

Geometry and Nonlinear Analysis*

Gang Tian[†]

Nonlinear analysis has played a prominent role in the recent developments in geometry and topology. The study of the Yang-Mills equation and its cousins gave rise to the Donaldson invariants and more recently, the Seiberg-Witten invariants. Those invariants have enabled us to prove a number of striking results for low dimensional manifolds, particularly, 4-manifolds. The theory of Gromov-Witten invariants was established by using solutions of the Cauchy-Riemann equation (cf. [RT], [LT], [FO], [Si], [Ru]). These solutions are often referred as pseudo-holomorphic maps which are special minimal surfaces studied long in geometry. It is certainly not the end of applications of nonlinear partial differential equations to geometry. In this talk, we will discuss some recent progress on nonlinear partial differential equations in geometry. We will be selective, partly because of my own interest and partly because of recent applications of nonlinear equations. There are also talks in this ICM to cover some other topics of geometric analysis by R. Bartnik, B. Andrew, P. Li and X.X. Chen, etc.

Standard partial differential equations in geometry are the Einstein equation, Yang-Mills equation, minimal surface equation as well as its close cousin, Harmonic map equation. There are also parabolic versions of these equations, leading to R. Hamilton's Ricci flow, the Yang-Mills flow and the mean curvature flow. The solutions, which played a fundamental role in geometry and topology, of these equations are their self-dual type ones. I will focus on self-dual type solutions in this talk. All these equations are in general hyperbolic equations if we allow Lorentz metrics on the underlying manifolds, but in differential geometry, so far, we only concern static solutions, that is, we assume that the metrics involved are Riemannian. I do believe that the study of this static case will be very important in our future understanding general Einstein equation.

1. Einstein equation

We will begin with the Einstein equation. We will always denote by M a differentiable manifold. A metric g on M is given by a non-degenerate matrix-

*Supported partially by NSF grants and a Simons fund.

[†]Department of Mathematics, Massachusetts Institute of Technology, USA and Beijing University, China. E-mail: tian@math.mit.edu

valued functions (g_{ij}) in local coordinates x_1, \dots, x_n , where n is the dimension of M . Recall that g is Riemannian if the matrices (g_{ij}) are positive definite.

Associated to each metric, there is a canonical connection, the Levi-Civita connection, ∇ characterized by the torsion freeness and $\nabla g = 0$, which means that g is parallel. In local coordinates,

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \Gamma_{ij}^k \frac{\partial}{\partial x_k}, \quad \Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l} \right), \tag{1.1}$$

where (g^{kl}) denotes the inverse of (g_{ij}) . Then the curvature (R_{ijkl}^i) is defined by

$$R_{ijkl}^i = \frac{\partial \Gamma_{jk}^i}{\partial x_l} - \frac{\partial \Gamma_{jl}^i}{\partial x_k} + \Gamma_{sk}^i \Gamma_{jl}^s - \Gamma_{sl}^i \Gamma_{jk}^s. \tag{1.2}$$

The curvature is completely determined by its sectional curvatures, that is, Gauss curvatures of surface cross sections. The Ricci curvature $\text{Ric} = (R_{ij})$ is given by taking trace of the curvature:

$$R_{ij} = \sum_k R_{ikj}^k. \tag{1.3}$$

The Ricci curvature essentially measures the variation of the volume form.

A metric g is called an Einstein metric if it satisfies the following Einstein equation

$$R_{ij} = \lambda g_{ij}, \tag{1.4}$$

where λ is a constant, usually called Einstein constant. (1.4) is the Euler-Lagrangian equation of the functional $\int_M s(g) dv$, where $s(g)$ denotes the scalar curvature of g , on the space of metrics with fixed volume. (1.4) is invariant under diffeomorphism group action. It is elliptic modulo diffeomorphisms when g is a Riemannian metric. From now on, I will assume that g is Riemannian.

The simplest examples of Einstein metrics include the euclidean metric on \mathbb{R}^n , the standard spherical metric on the unit sphere S^n and the hyperbolic metric on the unit ball $B^n \in \mathbb{R}^n$. In fact, all these metrics have constant sectional curvature 0 or 1 or -1 , consequently, their Einstein constant λ are 0, $n - 1$, $-n + 1$, respectively.

Every Riemannian 2-manifold (M, g) has a natural conformal structure. The classical Uniformization Theorem states that the universal covering of M together with the induced conformal structure is conformal to either \mathbb{C}^1 or S^2 or B^2 with canonical conformal structures. This implies that every Riemannian 2-manifold (M, g) admits a unique Einstein metric within its associated conformal class and with Einstein constant 0 or 1 or -1 .

Another way of proving this existence is to use partial differential equation. Given a Riemannian 2-manifold (M, g) , consider a metric \tilde{g} conformal to g , so it is of the form $e^\varphi g$. A simple computation shows that \tilde{g} is of constant curvature λ if and only if φ satisfies the following equation

$$\Delta \varphi + \frac{s(g)}{2} = \lambda e^\varphi. \tag{1.5}$$

This equation can be solved (cf. [Au], [KW]), so there is an Einstein metric in any given conformal class on any Riemannian 2-manifold. In early 1990's, R. Hamilton gave a heat flow proof of the Uniformization Theorem ([Ha], [Ch]). This new proof also yields a biproduct: The space of metrics on S^2 with positive curvature is contractible.

The uniformization for 2-manifolds led to two generalizations in higher dimensions. The one is the Yamabe problem (cf. [Au], [Sc]). The other is the Calabi's problem on Kähler-Einstein metrics which I will address more later.

For 3-manifolds, Einstein metrics are also of constant sectional curvature, so their universal coverings are either S^3 or \mathbb{R}^3 or hyperbolic 3-space H^3 . A major part of Thurston's program is to show the existence of metrics with constant sectional curvature on 3-manifolds which satisfy certain mild topological conditions. Thurston claimed long time ago that an atoroidal Haken manifold admits a complete hyperbolic metric. It will be interesting to have an analytic proof of this claim by solving the corresponding Einstein equation. In general, one hopes that any 3-manifold can be canonically split into some pieces of simple topological type and other pieces which admit Einstein metrics. There are at least two possible approaches to this: one is the variational method, trying to minimax certain functional involving curvatures, while the other is to use the Ricci flow, hoping that one can understand how the singularity is formed along the flow. So far none of them work yet.

When the dimension is higher than or equal to 4, an Einstein metric may not be of constant sectional curvature. It is still a very interesting question to find out topological constraints on Einstein manifolds. If a 4-manifold M admits an Einstein metric, then the Hitchin-Thorpe inequality says that $|\tau(M)| \leq \frac{2}{3}|\chi(M)|$. This implies that the connected sum of more than 4 copies of $\mathbb{C}P^2$ can not have any Einstein metric. More recently, C. Lebrun showed that a 4-manifold M with non-vanishing Seiberg-Witten invariant admits an Einstein metric only if $3\tau(M) \leq \chi(M)$. We do not know any constraints on compact Einstein manifolds of dimension higher than 4. It may be possible that any manifold of dimension ≥ 5 has an Einstein metric.

Examples of Einstein metrics can be constructed by exploring symmetries, such as, homogeneous Einstein metrics, cohomogeneity one Einstein metrics. One can also construct new Einstein metrics from known ones through certain intrigue constructions when the underlying manifolds are of special fibration structures (cf. [Wang], [BG]).

When $n \geq 4$, there is a special class of solutions of the Einstein equation, that is, Einstein metrics of special holonomy. If (M, g) is an irreducible Riemannian manifold, a well-known theorem of M. Berger states that either (M, g) is a locally symmetric space or its reduced holonomy is one of the following groups: $SO(n)$, $U(\frac{n}{2})$ ($n \geq 4$), $SU(\frac{n}{2})$ ($n \geq 4$), $Sp(1) \cdot Sp(\frac{n}{4})$ ($n \geq 8$), $Sp(\frac{n}{4})$ ($n \geq 4$) and two exceptional groups $Spin(7)$ and G_2 . We call (M, g) a Riemannian manifold with special holonomy if it is irreducible and its (reduced) holonomy is strictly contained in $SO(n)$. It can be shown that a Riemannian manifold of special holonomy is automatically Einstein if its holonomy is other than $U(\frac{n}{2})$. We have Kähler-Einstein

metrics for the $U(\frac{n}{2})$ case. In fact, these special Einstein metrics are of self-dual type. Each manifold (M, g) of special holonomy has a parallel $n - 4$ form defined as follows: Let $W \subset \Lambda^2 TM$ be the subspace generated by the Lie algebra of the holonomy group of (M, g) , then the curvatures lie in $S^2(W)$. Define a 4-form $\psi(W)$ by

$$\psi(W) = \sum w_i \wedge w_i, \tag{1.6}$$

where $\{w_i\}$ is an orthonormal basis of W . Clearly, it is independent of the choice of $\{w_i\}$ and is parallel. This 4-form induces a symmetric operator $T_{\psi(W)} : W \mapsto W$: $T_{\psi(W)}(v) = i_v \psi(W)$, where i_v denotes the interior product with v . One can check that $T_{\psi(W)}$ has at most two distinct eigenvalues. Moreover, there is a distinguished eigenspace W_0 of $T_{\psi(W)}$ of codimension 0, 1 and 3. Let β be the corresponding eigenvalue. Put

$$\Omega(W) = \frac{1}{\beta} * \psi(W). \tag{1.7}$$

Clearly, it is parallel. Denote by $S^2W = S^2W_0 + S^2W_1$ the decomposition according to eigenvalues, then the curvature $R(g)$ of g , which lies in S^2W , is decomposed into $R_0 \in S^2W_0$ and $R_1 \in S^2W_1$. Furthermore, we have

$$R_0 \wedge \Omega = *R_0 \tag{1.8}$$

and R_1 , which can be void, is completely determined by Ricci curvature of g . Therefore, manifolds of special holonomy are always self-dual. It is very important in the study of Einstein metrics of special holonomy. For example, the self-duality implies an a priori L^2 -bound on curvature: There is a uniform constant $C(p_1(M), \Omega, s(g))$, depending only on the first Pontrjagin class, Ω and the scalar curvature $s(g)$, such that for any Einstein metric g of special holonomy, we have

$$\int_M |R(g)|^2 dv = C(p_1(M), \Omega, s(g)). \tag{1.9}$$

Here Ω is the corresponding parallel form. In dimension 4, we can study self-dual Einstein metrics, that is, Einstein metrics with self-dual Weyl curvature. These self-dual metrics share similar properties as those with special holonomy do.

The special geometry we see most is the Kähler geometry. A Kähler manifold is a Riemannian manifold (M, g) whose holonomy lies in $U(\frac{n}{2})$, it is equivalent to saying that M has a compatible and parallel complex structure J , that is, $g(Ju, Jv) = g(u, v)$, where $u, v \in TM$ are arbitrary, and $\nabla J = 0$. So M is a complex manifold with induced complex structure by J . Usually, we denote g by its Kähler form $\omega_g = g(\cdot, J\cdot)$. In local complex coordinates z_1, \dots, z_m of M ($n = 2m$),

$$\omega_g = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^m g_{i\bar{j}} dz_i \wedge d\bar{z}_j, \tag{1.10}$$

where $(g_{i\bar{j}})$ is a positive Hermitian matrix-valued function. The self-duality simply means that the curvature of a Kähler metric has only components of type (1,1). A Kähler metric g is Einstein if and only if the trace of its curvature against ω_g is

constant, we call such a metric Kähler-Einstein. A necessary condition for the existence of Kähler-Einstein metric on M is that the first Chern class $c_1(M)$ is definite. Since the Ricci curvature of a Kähler metric g can be expressed as $-\partial\bar{\partial}\log\det(g_{i\bar{j}})$, the Einstein equation is reduced to solving the following complex Monge-Ampère equation

$$\det(g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z_i\partial\bar{z}_j}) = e^{h-\lambda\varphi} \det(g_{i\bar{j}}), \quad (g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z_i\partial\bar{z}_j}) > 0, \quad (1.11)$$

where φ is unknown and h is a given function depending only on g . This is a fully nonlinear elliptic equation and easier to solve.

A program initiated by E. Calabi in early 1950's is to study the existence and uniqueness of Kähler-Einstein metrics.¹ The uniqueness of Kähler-Einstein metrics was known in 1950's in the case that the first Chern class is nonpositive and was proved by Bando-Mabuchi [BM] in 1986 in the case that the first Chern class is positive. The difficult part of Calabi's program is about the existence. The celebrated solution of Yau [Ya] for the Calabi conjecture established the existence of a Ricci-flat metric, now named as Calabi-Yau metric, in each Kähler class on a compact Kähler manifold M with $c_1(M) = 0$. If $c_1(M) < 0$, the existence of Kähler-Einstein metrics was proved by Yau [Ya] and Aubin [Au], independently. There are further analytic obstructions to the existence of Kähler-Einstein metrics on M with $c_1(M) > 0$. Matsushima proved that M has a Kähler-Einstein metric only if the Lie algebra $\eta(M)$ of its holomorphic fields is reductive. Also if M has a Kähler-Einstein metric, then the Futaki invariant from [Fu] vanishes. The Futaki invariant is a character of $\eta(M)$. If M is a complex surface with $c_1(M) > 0$, then it admits a Kähler-Einstein metric if and only if the Lie algebra of holomorphic vector fields is reductive [Ti1]. For a general M with $c_1(M) > 0$, the existence of Kähler-Einstein metrics is equivalent to certain analytic stability [Ti2]. This analytic stability amounts to checking an nonlinear inequality of Moser-Trudinger type: Assume that $\eta(M) = \{0\}$.² If ω is a Kähler metric with $[\omega] = c_1(M)$ and φ with $\int_M \varphi \omega^n = 0$ and $\omega + \partial\bar{\partial}\varphi > 0$,

$$\log\left(\int_M e^{-\varphi}\omega^n\right) \leq J_\omega(\varphi) - f(J_\omega(\varphi)), \quad (1.12)$$

where f is some function bounded from below and satisfies $\lim_{t\rightarrow\infty} f(t) = \infty^3$ and J_ω is defined by

$$J_\omega(\varphi) = \sum_{i=0}^{n-1} \frac{i+1}{n+1} \frac{\sqrt{-1}}{2V} \int_M \partial\varphi \wedge \bar{\partial}\varphi \omega^i \wedge (\omega + \partial\bar{\partial}\varphi)^{n-i-1}, \quad (1.13)$$

where $V = \int_M \omega^n$. The inequality (1.12) has been checked for many manifolds, such as Fermat hypersurfaces. Furthermore, the analytic stability implies the asymptotic

¹Later in 1980's, E. Calabi extended this to extremal Kähler metrics, one can see X.X. Chen's paper in this proceeding for recent progresses on extremal metrics.

²If $\eta(M) \neq \{0\}$, then the inequality holds only for those functions perpendicular to functions induced by holomorphic vector fields.

³ f may depend on ω .

CM-stability of M introduced in [Ti2] in terms of Geometric Invariant Theory. If one proved the partial C^0 -estimate conjectured in [Ti3], this asymptotic stability in [Ti2] would imply the existence of Kähler-Einstein metrics. Very recently, by using the Tian-Yau-Zeldich expansion (cf. [Ti4], [Cat], [Zel]) and a result of Zhiqin Lu [Lu], S. Donaldson [Do1] proved the asymptotic Chow stability [Mu] of algebraic manifolds which admit Kähler-Einstein metrics [Do1]. This gives a partial answer to one conjecture of Yau: If $\eta(M) = 0$, then there is a Kähler-Einstein metric on M if and only if M is asymptotically Chow stable. It would be a very interesting problem in algebraic geometry to compare the Chow stability with the CM-stability introduced in [Ti2]. Both stabilities can be defined in terms of the Chow coordinate of M , but their corresponding polarizations are different (cf. [Paul]).

Kähler-Ricci solitons arose naturally from the study of the existence of Kähler-Einstein metrics and Hamilton's Ricci flow in Kähler geometry and generalize Kähler-Einstein metrics. A Kähler metric g is a Kähler-Ricci soliton if there is a holomorphic field X such that

$$\text{Ric}(g) - \lambda\omega_g = L_X\omega_g. \quad (1.14)$$

As before, this equation can be reduced to a slightly more complicated complex Monge-Ampere equation (cf. [TZ]). It was proved in [TZ] that Kähler-Ricci solitons are unique modulo automorphisms. In subsequent papers, we also gave an analytic criterion for the existence as one did in [Ti2]. It was conjectured that given any Kähler manifold M with $c_1(M) > 0$, either M has a Kähler-Einstein metric or there are diffeomorphisms ϕ_i and Kähler metrics g_i such that $\phi_i^*g_i$ converge to a unique Kähler-Ricci soliton on M' , which may be different from M . This conjecture was posed by R. Hamilton in studying the Ricci flow and myself in studying Kähler-Einstein metrics. When the complex dimension of M is 2, in view of the main result in [Ti1], it suffices to show that the blow-up of $\mathbb{C}P^2$ at two points admits a Kähler-Ricci soliton. This should be doable.

So far, the most successful method in proving the existence is the continuity method. The other possible approach is to use the Kähler-Ricci flow, which has only partial success (cf. X.X.Chen's talk at this ICM).

There remain many problems in studying Kähler-Einstein with prescribed singularities, though a lot has been done (cf. [CY], [TY1], [Ts], etc.). A given Kähler manifold M may not have definite first Chern class, so it does not admit any Kähler-Einstein metrics, but by blowing down certain subvarieties, the resulting manifold (possibly singular) may admit a canonical Kähler-Einstein metric. For instance, if M is an algebraic manifold of general type, can any given Kähler metric be deformed along the Kähler-Ricci flow to a unique Kähler-Einstein metric? Is the limiting metric independent of the initial metric? The answer to these questions seems to be affirmative in complex dimension 2 or in the case that minimal models exist. Another unsolved problem is Yau's conjecture: Every complete Calabi-Yau open manifold can be compactified such that the divisor at infinity is the zero-locus of a section of a line bundle proportional to the anti-canonical bundle. This is a hard problem. In [TY2], [TY3] and [BK], complete Calabi-Yau manifolds were constructed on complements of a smooth divisor which is a fraction of anti-canonical

divisor and satisfies certain positivity conditions (also see [Jo]). In view of these and [CT1], one is led to the following conjecture: a complete Calabi-Yau manifold M with quadratic curvature decay and euclidean volume growth is of the form $M = \overline{M} \setminus D$ such that D is ample near D and the anti-canonical bundle K_M^{-1} is $\alpha[D]$ for some $\alpha > 1$. This can be considered as the refinement of Yau's conjecture in a special case.

The next special holonomy is contained in $Sp(1)Sp(\frac{n}{4})$. Riemannian manifolds with such a holonomy are called quaternion-Kähler manifolds. They are automatically Einstein. The prototype is the quaternionic projective space $\mathbf{P}_{\mathbf{H}}^{\frac{n}{4}}$. There are many examples of quaternion-Kähler manifolds due to the works of many people, including S. Salamon, Galicki-Lawson, Lebrun, etc. Quaternion-Kähler manifolds with zero scalar curvature are hyperKähler, that is, its holonomy lies in $Sp(\frac{n}{4})$. The existence of hyperKähler manifolds follows from Yau's solution for the Calabi conjecture. However, we do not know yet if there are quaternion-Kähler manifolds with positive scalar curvature and which are not locally symmetric, while we do have a number of symmetric ones, the so-called Wolf spaces. It led Lebrun and S. Salamon to guess that the Wolf spaces are all complete quaternion-Kähler manifolds with positive scalar curvature. So far, it has been checked up to dimension $\frac{n}{4} \leq 3$. We would like to point out that there are many non-symmetric quaternion-Kähler orbifolds with positive scalar curvature due to Galicki-Lawson ([GL]).

Riemannian manifolds with holonomy G_2 and $Spin(7)$ must be Ricci-flat and of dimension 7 and 8, respectively. It took a long time to settle the question of whether such metrics exist, even locally. Local metrics with these holonomy were constructed by R. Bryant [Br]. Later, complete examples were constructed by Bryant and S. M. Salamon[BS]. Examples of compact 7- and 8-manifolds with holonomy G_2 and $Spin(7)$ were first constructed by D. Joyce in early 1990's (cf. [Jo]). D. Joyce's construction was inspired by the Kummer construction: metrics with holonomy $SU(2)$ on the $K3$ surface can be obtained by resolving the 16 singularities of the orbifolds T^4/Z_2 , where Z_2 acts on T^4 with 16 fixed points. In the case of G_2 , Joyce chooses a finite group $\Gamma \subset G_2$ of automorphisms of the torus T^7 . Then he resolves the singularities of T^7/Γ to get a compact 7-manifold M with holonomy G_2 . A similar construction can be implemented for the $Spin(7)$ case by choosing a finite group Γ of automorphisms of the torus T^8 and a flat Γ -invariant $Spin(7)$ -structure on T^8 . More recently, Kovalev gave a new construction of Riemannian metrics with special holonomy G_2 on compact 7-manifolds. The construction is based on gluing asymptotically cylindrical Calabi-Yau manifolds built up on the work in [TY2]. Examples of new topological types of compact 7-manifolds with holonomy G_2 were obtained.

So far, all Ricci-flat compact manifolds are of special holonomy. There should exist complete Ricci-flat manifolds with generic holonomy $SO(n)$. The question is how we can find them. Here is a possible example in 4-dimension: It was shown that there is a Calabi-Yau manifold with cylindrical end asymptotic to T^3 [TY2]. Complex analytically, this manifold can be obtained by blowing up the 9 base points of a generic elliptic pencil on $\mathbb{C}P^2$ and removing one smooth fiber. Now take two copies of such Calabi-Yau manifolds and glue them along the T^3 's at infinity. One

way of gluing them is to respect the complex structures, then we will get a $K3$ surface. Could one use different gluing maps which do not preserve the complex structures, so that one may obtain new Ricci-flat manifolds with generic holonomy? We can also ask if any complete Ricci-flat manifolds can be decomposed in some sense into a connected sum of Calabi-Yau manifolds. Similar things can be done in higher dimensions.

Geometry of moduli space of Einstein manifolds is extremely important. For example, the moduli space of Calabi-Yau manifolds provides the B-model in the Mirror Symmetry. If (M, g) is an Einstein manifold with special holonomy, then it was proved that nearby Einstein manifolds in the moduli space is also of special holonomy. The first analytic problem about the moduli is its compactness. The moduli space is very often noncompact, so we need to compactify it. Then we can consider what structures a compactified moduli space has.

We have pointed out before that for any Einstein manifold (M, g) with special holonomy, the L^2 -norm of its curvature depends only on the second Chern character and the Einstein constant λ (assuming that the volume of M is normalized, say 1). One can first give a weak compactification $\overline{\mathcal{M}}$ of the moduli space \mathcal{M} of Einstein manifolds with special holonomy in the Gromov-Hausdorff topology. A basic problem is the regularity of a limit in $\overline{\mathcal{M}} \setminus \mathcal{M}$. There are two cases of the limit, one is when the limit is still compact, while the other has infinite diameter as a length space. Here let us consider only the first case, since we know much more in this case and it is necessary for studying the second. If M_∞ is a compact limit, then there is a sequence of Einstein manifolds (M_i, g_i) with special holonomy and bounded diameter converging to M_∞ in the Gromov-Hausdorff topology. When the dimension is 2, it was proved in [Ti1] that M_∞ is a Kähler-Einstein orbifold with isolated singularities. Its real version was done by M. Anderson in [An]. The compactness theorem played a very important role in the resolution of the Calabi problem for complex surfaces (cf. [Ti1]). In [CT2], Cheeger and I proved

Theorem 1.1. ⁴ *Let M_∞ be the above limit of a sequence of Einstein manifolds (M_i, g_i) with the same special holonomy and uniformly bounded diameter. Then there is a rectifiable closed subset $S \subset M_\infty$ such that $M_\infty \setminus S$ is a smooth manifold which admits an Einstein metric g_∞ with the same holonomy as (M_i, g_i) do. Furthermore, M_∞ is the metric completion of $M_\infty \setminus S$ with respect to the distance induced by g_∞ .*

This is based on deep works of Cheeger-Colding [CC] on spaces which are limits of manifolds with Ricci curvature bounded from below, my joint work with Cheeger and Colding on structure of the singular sets of limits of Einstein manifolds with L^2 curvature bounds [CCT] and Cheeger's work on rectifiability of singular sets of the limits [Che]).

Remark 1.2. In fact, the convergence can also be strengthened: There is an exhaustion of $M_\infty \setminus S$ by compact sets $K_i \subset K_{i+1} \cdots$ and diffeomorphisms $\phi_i : K_i \rightarrow M_i$ such that $\phi_i(K_i)$ converge to S in the Gromov-Hausdorff topology and $\phi_i^* g_i$ converge to g_∞ in the C^∞ -topology.

⁴The Kähler case of this theorem was proved in [CCT].

The most fundamental problem left is the regularity of S or structure of M_∞ along S . The conjecture is that S can be stratified into $\coprod_{i \leq n-4} S_i$ such that each stratum S_i is a smooth manifold of dimension $\leq i$. If (M_i, g_i) are Kähler-Einstein, then $S_{2j+1} = S_{2j}$. We know (cf. [CCT], [CT2]) that tangent cones at almost all points of S_{n-4} are of the form $\mathbb{R}^{n-4} \times C(S^3/\Gamma)$, where $C(S^3/\Gamma)$ is the cone over S^3/Γ and $\Gamma \subset SO(4)$ is a finite group. It makes us conjecture that M_∞ should be homeomorphic to an open set of $\mathbb{R}^{n-4} \times C(S^3/\Gamma)$ locally along $S_{n-4} \setminus \coprod_{i \leq n-5} S_i$. It is plausible that M_∞ is actually smooth along $S_{n-4} \setminus \coprod_{i \leq n-5} S_i$ in a suitable sense. We also believe that the $(n-4)$ -form Ω_∞ associated to the special holonomy of g_∞ extends to S in a suitable sense and its restriction to S is the same as the volume, i.e., S is calibrated by Ω_∞ in a suitable sense.

When (M_i, g_i) are Kähler-Einstein manifolds with positive scalar curvature, it was conjectured by the author long time ago that a multiple of the anticanonical bundle of $M_\infty \setminus S$ extends to be a line bundle across the singular set S . This is of course true if one can show that M_∞ has only quotient singularities. The affirmation of this conjecture will enable us to prove the converse of a result in [Ti2], that is, the algebraic stability of a Kähler manifold with positive first Chern class assures the existence of Kähler-Einstein metrics.

Finally, we shall refer the readers to [CT2] for detailed study of tangent cones at any singularity of M_∞ .

2. Yang-Mills equation

Next we discuss the Yang-Mills equation. The Yang-Mills equation has played a fundamental role in our study of physics and geometry and topology in last few decades. In the following, unless specified, we assume that (M, g) is a Riemannian manifold of dimension n and \mathbf{G} is a compact subgroup in $\mathbf{SO}(r)$ and \mathfrak{g} is its Lie algebra. Let E be a \mathbf{G} -bundle over M .

First we recall that a connection of E over M is locally of the form

$$A = A_i dx_i, \quad A_i \in \mathfrak{g} \tag{2.1}$$

where x_1, \dots, x_n are euclidean coordinates of \mathbb{R}^n and A_i are matrices in \mathfrak{g} . Its curvature can be computed as follows:

$$F_A = dA + A \wedge A. \tag{2.2}$$

The Yang-Mills functional is defined on the space of connections and given by

$$\mathcal{Y}(A) = \frac{1}{4\pi^2} \int_M |F_A|_g^2 dV_g. \tag{2.3}$$

The Yang-Mills equation is simply its Euler-Lagrange equation

$$D_A^* F_A = 0, \tag{2.4}$$

where D_A denotes the covariant derivative of A and D_A^* is its adjoint. On the other hand, being the curvature of a connection, A automatically satisfies the second

Bianchi identity $D_A F_A = 0$. We will call A a Yang-Mills connection if it satisfies (2.4).

The gauge group \mathcal{G} consists of all sections of $Ad(E)$ over M , locally, they are just maps into $\mathbf{G} \subset \mathbf{SO}(r)$. It acts on the space of connections by assigning A to $\sigma(A) = \sigma A \sigma^{-1} - \sigma d\sigma^{-1}$ for each $\sigma \in \mathcal{G}$. Clearly, the Yang-Mills functional is invariant under the action of \mathcal{G} , so does the Yang-Mills equation. In particular, it implies that the Yang-Mills equation is not elliptic. A difficult problem is to construct good gauges which can be controlled by curvatures. So called Coulomb gauges have been constructed by Uhlenbeck [Uh2] in $L^{n/2}$ -norms and more recently, by Tao-Tian and Meyer-Riviere in Morrey norms (cf. [TT]).

The simplest Yang-Mills connections are provided by harmonic one forms: If $G = U(1)$, then $\mathfrak{g} = i\mathbb{R}$ and A is simply a one-form and the Yang-Mills equation is $d^*dA = 0$, the gauge transformation is given by $\sigma = e^{ia} \mapsto A + ida$. It follows that modulo gauge transformations, abelian Yang-Mills connections are in one-to-one correspondence with harmonic one forms.

Now we assume that (M, g) is of special holonomy. Let Ω be the associated closed form of degree $n - 4$. We say that a connection A is Ω -self-dual if

$$*(\Omega \wedge F_A) = F_A, \quad (2.5)$$

where $*$ is the Hodge operator ⁵

One can show that an Ω -self-dual connection is a Yang-Mills connection. Clearly, the self-duality is invariant under gauge transformations. So self-dual connections provide a special class of Yang-Mills solutions.

There are many examples of Ω -self-dual connections. First, the Levi-Civita connection of the underlying Riemannian metric is Ω -self-dual. In this sense, the Yang-Mills equation is a semi-linear version of the Einstein equation. Secondly, if E is a stable holomorphic vector bundle, then the Donaldson-Yau-Uhlenbeck theorem ([Do2], [UY]) states that E has a unique Hermitian-Yang-Mills connection, an easy computation shows that a connection is Hermitian-Yang-Mills if and only if it is Ω -self-dual, where $\Omega = -\frac{\omega^{n/2-2}}{(n/2-2)!}$ and ω is the underlying Kähler form. Thirdly, if (M, g) is a Calabi-Yau 4-fold and Ω is its associated $(n - 4)$ -form induced by the $SU(4) \subset Spin(7)$ -structure, then Ω -self-dual connections are just complex self-dual instantons of Donaldson-Thomas [DT]. Also, one may construct Ω -self-dual instantons from Ω -calibrated submanifolds of M .

When $n = 4$, self-dual instantons were used to construct the Donaldson invariants for 4-manifolds. This eventually led to the Seiberg-Witten invariants, which is much easier to compute. The construction goes roughly as follows: Let M be a 4-manifold and g is a generic metric. Let E be an $SU(2)$ -bundle over M . Consider the moduli space \mathcal{M}_E of self-dual instantons of E , that is, solutions of

$$F_A = *F_A \quad (2.6)$$

⁵If $\mathbf{G} \subset U(r)$, one can consider more general self-dual equation: A is an Ω -self-dual connection if $\text{tr}(F_A)$ is harmonic and $*(\Omega \wedge F_A^0) = F_A^0$, where $F_A^0 = F_A - \frac{1}{r}\text{tr}(F_A)Id$. If $\mathbf{G} = SU(r)$, this coincides with (2.5).

modulo gauge transformations. A generalized instanton consists of an anti-self-dual instanton and a tuple of points of M such that the second Chern class of the instanton and totality of the points sum up to represent $c_2(E)$. Let $\overline{\mathcal{M}}_E$ be the moduli space of all generalized instantons of E . Then Uhlenbeck compactness theorem states that $\overline{\mathcal{M}}_E$ is compact. Also $\overline{\mathcal{M}}_E$ is a stratified space with \mathcal{M}_E as its main stratum. If $b_2^+(M) \geq 3$ and g is generic, then each stratum has expected dimension which can be easily computed by the Atiyah-Singer index theorem, so $\overline{\mathcal{M}}_E$ can be taken as a fundamental class. The Donaldson invariants are obtained by integrating pull-backs of cohomology classes of M on this fundamental class.

Similarly, one can define the Seiberg-Witten invariant by using the Seiberg-Witten equation. Technically, it is much easier since the moduli space is already compact. Sometimes, it was said that the Seiberg-Witten invariant does not need hard analysis, in fact, it is false. Taubes' deep theorem on equivalence of Seiberg-Witten and Gromov-Witten invariants requires hard analysis.

What about higher dimensional cases? Can we construct new deformation invariants by using Ω -self-dual connections? In order to achieve it, one has to consider the following issues: 1. Is the corresponding self-dual equation elliptic? Indeed, the self-dual equation on a $Spin(7)$ -manifold is elliptic. 2. Can we compactify the moduli space? If so, how do we stratify the compactified moduli space? 3. Does each stratum have right dimension which can be predicted by the index theorem? If we solve these issues, we can define new invariants, then we can study how to compute them.

The first issue is easy to check. We just need to linearize the self-dual equation and see if it is elliptic. There are examples, such as, self-dual equations on 4-manifolds and Donaldson-Thomas complex self-dual on Calabi-Yau 4-manifolds. It will be very useful to construct deformation invariants by using complex self-dual instantons. The success of it will provide a powerful tool of constructing holomorphic cycles of codimension 4, which are pretty much evading us.

Next we consider the compactification. Having a good compactification, we will be able to get property 3 in the above. Let (M, g) be a compact Riemannian manifold of dimension n and with special holonomy. Let Ω be the associated closed form of degree $n - 4$. Let E be a unitary vector bundle over M . Recall that $\mathfrak{M}_{\Omega, E}$ consists of all gauge equivalence classes of Ω -asd instantons of E over M . In general, $\mathfrak{M}_{\Omega, E}$ may not be compact. So we will compactify it.

An admissible Ω -self-dual instanton is simply a smooth connection A of E over $M \setminus S(A)$ for a closed subset $S(A)$ of Hausdorff dimension $n - 4$ such that $\int_M |F_A|^2 < \infty$. A generalized Ω -self-dual instanton is made of an admissible Ω -self-dual instanton A of E and a closed integral current $C = C_2(S, \Theta)$ calibrated by Ω , such that cohomologically,

$$[C_2(A)] + \text{PD}[C_2(S, \Theta)] = C_2(E), \tag{2.7}$$

where $C_2(A)$ denotes the Chern-Weil form of A^6 and $C_2(E)$ denotes the second Chern class of E . Two generalized Ω -self-dual instantons $(A, C), (A', C')$ are equiv-

⁶One can show this form, which was originally defined on $M \setminus S(A)$, extends to a well-defined current on M .

alent if and only if $C = C'$ and there is a gauge transformation σ on $M \setminus S(A) \cup S(A')$, such that $\sigma(A) = A'$ on $M \setminus S(A) \cup S(A')$. We denote by $[A, C]$ the gauge equivalence class of (A, C) . We identify $[A, 0]$ with $[A]$ in $\mathcal{M}_{\Omega, E}$ if A extends to a smooth connection of E over M modulo a gauge transformation. We define $\overline{\mathfrak{M}}_{\Omega, E}$ to be set of all gauge equivalence classes of generalized Ω -self-dual instantons of E over M .

The topology of $\overline{\mathfrak{M}}_{\Omega, E}$ can be defined as follows: a sequence $[A_i, C_i]$ converges to $[A, C]$ in $\overline{\mathfrak{M}}_{\Omega, E}$ if and only if there are representatives (A_i, C_i) such that their associated currents $C_2(A_i, C_i)$ converge weakly to $C_2(A, C)$ as currents, where

$$C_2(A', C') = C_2(A') + C_2(S', \Theta'), \quad C' = (S', \Theta'). \tag{2.8}$$

It is not hard to show that by taking a subsequence if necessary, $\tau_i(A_i)$ converges to A outside $S(A)$ and the support of C for some gauge transformations τ_i .

The following was proved in [Ti5] and provides a compactification for the moduli space of Ω -self-dual connections.

Theorem 2.1. *For any M, g, Ω and E as above, $\overline{\mathfrak{M}}_{\Omega, E}$ is compact with respect to this topology.*

Of course, $\overline{\mathfrak{M}}_{\Omega, E}$ admits a natural stratification. The remaining problem, which is also important for issue 3, is about regularity of a generalized Ω -self-dual instanton. Another interesting problem is to develop a deformation theory of smoothing singular self-dual instantons. Are there any constraints on a singular self-dual instanton which is the limit of smooth self-dual instantons? We do not even know any example of a Hermitian-Yang-Mills connection with an isolated singularity and which can be approximated by smooth Hermitian-Yang-Mills connections.

Let us give an example. Assume that (M, g) is a Kähler manifold with Kähler form ω . Put $\Omega = \omega^{m-2}/(m-2)!$, where $n = 2m$. Then an Ω -self-dual instanton A is simply a Hermitian-Yang-Mills connection, that is $F_A^{0,2} = 0$ and $F_A^{1,1} \cdot \omega = 0$, where $F_A^{k,l}$ is the (k, l) -part of F_A . If (A, C) is a generalized Ω -self-dual instanton, it follows from a result of J. King that there are positive integers m_a and irreducible complex subvarieties V_a such that for any smooth φ with compact support in M ,

$$C_2(S, \Theta)(\varphi) = \sum_a m_a \int_{V_a} \varphi.$$

On the other hand, using a result of Bando-Siu, one can show [TYa] that there is a gauge transformation τ such that $\tau(A)$ extends to be a smooth connection outside a complex subvariety of codimension greater than 2.

We expect that general self-dual connections have analogous properties. If (A, C) is a generalized Ω -self-dual connection, we would like to have (1) the regularity of the current C , that is, C is presented by finitely many Ω -calibrated subvarieties with integral multiplicity; (2) There is a gauge transformation σ such that $\sigma(A)$ extends to a smooth connection outside a subvariety of codimension at least 6. In [Uh2], any isolated singularity of a Yang-Mills connection in dimension 4 can be removed. In [TT], a removable singularity theorem was established for stationary Yang-Mills connections in higher dimensions. Using this, we can conclude

that $\sigma(A)$ extends to a smooth connection outside a closed subset S with vanishing $(n-4)$ -Hausdorff measure. Further understanding on S is needed. We will discuss regularity of C in the next section.

A particularly interesting case is the complex self-dual instanton. We do expect to construct new invariants for Calabi-Yau 4-folds by showing that the above moduli space of generalized complex self-dual instantons gives rise to a fundamental class. The main problem left is the regularity of generalized instantons. A special case of this can be done nicely. If the underlying Calabi-Yau 4-fold M is of the form $Y \times T_{\mathbb{C}}^1$, where Y is a Calabi-Yau 3-fold and $T_{\mathbb{C}}^1$ is a complex 1-torus, then a $T_{\mathbb{C}}^1$ -invariant complex self-dual instanton is given by a Hermitian Yang-Mills connection A on Y and a $(0,3)$ -form f with $\bar{\partial}^* f = 0$. The expected dimension of its moduli space is zero. Counting them with sign gives rise to the holomorphic Casson invariant, which was constructed previously by R. Thomas using the virtual moduli cycle construction in algebraic geometry[Th].

Other analytic problems on the Yang-Mills equation include whether or not the Yang-Mills flow develops singularity at finite time. It was proved by Donaldson that the Yang-Mills flow along Hermitian metrics of a holomorphic bundle has global solution. Of course, if the dimension of the underlying manifold is less than 4, the Yang-Mills flow has a global solution. In general, it is still open. If a singularity forms at finite time, how does it look like?

3. Minimal submanifolds

The study of minimal submanifolds is a classical topic. We will not intend to cover all aspects of this topic. We will only discuss issues related to previous discussions and particularly self-dual type solutions of the minimal submanifold equation.

Let (M, g) be an n -dimensional Riemannian manifold and S be a submanifold in M . Recall that S is minimal if its mean curvature H_S vanishes. The mean curvature arises from the first variation of volume of submanifolds. Minimal submanifolds are closely related to the Yang-Mills equation. In fact, it was shown in [Ti5] that if a Yang-Mills connection has its curvature concentrated along a submanifold, then this submanifold must be minimal and of codimension 4. Motivated by this, recently, S. Brendle, etc. developed a deformation theory of constructing Yang-Mills connections from minimal submanifolds.

Now assume that M has a closed differential form Ω with its norm $|\Omega| \leq 1$. A submanifold S is calibrated by Ω if $\Omega|_S$ coincides with the induced volume form. Calibrated submanifolds are minimal (cf. [HL]). The study of calibrated submanifolds was pioneered in the seminal work [HL] of Harvey and Lawson. It now becomes extremely important in the string theory. As we have seen in the above, they also appear in formation of singularity in the Einstein equation. In particular, when a self-dual connection has its curvature concentrated along a submanifold, this submanifold is calibrated [Ti5], so a calibrated submanifold can be regarded as a self-dual solution of the minimal submanifold equation.

Let (M, ω) be a symplectic manifold and J be a compatible almost complex

structure, that is, $\omega(Ju, Jv) = \omega(u, v)$ and $\omega(u, Ju) > 0$ for any non-zero tangent vectors u and v . Define a compatible metric g by $g(u, v) = \omega(u, Jv)$. Any ω -calibrated submanifolds are J -holomorphic curves, that is, each tangent space is a J -invariant subspace in TM . They are particularly minimal with respect to g . Holomorphic curves have been used to establish a mathematical foundation of the quantum cohomology, the mirror symmetry, etc. (cf. [RT]). The key of it is to construct the Gromov-Witten invariants by showing the moduli space of J -holomorphic curves can be taken as a fundamental class in a suitable sense. This was proven by first constructing a “nice” compactification of the moduli space of J -holomorphic curves and then applying appropriate transversality theory.

I do believe that there should be new invariants by using other calibrated submanifolds. A particularly interesting case is the Cayley cycles in a $\text{Spin}(7)$ -manifold. Note that a Calabi-Yau 4-fold is a special $\text{Spin}(7)$ -manifold. Again the problem is about the structure of singular Cayley cycles. This proposed new invariants will provide a powerful tool of constructing Cayley cycles in a $\text{Spin}(7)$ -manifold, particularly, holomorphic and special Lagrangian cycles in a Calabi-Yau 4-fold. A related problem is to construct new invariants for hyperKähler manifolds by using tri-holomorphic maps. A good compactification for the moduli of tri-holomorphic maps is needed, but this should be technically easier. Partial results have been obtained in [LiT] and [CL1].

Another possible invariant may exist for Calabi-Yau manifolds. Let (M, ω) be a Calabi-Yau n -fold with a holomorphic n -form ω such that

$$\omega^n (-1)^{\frac{n(n-1)}{2}} n! \left(\frac{\sqrt{-1}}{2} \right)^n \Omega \wedge \bar{\Omega}. \quad (3.1)$$

A special Lagrangian submanifold is a submanifold $L \subset M$ such that $\omega|_L = 0$ and Ω restricts to the induced volume form of L . If one has a good compactification theorem for special Lagrangian submanifolds, then one can count them to obtain a new invariant for M . A particularly important case is for Calabi-Yau 3-folds.

The minimal equation is nonlinear and does not have singular solutions. So one has to introduce weak solutions. An integral k -dimensional current $C = (S, \Theta, \xi)$ consists of a k -dimensional rectifiable set S ⁷ of locally finite Hausdorff measure, an H^k -integrable integer-valued function Θ and a k -form $\xi \in \wedge^k TS$ with unit norm. Each current induces a natural functional Φ_C on smooth forms with compact support: For any smooth form φ ,

$$\Phi_C(\varphi) = \int_S \langle \varphi, \xi \rangle dH, \quad (3.2)$$

where dH^k denotes the k -dimensional Hausdorff measure. We say C has no boundary if $\Phi_C(d\psi) = 0$ for any ψ . One can define the generalized mean curvature of C as the variation of volume. A current C is minimal if its mean curvature vanishes. A current C is calibrated by a k -form Ω if $\Omega|_{T_x S}$ coincides with the induced volume form whenever the tangent space $T_x S$ exists. Of course, a calibrated current is minimal, provided that $d\Omega = 0$ and $|\Omega| \leq 1$.

⁷This implies that there is a unique tangent space at a.e. point of S .

A fundamental problem in the regularity theory of minimal surfaces is the regularity of minimizing currents. A result of F. Almgren claims that an area minimizing current is regular outside a subset of Hausdorff codimension two [Alm]. In many geometric applications, we will encounter with calibrated currents, for example, in the famous work of Taubes on equivalence of the Seiberg-Witten invariants and the Gromov invariants, the key technical point is to show that any ω -calibrated current in a symplectic 4-manifold (M, ω) is a classical minimal surface, i.e., the image of a pseudo-holomorphic map from a smooth Riemann surface ([Ta], also see [RiT]).⁸ This current is obtained as an adiabatic limit of curvature forms of solutions of deformed Seiberg-Witten equations. The problems of this type should also occur when we study the Calabi-Yau manifolds near large complex limits. Of course, this regularity problem also appears in compactifying moduli spaces of calibrated cycles.

Here is what we think should be true: If $C = (S, \Theta, \xi)$ is a k -dimensional calibrated current, then S can be stratified into $\coprod_i S_i$ such that each stratum S_i is a smooth manifold of dimension i , which is at most $k - 2$, and Θ is constant on each stratum.

If the calibrating form Ω is $\omega^l/l!$ ($k = 2l$) on a symplectic manifold (M, ω) with a compatible metric g , then Ω -calibrated current is pseudo-holomorphic, that is, any tangent space is invariant under the almost complex structure induced by ω and g . In this special case, the conjecture is that S is stratified into pseudo-holomorphic strata S_{2j} . If the dimension of C is 2, then the conjecture claims that C is induced by a pseudo-holomorphic curve. This conjecture follows from a result of King when (M, g) is Kähler, i.e., the corresponding almost complex structure is integrable. Very recently, Riviere and I can prove this conjecture when $\dim C = 2$. When $\dim M = 4$, it was already known (cf. [Ta], [RiT]).

A nice way of deforming a surface into a minimal one is to use the mean curvature flow. If (M, g) is a compact Kähler-Einstein surface and S_0 is a symplectic surface with respect to the Kähler form, then surfaces along the mean curvature flow starting from S_0 are also symplectic [ChT]. If the flow has a global solution, then S_0 can be deformed to a symplectic minimal surface. In particular, S_0 is isotopic to a symplectic minimal surface. However, it is highly nontrivial to show that the flow has a global solution. Partial results have been obtained ([CL2], [WaM]). Nevertheless, it was conjectured in [Ti6] that any symplectic surface in a Kähler-Einstein surface is isotopic to a symplectic minimal surface. This has been checked for a quite big class of symplectic surfaces in a Kähler-Einstein surfaces with positive scalar curvature (cf. [ST]) by using pseudo-holomorphic curves. One can also ask similar questions for the mean curvature flow along Lagrangian submanifolds. It was proved first by Smoczyk [Sm] that the mean curvature flow preserves the Lagrangian property. We refer [ThY] for more discussions on the mean curvature flow for Lagrangian submanifolds.

⁸This is a special case of the main result in [Sch], which states that a 2-dimensional area minimizing is a classical minimal surface. But Chang's proof relies on some hard techniques of [Alm], so a self-contained proof is very desirable.

References

- [Alm] F. Almgren, *Almgren's big regularity paper. Q -valued functions minimizing Dirichlet's integral and the regularity of area-minimizing rectifiable currents up to codimension 2*, World Scientific Monograph Series in Mathematics, 1. World Scientific Publishing Co., Inc., 2000.
- [An] M. Anderson, *Ricci curvature bounds and Einstein metrics on compact manifolds*, J. Amer. Math. Soc., 2 (1989), no. 3, 455–490.
- [Au] T. Aubin, *Some nonlinear problems in Riemannian geometry*. Springer, Berlin-Heidelberg-New York, (1997).
- [BG] C. Boyer and K. Galicki, *3-Sasakian manifolds*, Surveys in differential geometry: Essays on Einstein manifolds, 123–184, Surv. Differ. Geom., VI, Int. Press, Boston, MA, 1999. S. Bando and R. Kobayashi, *Ricci-flat Kähler metrics on affine algebraic manifolds*, Math. Ann., 287 (1990), no. 1, 175–180.
- [BK] S. Bando and R. Kobayashi, *Ricci-flat Kähler metrics on affine algebraic manifolds. II*, Math. Ann., 287 (1990), no. 1, 175–180.
- [BM] S. Bando and T. Mabuchi, *Uniqueness of Einstein Kähler metrics modulo connected group actions*, Algebraic Geometry, Adv. Studies in Pure Math., 10 (1987).
- [Br] R. Bryant, *Metrics with exceptional holonomy*, Ann. of Math., (2) 126 (1987), no. 3, 525–576.
- [BS] R. Bryant and S. Salamon, *On the construction of some complete metrics with exceptional holonomy*, Duke Math. J., 58 (1989), no. 3, 829–850.
- [Cat] D. Catlin, *The Bergman kernel and a theorem of Tian*, Analysis and geometry in several complex variables (Katata, 1997), 1–23, Trends Math., Birkhuser Boston, Boston, MA, 1999.
- [CC] J. Cheeger and T. Colding, *On the structure of spaces with Ricci curvature bounded below, I-III*, J. Differential Geom., 46 (1997), no. 3, 406–480 and 54 (2000), no. 1, 13–74.
- [CCT] J. Cheeger, T. Colding and G. Tian, *Constraints on singularities under Ricci curvature bounds*, C. R. Acad. Sci. Paris Sr. I Math. 324 (1997), no. 6, 645–649 (Its full version will appear in GAFA).
- [Ch] B. Chow, *The Ricci flow on the 2-sphere*, J. Differential Geom., 33 (1991), no. 2, 325–334.
- [Che] J. Cheeger, *L_p -bounds on curvature, elliptic estimates and rectifiability of singular sets*, C. R. Math. Acad. Sci. Paris 334 (2002), no. 3, 195–198.
- [ChT] J.Y. Chen and G. Tian, *Moving symplectic curves in Kähler-Einstein surfaces*, Acta Math. Sin. (Engl. Ser.), 16 (2000), no. 4, 541–548.
- [CL1] J.Y. Chen and Jiayu Li, *Quaternionic maps between hyperkähler manifolds*, J. Differential Geom., 55 (2000), no. 2, 355–384.
- [CL2] J.Y. Chen and Jiayu Li, *Mean curvature flow of surface in 4-manifolds*, Adv. Math., 163 (2001), no. 2, 287–309.
- [CT1] J. Cheeger and G. Tian, *On the cone structure at infinity of Ricci flat manifolds with Euclidean volume growth and quadratic curvature decay*, Invent. Math., 118 (1994), no. 3, 493–571.
- [CT2] J. Cheeger and G. Tian, *Compactifying moduli of special Einstein manifolds*,

in preparation.

- [CY] S.Y. Cheng and S.T. Yau, *On the existence of a complete Kähler metric on noncompact complex manifolds and the regularity of Fefferman's equation*, Comm. Pure Appl. Math., 33 (1980), no. 4, 507–544.
- [Do1] S. Donaldson, *Scalar curvatures and projective imbeddings, I*, preprint, 2001.
- [Do2] S. Donaldson, *Infinite determinants, stable bundles and curvature*, Duke Math. J., 54 (1987), no. 1, 231–247.
- [DT] S. Donaldson and R. Thomas, *Gauge theory in higher dimensions*, The geometric universe (Oxford, 1996), 31–47, Oxford Univ. Press, Oxford, 1998.
- [Fe] H. Federer, *Geometric measure theory*. Springer. Berlin-Heidelberg-New York, (1969).
- [FO] K. Fukaya and K. Ono, *Arnold conjecture and Gromov-Witten invariants*, Topology 38 (1999), no. 5, 933–1048.
- [Fu] A. Futaki, *An obstruction to the existence of Einstein-Kähler metrics*, Inv. Math., 73 (1983), 437–443.
- [GL] K. Galicki and B. Lawson, *Quaternionic reduction and quaternionic orbifolds*, Math. Ann., 282 (1988), no. 1, 1–21.
- [Ha] R. Hamilton, *The Ricci flow on surfaces*, Mathematics and general relativity (Santa Cruz, CA, 1986), 237–262, Contemp. Math., 71, Amer. Math. Soc., Providence, RI, 1988.
- [HL] R. Harvey and H.B. Lawson, *Calibrated geometries*, Acta. Math., 148 (1982), 47–157.
- [Jo] D. Joyce, *Compact manifolds with special holonomy*. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2000.
- [KW] J. Kazdan and F. Warner, *Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvatures*, Ann. of Math. (2) 101 (1975), 317–331.
- [LiT] Jiayu Li and G. Tian, *A blow-up formula for stationary harmonic maps*, Internat. Math. Res. Notices, 1998, no. 14, 735–755.
- [LT] Jun Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties*, J. Amer. Math. 11(1998), no. 1, 119–174.
- [Lu] Z.Q. Lu, *On the lower order terms of the asymptotic expansion of Tian-Yau-Zelditch*, Amer. J. Math., 122 (2000), no. 2, 235–273.
- [Ma] Y. Matsushima, *Sur la structure du groupe homéomorphismes analytiques d'une certaine variété Kählérienne*, Nagoya Math. J., 11 (1957), 145–150.
- [Mu] D. Mumford, *Stability of projective varieties*, Enseignement Math., (2) 23 (1977), no. 1-2, 39–110.
- [Na] H. Nakajima, *Compactness of the moduli space of Yang-Mills connections in higher dimensions*, J. Math. Soc. Japan, 40 (1988).
- [Paul] S. Paul and G. Tian, preprint, 2002.
- [Pr] P. Price, *A monotonicity formula for Yang-Mills fields*, Manuscripta Math., 43 (1983), 131–166.
- [RiT] T. Riviere and G. Tian, *The singular set of J-holomorphic maps into algebraic varieties*, preprint, 2001.
- [RT] Y.B. Ruan and G. Tian, *A mathematical theory of quantum cohomology*, J.

- Diff. Geom., 42 (1995), no. 2, 259–367.
- [Ru] Y.B. Ruan, *Virtual neighborhoods and pseudo-holomorphic curves*, preprint, 1996.
- [Sc] R. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geom., 20 (1984), no. 2, 479–495.
- [SCh] S. Chang, *Two-dimensional area minimizing integral currents are classical minimal surfaces*, J. Amer. Math. Soc., 1 (1988), no. 4, 699–778.
- [Sm] K. Smoczyk, *Harnack inequality for the Lagrangian mean curvature flow*, Calc. Var. Partial Differential Equations, 8 (1999), no. 3, 247–258.
- [Si] B. Siebert, *Gromov-Witten invariants for general symplectic manifolds*, preprint, 1996.
- [ST] B. Siebert and G. Tian, *On the holomorphicity of genus two Lefschetz fibrations*, preprint, 2001.
- [Ta] C. Taubes, *Seiberg Witten and Gromov invariants for symplectic 4-manifolds*. Edited by Richard Wentworth. First International Press Lecture Series, 2. IP, 2000.
- [Th] R. Thomas, *A holomorphic Casson invariant for Calabi-Yau 3-folds and bundles on K3 fibrations*, J. Differential Geom., 54 (2000), no. 2, 367–438.
- [ThY] R. Thomas and S.T. Yau, *Special Lagrangians, stable bundles and mean curvature flow*, preprint, 2001.
- [Ti1] G. Tian, *On Calabi's conjecture for complex surfaces with positive first Chern class*, Inv. Math., 101 (1990), no. 1, 101–172.
- [Ti2] G. Tian, *Kähler-Einstein metrics with positive scalar curvature*, Invent. Math., 130 (1997), 1–39.
- [Ti3] G. Tian, *Kähler-Einstein metrics on algebraic manifolds*, Proceedings of the International Congress of Mathematicians, Vol. I (Kyoto, 1990), 587–598.
- [Ti4] G. Tian, *On a set of polarized Kähler metrics on algebraic manifolds*, J. Diff. Geom., 32 (1990), 99–130.
- [Ti5] G. Tian, *Gauge theory and calibrated geometry. I*, Ann. of Math., (2) 151 (2000), no. 1, 193–268.
- [Ti6] G. Tian, *Symplectic isotopy in four dimension*, First International Congress of Chinese Mathematicians (Beijing, 1998), 143–147, AMS/IP Stud. Adv. Math., 20, Amer. Math. Soc., 2001.
- [Ts] H. Tsuji, *Existence and degeneration of Kähler-Einstein metrics on minimal algebraic varieties of general type*, Math. Ann., 281 (1988), no. 1, 123–133.
- [TT] T. Tao and G. Tian, *A singularity removal theorem for Yang-Mills fields in higher dimensions*, preprint, 2001.
- [TY1] G. Tian and S.T. Yau, *Existence of Kähler-Einstein Metrics on complete Kähler manifolds and their applications to algebraic geometry*, Mathematical aspects of string theory (San Diego, Calif., 1986), 574–628, Adv. Ser. Math. Phys., 1, World Sci. Publishing, Singapore, 1987.
- [TY2] G. Tian and S.T. Yau, *Complete Kähler manifolds with zero Ricci curvature. I*, J. Amer. Math. Soc., 3 (1990), no. 3, 579–609.
- [TY3] G. Tian and S.T. Yau, *Complete Kähler manifolds with zero Ricci curvature. II*, Invent. Math., 106 (1991), no. 1, 27–60.

- [TYa] G. Tian and B.Z. Yang, *Compactification of the moduli spaces of vortices and coupled vortices*, preprint, 2002.
- [TZ] G. Tian and X.H. Zhu, *Uniqueness of Kähler-Ricci solitons*, *Acta Math.*, 184 (2000), no. 2, 271–305.
- [Uh1] K.K. Uhlenbeck, *Connections with L^p Bounds on Curvature*, *Comm. in Math. Phys.*, 83 (1982), 31–42.
- [Uh2] K.K. Uhlenbeck, *Removable Singularities in Yang-Mills Fields*, *Comm. in Math. Phys.*, 83 (1982), 11–29.
- [UY] K. Uhlenbeck and S.T. Yau, *On the existence of Hermitian-Yang-Mills connections in stable vector bundles*, *Comm. Pure Appl. Math.*, 39 (1986), no. S, suppl., S257–S293.
- [WaM] M.T. Wang, *Mean curvature flow of surfaces in Einstein four-manifolds*, *J. Differential Geom.*, 57 (2001), no. 2, 301–338.
- [Wang] McKenzie Y. Wang, *Einstein metrics from symmetry and bundle constructions*, *Surveys in differential geometry: Essays on Einstein manifolds*, 287–325, *Surv. Differ. Geom.*, VI, Int. Press, Boston, MA, 1999.
- [Ya] S.T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I**, *Comm. Pure Appl. Math.*, 31 (1978), 339–441.
- [Zel] S. Zelditch, *Szegő kernels and a theorem of Tian*, *Internat. Math. Res. Notices*, 1998, no. 6, 317–331.

