Motion of the Oloid-toy

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Summary. We present a kinematic analysis and dynamic simulation of the toy known as the Oloid. The Oloid is defined by the convex hull of two equal radius disks whose symmetry planes are at right angles with the distance between their centers equal to their radius. The no-slip constraints of the Oloid are integrable, hence the system is essentially holonomic. In this paper we present analytic expressions for the trajectories of the ground contact points, basic dynamic analysis, and observations on the unique behavior of this system.

Introduction.

The geometry of the Oloid is described by the disc radii R. There are five points of interest on the Oloid: A and B denote the contact points of the first and second disc with the ground plane, C_1 and C_2 denote the two disc centers, G denotes the midpoint of the line segment C_1C_2. We introduce a coordinate system $Gx_1x_2x_3$ centered at G with x_1 in the plane of the first disc, x_3 in the plane of the second, and x_2 along connecting axis.

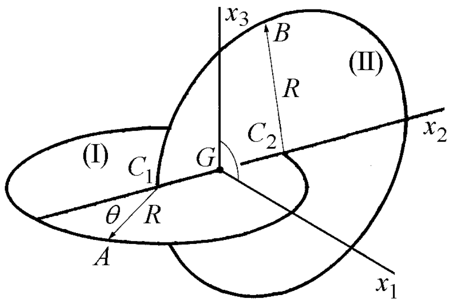

Figure 1: The coordinate system $Gx_1x_2x_3$.

The Oloid was constructed for the first time by Paul Schatz [2,3]. The geometric properties of the surface of the Oloid have been discussed in [1]. The Oloid is also used for technical applications. Special mixing-machines are constructed using such bodies [5]. In our paper we make the complete kinematical analysis of motion of this object on the horizontal plane. Further we briefly describe basic facts from Kinematics and Differential Geometry which we will use in our investigation. We conclude with some observations on the dynamic behavior of the rolling Oloid.

Methods

The Frenet - Serret formulas

Consider a particle which moves along a continuous differentiable curve in three - dimensional Euclidean Space $\mathcal{R}^3$. We can introduce the following coordinate system: the origin of this system is in the moving particle, $\boldsymbol{\tau}$ is the unit vector tangent to the curve, pointing in the direction of motion, $\boldsymbol{\nu}$ is the derivative of $\boldsymbol{\tau}$ with respect to the arc-length parameter of the curve, divided by its length and $\boldsymbol{\beta}$ is the cross product of $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$: $\boldsymbol{\beta} = [\boldsymbol{\tau} \times \boldsymbol{\nu}]$.
Then the Frenet - Serret formulas for the derivatives of $\boldsymbol{\tau}$, $\boldsymbol{\nu}$ and $\boldsymbol{\beta}$ are:

$$\frac{d\boldsymbol{\tau}}{dt} = k\dot{s}\boldsymbol{\nu}, \quad \frac{d\boldsymbol{\nu}}{dt} = -k\dot{s}\boldsymbol{\tau} + æ\dot{s}\boldsymbol{\beta}, \quad \frac{d\boldsymbol{\beta}}{dt} = -æ\dot{s}\boldsymbol{\nu}.$$

Here d/dt is the derivative with respect to the time, k is the curvature and æ is the torsion of the curve. The tangent $\boldsymbol{\tau}$, the normal $\boldsymbol{\nu}$ and the binormal $\boldsymbol{\beta}$ unit vectors are known as the Frenet - Serret frame.

The Poisson formulas

Let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ any moving coordinate system with angular velocity $\boldsymbol{\omega}$. Then the derivatives of $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ satisfy the Poisson formulas:

$$\frac{d\boldsymbol{e}_i}{dt} = [\boldsymbol{\omega} \times \boldsymbol{e}_i], \quad i = 1, 2, 3.$$

Thus we can rewrite the Frenet - Serret formulas as:

$$\frac{d\boldsymbol{\tau}}{dt} = [\boldsymbol{\Omega} \times \boldsymbol{\tau}], \quad \frac{d\boldsymbol{\nu}}{dt} = [\boldsymbol{\Omega} \times \boldsymbol{\nu}], \quad \frac{d\boldsymbol{\beta}}{dt} = [\boldsymbol{\Omega} \times \boldsymbol{\beta}],$$

where $\boldsymbol{\Omega} = k\dot{s}\boldsymbol{\beta} + æ\dot{s}\boldsymbol{\tau}$ is the angular velocity of the Frenet - Serret frame known also as the Darboux vector. In particular, for a plane curve we have $æ = 0$ and therefore

$$\boldsymbol{\Omega} = k\dot{s}\boldsymbol{\beta}. \tag{1}$$

Two contact points

Consider an Oloid rolling without slip on a fixed horizontal plane. Let G be the center of mass of the moving body and K_1, K_2 be two contact points of the body with the plane. From the rolling conditions for the two contact points follows that the angular velocity is always parallel to the line between the contact points, i.e.

$$\left[\boldsymbol{\omega} \times \overrightarrow{K_1K_2}\right] = 0.$$

Now we introduce four coordinate systems. The first coordinate system is the fixed system $Oxyz$ with origin at any point O of the fixed plane and with the Oz - axis directed upward. The second is $Gx_1x_2x_3$ with the Gx_2 axis along the common axis of symmetry of the two lamina and Gx_3 - axis perpendicular to the plane of the first lamina and the Gx_1 - axis is perpendicular to the plane of the second lamina. The third coordinate system is the Frenet - Serret frame $K_1\boldsymbol{\tau\nu\beta}$ for the boundary of the first lamina: $\boldsymbol{\tau}$ - vector is the tangent vector to the boundary of the lamina, $\boldsymbol{\nu}$ - vector is in the plane of the lamina and the $\boldsymbol{\beta}$ - vector coincides in the unit vector of Gx_3 axis (i.e. $\boldsymbol{\beta}$ is perpendicular to the plane of lamina). The fourth coordinate system is also a Frenet - Serret frame. The origin of this system is also in K_1, the first unit vector $\boldsymbol{\tau}_1$ of this system coincides with the vector $\boldsymbol{\tau}$ of the system $K_1\boldsymbol{\tau\nu\beta}$. The second vector $\boldsymbol{\nu}_1$ of this system is in the plane of motion and the third vector $\boldsymbol{\beta}_1$ is perpendicular to the plane of motion (i.e. the vector $\boldsymbol{\beta}_1$ coincides with the vector $\boldsymbol{e}_z$ of the system $Oxyz$). Therefore we can conclude that the coordinate system $K_1\boldsymbol{\tau}_1\boldsymbol{\nu}_1\boldsymbol{\beta}_1$ is the Frenet - Serret frame for the curve which is obtained by the motion of the contact point K_1 (i.e. it is a "trace" of the point of contact on the plane).
Let us find the angular velocity of the moving body. We will use the law of composition of angular velocities. Let us consider the systems $Oxyz$, $K_1\boldsymbol{\tau\nu\beta}$ and $Gx_1x_2x_3$. The absolute angular velocity of $Gx_1x_2x_3$ system with respect to $Oxuz$ system is the sum of angular velocity of $Gx_1x_2x_3$ with respect to $K_1\boldsymbol{\tau\nu\beta}$ and the angular velocity of $K_1\boldsymbol{\tau\nu\beta}$ with respect to $Oxyz$. But the angular velocity of $Gx_1x_2x_3$ system with respect to $K_1\boldsymbol{\tau\nu\beta}$ system can be found easily: it is equal to $-k\dot{s}\boldsymbol{\beta}$. So we need calculate now the angular velocity of $K_1\boldsymbol{\tau\nu\beta}$ system with respect to $Oxyz$ system.
Let us denote by φ the angle between two unit vectors $\boldsymbol{\beta} = \boldsymbol{e}_3$ and $\boldsymbol{\beta}_1 = \boldsymbol{e}_z$. Then we can represent the angular velocity of $K_1\boldsymbol{\tau\nu\beta}$ system with respect to $Oxyz$ system as the sum of the angular velocity of the $K_1\boldsymbol{\tau\nu\beta}$ system with respect to $K_1\boldsymbol{\tau}_1\boldsymbol{\nu}_1\boldsymbol{\beta}_1$ system and the angular velocity of the $K_1\boldsymbol{\tau}_1\boldsymbol{\nu}_1\boldsymbol{\beta}_1$ system with respect to $Oxyz$ system. But the angular velocity of $K_1\boldsymbol{\tau\nu\beta}$ with respect to $K_1\boldsymbol{\tau}_1\boldsymbol{\nu}_1\boldsymbol{\beta}_1$ is $\dot{\varphi}\boldsymbol{\tau}$ and the angular velocity of $K_1\boldsymbol{\tau}_1\boldsymbol{\nu}_1\boldsymbol{\beta}_1$ with respect to $Oxyz$ system is $K\dot{s}\boldsymbol{\beta}_1$. Therefore finally we obtain:

$$\boldsymbol{\omega} = \dot{\varphi}\boldsymbol{\tau} - k\dot{s}\boldsymbol{\beta} + K\dot{s}\boldsymbol{\beta}_1.$$

Since $\boldsymbol{\beta} = -\boldsymbol{\nu}_1 \sin\varphi + \boldsymbol{\beta}_1 \cos\varphi$ we have

$$\boldsymbol{\omega} = \dot{\varphi}\boldsymbol{\tau}_1 + k\dot{s}\sin\varphi\boldsymbol{\nu}_1 - k\dot{s}\cos\varphi\boldsymbol{\beta}_1 + K\dot{s}\boldsymbol{\beta}_1.$$

But from the rolling conditions already obtained, the vector $\boldsymbol{\omega}$ is parallel to $\overrightarrow{K_1K_2}$ vector, i.e. $\boldsymbol{\omega}$ is always in the plane of motion

$$\boldsymbol{\omega} = \omega_1\boldsymbol{\tau}_1 + \omega_2\boldsymbol{\nu}_1.$$

This means that

$$K = k\cos\varphi \quad \text{or} \quad \rho\cos\varphi = r, \tag{2}$$

where ρ and r are radii of curvature of the curve on the fixed plane (trajectory of point K_1) and the bound of the first lamina.

Natural equations of a curve

Let us consider the planar curve, defined by its parametric equations:

$$\boldsymbol{r} = \boldsymbol{r}\,(s) = x(s)\boldsymbol{e}_x + y(s)\boldsymbol{e}_y,$$

with arc-length s as a parameter. We denote by $\alpha(s)$ the angle between the unit tangent vector $\boldsymbol{\tau}$ to the given curve

$$\boldsymbol{\tau} = \frac{d\boldsymbol{r}}{ds} = \frac{dx}{ds}\boldsymbol{e}_x + \frac{dy}{ds}\boldsymbol{e}_y$$

and the unit vector $\boldsymbol{e}_x$ of the Ox - axis. The initial value of $\alpha(s)$ at $s = 0$ can be chosen to be a value divisible by 2π. For other points the angle $\alpha(s)$ is defined explicitly.
Since $\boldsymbol{\tau}(s)$ is the unit vector its projections on the Ox and Oy axes are $\cos\alpha$ and $\sin\alpha$ respectively. From the other side

$$\boldsymbol{\tau}(s) = \cos\alpha\boldsymbol{e}_x + \sin\alpha\boldsymbol{e}_y = \frac{dx}{ds}\boldsymbol{e}_x + \frac{dy}{ds}\boldsymbol{e}_y,$$

and therefore

$$dx/ds = \cos\alpha, \quad dy/ds = \sin\alpha. \tag{3}$$

Moreover, using the first Frenet – Serret formula we have

$$\frac{d\boldsymbol{\tau}}{ds} = (-\sin\alpha \boldsymbol{e}_x + \cos\alpha \boldsymbol{e}_y)\frac{d\alpha}{ds}\boldsymbol{\nu} = k(s)\boldsymbol{\nu},$$

and hence

$$d\alpha/ds = k(s). \tag{4}$$

This means that we can find the parametric equations of the curve if we know its curvature $k(s)$. Thus we derived all the necessary facts for the investigation of the Oloid motion.

Coordinate frames and parametrization

We consider now the motion of Oloid. Let two circles of the same radius R in perpendicular planes be given such that each circle contains the center of the other. Then the convex hull of these circles is called Oloid. Let (I) and (II) be two circles of the same radius R in perpendicular planes Π_1 and Π_2 such that (I) passes through the center C_2 of (II) and k_B passes through the center C_1 of (I) (Fig. 1).

According to the previous theory let us introduce the moving coordinate system $Gx_1x_2x_3$. The origin of this system will be at the midpoint G of C_1C_2 (i.e. G is the center of mass of the system). The Gx_3 - axis is perpendicular to the plane Π_1 of the first circle, Gx_1 - axis is perpendicular to the plane Π_2 of the second circle and Gx_2 axis is directed along the common axis of symmetry of two circles (Fig. 1).

We will parametrize the first circle by the angle θ between the negative direction of Gx_2 axis and the direction to the point of contact. Note that this parametrization is proportional to the arc-length parametrization s: $s = R\theta$. We introduce also the angle ψ for the parametrization of the second circle: let ψ be the angle between the positive direction of Gx_2 axis and the direction to the point of contact B. Then the radius - vector of the point A can be written as follows:

$$\overrightarrow{GA} = \boldsymbol{r}_1 = R\sin\theta \boldsymbol{e}_1 - (R/2 + R\cos\theta)\,\boldsymbol{e}_2.$$

The radius-vector of the point B has the form:

$$\overrightarrow{GB} = \boldsymbol{r}_2 = (R/2 + R\cos\psi)\,\boldsymbol{e}_2 + R\sin\psi \boldsymbol{e}_3.$$

When the Oloid rolls on a fixed plane the three vectors $\boldsymbol{r}_1 - \boldsymbol{r}_2$, $\boldsymbol{r}_1'$ and $\boldsymbol{r}_2'$ should be situated in this plane. We can write this condition as:

$$\begin{vmatrix} R\sin\theta & -R - R\cos\theta - R\cos\psi & -R\sin\psi \\ R\cos\theta & R\sin\theta & 0 \\ 0 & -R\sin\psi & R\cos\psi \end{vmatrix} = 0.$$

As a result we have the following constraint between two parametrizations:

$$\cos\psi + \cos\theta\cos\psi + \cos\theta = 0.$$

and now can obtain the radius vector $\overrightarrow{GB} = \boldsymbol{r}_2$ in the θ parametrization:

$$\overrightarrow{GB} = \boldsymbol{r}_2 = \left(\frac{R}{2} - \frac{R\cos\theta}{1+\cos\theta}\right)\boldsymbol{e}_2 - \frac{R\sqrt{1+2\cos\theta}}{1+\cos\theta}\boldsymbol{e}_3. \tag{5}$$

Note here the interesting fact that the length of the vector $\overrightarrow{AB}$

$$\overrightarrow{AB} = \overrightarrow{GB} - \overrightarrow{GA} = -R\sin\theta \boldsymbol{e}_1 + \left(R + \frac{R\cos^2\theta}{1+\cos\theta}\right)\boldsymbol{e}_2 - \frac{R\sqrt{1+2\cos\theta}}{1+\cos\theta}\boldsymbol{e}_3$$

will be a constant

$$AB = R\sqrt{3}.$$

This feature is used in construction of the Oloid. The expression (5) for the vector $\boldsymbol{r}_2$ then should give:

$$0 \le 1 + 2\cos\theta \quad \forall\, \theta, \psi \in [-\frac{2\pi}{3}, \frac{2\pi}{3}]$$

Results

Trajectories of the points of contact

We derive now the equation of the fixed plane in the $Gx_1x_2x_3$ coordinate system, writing it in the form:

$$LX + MY + NZ + P = 0.$$

Indeed points A, B and the tangent vector to the first circle at A are always in this plane. Therefore the following condition is valid:

$$\begin{vmatrix} X - R\sin\theta & Y + \dfrac{R}{2} + R\cos\theta & Z \\ -R\sin\theta & R\left(1 + \dfrac{\cos^2\theta}{1+\cos\theta}\right) & -\dfrac{R\sqrt{1+2\cos\theta}}{1+\cos\theta} \\ R\cos\theta & R\sin\theta & 0 \end{vmatrix} = 0.$$

From this condition, after some simplifications, we have the following expression for the plane of motion

$$-\sin\theta X + \cos\theta Y + \sqrt{1+2\cos\theta}Z + \frac{R}{2}\left(2+\cos\theta\right) = 0.$$

The unit vector

$$\boldsymbol{n} = -\sin\frac{\theta}{2}\boldsymbol{e}_1 + \left(\frac{1}{2\cos\frac{\theta}{2}} - \cos\frac{\theta}{2}\right)\boldsymbol{e}_2 + \frac{\sqrt{1+2\cos\theta}}{2\cos\frac{\theta}{2}}\boldsymbol{e}_3.$$

is the normal vector to this plane. Therefore the angle between the plane of the first circle and the fixed plane is defined as follows:

$$\cos\varphi = (\boldsymbol{n}\cdot\boldsymbol{e}_3) = \frac{\sqrt{1+2\cos\theta}}{2\cos\frac{\theta}{2}}.$$

The radius of curvature of a circle at any point is equal to R. Thus using (2) we can calculate the radius of curvature of a curve drawn by the point of contact A on the fixed plane:

$$\rho = \frac{R}{\cos\varphi} = \frac{2R\cos\frac{\theta}{2}}{\sqrt{1+2\cos\theta}} \quad \text{and} \quad K = \frac{1}{\rho} = \frac{\sqrt{1+2\cos\theta}}{2R\cos\frac{\theta}{2}}.$$

Having an expression for K, we can find the parametric equations of the trajectory of the point A on the fixed plane.
For this purpose let us introduce the fixed coordinate system $Oxyz$, whose origin O coincides with the point of contact of the first circle with the plane at $\theta = 0$. The Ox - axis is tangent to the first circle, the Oz - axis is directed upwards. The Oy - axis forms a right triple with the Ox and Oz axes. Then we obtain:

$$\frac{d\alpha}{ds} = \frac{d\alpha}{Rd\theta} = K\left(\theta\right), \quad \text{i.e.} \quad \frac{d\alpha}{d\theta} = \frac{\sqrt{1+2\cos\theta}}{2\cos\frac{\theta}{2}}.$$

Integration of this equation gives the following expression for α:

$$\alpha = 2\arcsin\left(\frac{2}{\sqrt{3}}\sin\frac{\theta}{2}\right) - \arcsin\left(\frac{\sin\frac{\theta}{2}}{\sqrt{3}\cos\frac{\theta}{2}}\right).$$

Then

$$\sin\alpha = \frac{\sqrt{3}\sin\frac{\theta}{2}}{9\cos\frac{\theta}{2}}\left(5+4\cos\theta\right), \quad \cos\alpha = \frac{\sqrt{3}}{9}\frac{\left(1+2\cos\theta\right)^{\frac{3}{2}}}{\cos\frac{\theta}{2}}.$$

Using these formulas together with (3), (4) we find after some trigonometric simplifications

$$x_A\left(\theta\right) = \frac{2R\sqrt{3}}{9}\left(\arcsin\left(\frac{2}{\sqrt{3}}\sin\frac{\theta}{2}\right) + \arcsin\left(\frac{\sin\frac{\theta}{2}}{\sqrt{3}\cos\frac{\theta}{2}}\right) + 2\sin\frac{\theta}{2}\sqrt{1+2\cos\theta}\right),$$

$$y_A\left(\theta\right) = \frac{8R\sqrt{3}}{9}\sin^2\left(\frac{\theta}{2}\right) - \frac{2R\sqrt{3}}{9}\ln\left(\cos\left(\frac{\theta}{2}\right)\right), \quad -\frac{2\pi}{3} < \theta < \frac{2\pi}{3}.$$

These equations give a parametric representation for the trajectory of the point A on the fixed plane.
We can use a similar method in finding the trajectory of point B, represented in the scalar form as:

$$x_B\left(\theta\right) = x_A\left(\theta\right) + R\sqrt{3}\cos\left(\alpha+\gamma\right), \quad y_B\left(\theta\right) = y_A\left(\theta\right) + R\sqrt{3}\sin\left(\alpha+\gamma\right).$$

Here γ is the angle between the vector $\boldsymbol{e}_{AB}$ and the tangent vector $\boldsymbol{\tau}$ to the first circle at A. The tangent vector $\boldsymbol{\tau}$ has the form:

$$\boldsymbol{\tau} = \cos\theta\boldsymbol{e}_1 + \sin\theta\boldsymbol{e}_2$$

and therefore its scalar product with $\overrightarrow{AB}$ leads to:

$$\left(\overrightarrow{AB}\cdot\boldsymbol{\tau}\right) = AB\left(\boldsymbol{e}_{AB}\cdot\boldsymbol{\tau}\right) = AB\cos\gamma = R\sqrt{3}\cos\gamma = \frac{R\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}},$$

$$\cos\gamma = \frac{\sin\frac{\theta}{2}}{\sqrt{3}\cos\frac{\theta}{2}}, \quad \sin\gamma = \frac{\sqrt{1+2\cos\theta}}{\sqrt{3}\cos\frac{\theta}{2}}.$$

Further, it is easy to find

$$\sin(\alpha+\gamma) = \frac{7}{9} + \frac{4\cos^2\theta}{9(1+\cos\theta)}, \quad \cos(\alpha+\gamma) = -\frac{4(2+\cos\theta)\sin\frac{\theta}{2}}{9(1+\cos\theta)}\sqrt{1+2\cos\theta}.$$

Finally in the explicit form we have the following expressions for x_B and y_B:

$$x_B(\theta) = \frac{2R\sqrt{3}}{9}\left(\arcsin\left(\frac{2}{\sqrt{3}}\sin\frac{\theta}{2}\right) + \arcsin\left(\frac{\sin\frac{\theta}{2}}{\sqrt{3}\cos\frac{\theta}{2}}\right) - \frac{2\sin\frac{\theta}{2}}{(1+\cos\theta)}\sqrt{1+2\cos\theta}\right),$$

$$y_B(\theta) = \frac{7R\sqrt{3}}{9} + \frac{2R\sqrt{3}}{9\cos^2\left(\frac{\theta}{2}\right)} - \frac{2R\sqrt{3}}{9}\ln\left(\cos\left(\frac{\theta}{2}\right)\right), \quad -\frac{2\pi}{3} < \theta < \frac{2\pi}{3}.$$

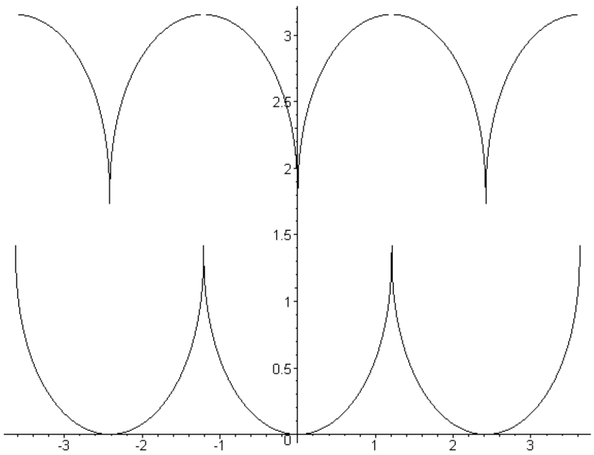

Figure 2: Trajectories of points A (bottom curve) and B (upper curve) on the supporting plane.

These equations give a parametric representation for the trajectory of the point B on the fixed plane. Figure 2 shows both trajectories on the fixed plane Oxy.

Non-Obvious Dynamic Behaviors

Unusual behavior was observed when simulating the dynamic motion of the Oloid. We found that the waveform of the generalized speed changed qualitatively with different initial speeds. Figure 3 below demonstrates this. The speeds and times were normalized to the initial values and shown for one cycle of the output. It can be seen at the lower initial speeds that the speed starts by decreasing and at a higher initial speed it starts by rising. We believe this behavior to be unusual for such a simple, deterministic, one degree of freedom system where energy is conserved. Also visible is the change in shape of waveform throughout the cycle from two large speed minima to four smaller speed minima.

Conclusions

We investigate here the motion of the Oloid toy on the fixed horizontal plane. Parametric equations for the trajectories of points of contact of the Oloid with the plane are derived.

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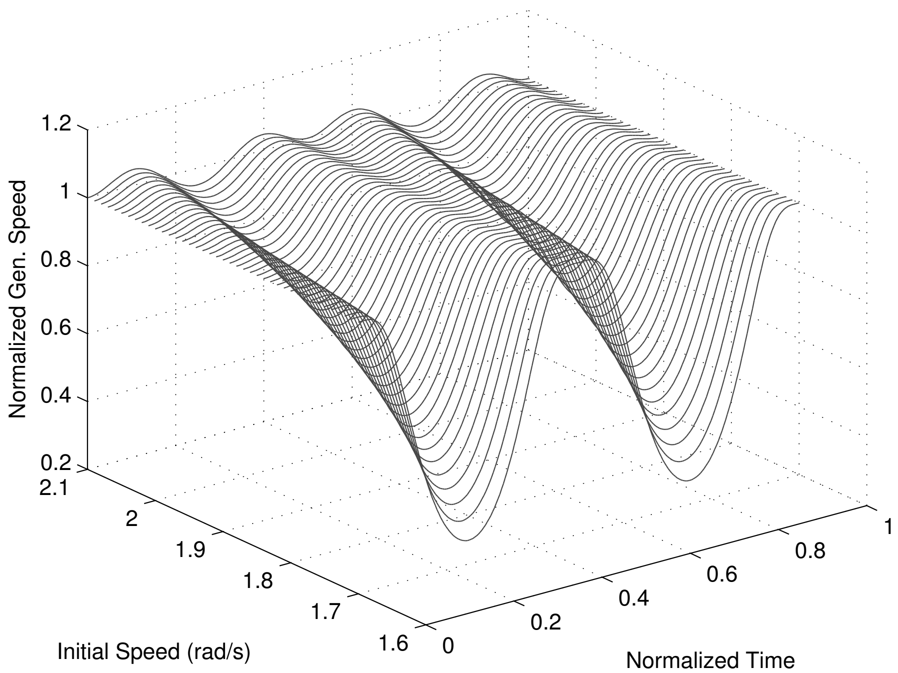

Figure 3: Qualitative Analysis of Generalized Speed Waveform (R = .1, G=9.8)