so is the solution for all t.

(iv) Energy is conserved, where

$$E = H(u_t) - H(u, Lu) = H(u_t) + D(u) - \left(\frac{n-1}{2}\right)^2 H(u).$$

Finally we have the special property

(v) For n odd the Huygens property holds, i.e. signals propagate with speed $= 1$.

It is not hard to show by integration by parts that for C_0^∞ data in $\boldsymbol{H}_n$, the energy E is positive. We denote by H the completion in the energy norm of C_0^∞ data. It follows from conservation of energy that $U(t)$ is unitary for the energy norm.

We define the *Radon transform* of a function u in $\boldsymbol{H}_n$ by

$$\hat{u} = \int_{\xi(s,\beta)} u\, dS.$$

Here $\xi(s, \beta)$ is the horosphere centered at the point β at ∞, whose distance from the origin is s. It is well known that

$$\widehat{Lu} = e^{-\frac{n-1}{2}s}\, \partial_s^2 e^{\frac{n-1}{2}s}\, \hat{u}.$$

Now take the Radon transform of the wave equation:

$$0 = \hat{u}_{tt} - \widehat{Lu} = \hat{u}_{tt} - e^{-\frac{n-1}{2}s}\, \partial_s^2 e^{\frac{n-1}{2}s}\, \hat{u}.$$

Introduce

$$\hat{u} = e^{\frac{n-1}{2}s} v;$$

then

$$0 = v_{tt} - v_{ss} = (\partial_t + \partial_s)(v_t - v_s),$$

from which it follows that $v_t - v_s$ is a function of $s - t$. We define now

$$R_+\{u, u_t\} = P(v_t - v_s) = P(e^s \hat{u}_t - \partial_s e^s \hat{u}),$$

P an appropriately chosen operator in s that commutes with translation.

R_+ is a translation representation of H for $U(t)$, i.e.

(i) $R_+\{u(t), u_t(t)\}(s) = R_+\{u(0), u_t(0)\}(s-t)$,

(ii) $\int (R_+\{u, u_t\})^2 ds\, dm(\beta) = E(u)$,

(iii) R_+ maps H onto $L_2(\boldsymbol{R}, B)$.

Of course (i) follows from the way R_+ was constructed; for (ii) and (iii), see Theorem 3.4 [25] when $n = 3$ and $P = c\partial_s$.

We turn now to the automorphic case. Here energy, defined as before by

$$E_F(u) = H_F(u_t) + D_F(u) - \left(\frac{n-1}{2}\right)^2 H_F(u),$$

is not necessarily positive. It was shown in Section 4 of [26] that one can add a quadratic function $K(u)$ to the energy so that $G(u) = E_F(u) + K(u)$ is positive, and that K is compact with respect to G. It follows from this that if E_F is negative on a subspace, that subspace is finite-dimensional. Since for $u_t = 0$ the energy is $-H_F(u, Lu)$, it follows that the positive spectrum of L consists of a finite number of eigenvalues. It can be further shown, using the fact that F contains full neighborhoods of points at ∞, that L has no negative eigenvalues, see Theorem 4.8 of [26]; this is a non-Euclidean version of a classical result of Rellich and Vekua.

It follows from the form $E_F = H_F(u_t) - H_F(u, Lu)$ that if u is orthogonal to all the eigenfunctions of L, then $E_F \geq 0$. If $u(0)$ and $u_t(0)$ are both orthogonal to all the eigenfunctions it follows that so is $u(t)$ for any t. We denote this space of initial data by H_c. Clearly H_c is invariant under the solution operators $U_F(t)$ for the automorphic solutions of the wave equation.

We define now a translation representation R_+^F of H_c for the group $U_F(t)$; it has $M+1$ components, M being the number of cusps of maximal rank. The zeroth component of R_+^F is R_+, defined as before; each of the remaining components are associated with cusps of maximal rank as follows:

Map the cusp to the point ∞, so that it has the form $F_\infty \times (a, \infty)$, F_∞ a fundamental polyhedron in Euclidean space of the parabolic subgroup keeping ∞ fixed. Since the cusp is of maximal rank, F_∞ has finite volume. Denote by $\bar{u}$ the mean value of u:

$$\bar{u}(g) = \frac{1}{|F_\infty|} \int\limits_{F_\infty} u(x, y)\, dx.$$

Note that the integration is over a part of a horosphere centered at ∞. Now integrate the wave equation over F_∞:

$$\bar{u}_{tt}-y^2\bar{u}_{yy}+(n-2)y\bar{u}_y-\left(\frac{n-1}{2}\right)^2\bar{u} = 0.$$

Introducing $w = y^{(n-1)/2}\,\bar{u}$ and $y = e^s$ as new variables we obtain

$$w_{tt}-w_{ss} = 0.$$

We now define the jth component of the translation representation as w_t-w_s, i.e.

$$R_+^j\{u, u_t\} = c\left(e^{\frac{n+1}{2}s}\,\bar{u}_t-\partial_s e^{\frac{n-1}{2}s}\,\bar{u}\right).$$

In Part I of [27] we show that R_+^F is a partial translation representation of H_c for the group $U_F(t)$; in Part II we prove the completeness of this representation. R_+^F is called the *outgoing* representation. One can define quite analogously the incoming representation R_-^F. The relation of the two is the scattering operator S_F, a notion introduced by Faddeev and Pavlov [14]. As pointed out in Section 4 of [25], the scattering operator is nontrivial already for the case of $\Gamma = \mathrm{id}$, i.e. for the translation representations R_S over all of H_n. This is in sharp contrast to the Euclidean case.

We wish to emphasize that the translation representations are constructed here in purely geometrical terms, i.e. in terms of integrals over horospheres.

3. The zero dispersion limit for the KdV equation

The equation in question is

$$u_t-6uu_x+\varepsilon^2u_{xxx} = 0,$$

and the question under discussion is this: if the initial values of u are fixed,

$$u(x, 0; \varepsilon) = u(x),$$

how does the solution $u(x, t; \varepsilon)$ behave as ε tends to 0?

When we set $\varepsilon = 0$ in the equation, we obtain the reduced equation

$$u_t-6uu_x = 0.$$

This equation has no solution for all t, only in the interval (t_b^{-1}, t_b^{+}),

$$t_b^- = \left(6\operatorname*{Min}_x u_x(x)\right)^{-1}, \quad t_b^+ = \left(6\operatorname*{Max}_x u_x(x)\right)^{-1};$$

here $u(x)$ is the initial value of u. It is reasonable to surmise that for t in (t_b^-, t_b^+), $u(x, t, \varepsilon)$ tends as $\varepsilon\to 0$ to the solution of the reduced equation. What happens when t lies outside this interval? Numerical experiments indicate that over some part of the x-axis, $u(x, t, \varepsilon)$ is oscillatory. As ε tends to 0, the amplitudes of these oscillations remain finite, their wavelengths are of order ε. Clearly, if we can talk of a limiting behavior as ε tends to 0, this limit can exist only in the weak sense, e.g. in the sense of distributions. This indeed is the case; the speaker and C. D. Levermore have shown, [23], that $u(x, t, \varepsilon)$ tends in the sense of distribution to a limit $\bar{u}$, provided that the initial value $u(x)$ is nonpositive and tends to zero so fast that $\int x u(x)\,dx$ is finite. These papers show not only the existence of a limit but give a fairly explicit formula for the limit $\bar{u}$. For simplicity we take the case when $u(x)$ has a single minimum; then

$$\bar{u}(x, t) = \partial_x^2 Q^*(x, t),$$

where

$$Q^*(x, t) = \operatorname*{Min}_{0\leqslant\psi\leqslant\varphi} Q(\psi; x, t).$$

Here Q is a quadratic functional of u:

$$Q(\psi;\ x, t) = \frac{4}{\pi}(a, \psi) - \frac{1}{\pi}(L\psi, \psi),$$

L the linear integral operator

$$L\psi(\nu) = \frac{1}{\pi}\int \log\left|\frac{\nu-\mu}{\nu+\mu}\right|\psi(\mu)\,d\mu.$$

The functions admissible in the minimum problem are restricted to lie between 0 and φ, where φ is defined in terms of the initial data as follows:

$$\varphi(\eta) = \operatorname{Re}\int \frac{\eta\,dy}{\left(|u(y)| - \eta^2\right)^{1/2}}, \quad 0 \leqslant \eta.$$

The function a appearing in the linear term in Q depends linearly on x and t, and is defined as follows:

$$a(\eta, x, t) = \eta x - 4\eta^3 t - \theta_+(\eta),$$

where $\theta_+(\eta)$ is a function defined in terms of the inital data. Using the KdV equation it follows that also $u^2(x, t; \varepsilon)$ has a limit $\overline{u^2}$ in the distribution sense, and that

$$\bar{u}_t = \overline{3u_x^2}.$$

Multiplying the KdV equation by u and rewriting the resulting equation as a conservation law shows that

$$\lim_{\varepsilon \to 0} (u^3 + \tfrac{3}{4}\varepsilon^2 u_x^2) = \overline{u^3}$$

exists in the distribution sense, and that

$$\overline{u_t^2} = 4\overline{u_x^3}.$$

Combining this with the explicitly form the $\bar{u}$ leads easily to the formulas

$$\overline{u^2} = \tfrac{1}{3}\partial_x \partial_t Q^*, \qquad \overline{u^3} = \tfrac{1}{12}\partial_t^2 Q^*.$$

The minimum problem defining Q^* is a so-called quadratic programming problem; it turns out that it can be solved explicitly. To see this it is convenient to extend the functions ψ admitted in the minimum problem for all real η as odd functions; as a result we may replace the kernel of the operator L by $\log|\eta - \mu|$. One can show that L is negative definite, and that it is related to the Hilbert transform H as follows:

$$\partial_\eta L = H.$$

We extend now ψ to the upper half of the complex η plane as a harmonic function that vanishes at ∞; ψ can be regarded as the real part of an analytic function of Hardy class. The variational condition for the minimum problem can be regarded as prescribing the real and imaginary parts of the function on complementary subsets of the real axis; for details we refer the reader to [23]. Suffice it here to say that the resulting formulas for $\bar{u}$ show:

(i) For t in (t_b^-, t_b^+), $\bar{u}$ is a solution of the reduced equation, and that in this time interval the convergence of u to $\bar{u}$ takes place not only in the sense of distributions but for each t in the L^2 sense in x

(ii) For t outside the interval (t_b^-, t_b^+), $\bar{u}$ can be described by Whitham's averaged equations, or by the more general equations of Flaschka, Forest and McLaughlin based on multiphase averaging.

(iii) For t tending to ∞, $\bar{u}$ decays like t^{-1}; more precisely

$$\bar{u}(x,t) = -\frac{1}{2\pi}\varphi\left(\frac{1}{2}\sqrt{\frac{x}{t}}\right) + o(t^{-1})$$

for $0 < x/t < 4m$, where $m = \max_x[-u(x)]$. Outside this range $\bar{u}(x,t)$ is $O(t^{-2})$.

The formula for $u(x,t;\varepsilon)$ is obtained from Gardner, Greene, Kruskal and Miura's solution of the KdV equation by the scattering transform. We trace carefully the manner in which this solution depends on ε, and show that as ε tends to zero, it has a limit in the sense of distributions. The nonpositivity of the inital data makes the GGKM solution of the KdV equation particularly simple. A more difficult case has been handled by Venakides [40].

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