# L-Functions

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## Section 1. The Fundamental Conjectures

Since Hecke's work [0], the theories of $L$-functions and of automorphic forms have been closely interwoven. In this talk, we review some recent developments concerning the analytic aspects of these topics. In the case of the Riemann Zeta Function $\zeta(s)$ and Dirichlet's $L$-functions $L(s,\chi)$ (that is "$GL_1$ over $\mathbb{Q}$" $L$-functions) developments during the 1960's and 1970's see [1,2] offer a large body of techniques and results with many striking applications to classical number theory. Today the same can be said about $L$-functions of modular forms on the upper half plane $\mathbb{H}$ (that is "$GL_2$ $L$-functions") these being the main concern below. We begin however with the general $L$-function which in any case has important impact on $GL_2$ $L$-functions.

Fix $m \geq 1$ and let $\pi$ be an automorphic *cusp* form (or representation) for $GL_m(\mathbb{Q})$ (later in connection with Conjecture II below we also allow $GL_m(K)$, where $K$ is a number field). That is $\pi$ is an irreducible unitary representation of $GL_m(\mathbb{A})$ (which we assume has a unitary central character) which appears in its regular representation on $GL_m(\mathbb{Q})\backslash GL_m(\mathbb{A})$, $\mathbb{A}$ being the adele ring of $\mathbb{Q}$. Then $\pi \cong \otimes \pi_p$, where $\pi_p$ is an irreducible unitary representation of $GL_m(\mathbb{Q}_p)$ if $p < \infty$ and of $GL_m(\mathbb{R})$ if $p = \infty$. Moreover, for all but a finite number of places $p$, $\pi_p$ is unramified. The (standard) $L$-function, $L(s,\pi)$ associated with such a $\pi$ is an Euler product of degree $m$:

$$L(s,\pi) = \prod_{p<\infty} L(s,\pi_p) \tag{1}$$

where

$$L(s,\pi_p) = \prod_{j=1}^{m} (1 - \alpha_{j,\pi}(p)p^{-s})^{-1} \tag{2}$$

The numbers $\{\alpha_{j,\pi}(p)\}_{j=1}^{m}$ are determined from the local representation $\pi_p$. At the place $\infty$ the local factor $L(s,\pi_\infty)$ is a product of Gamma functions which if $\pi_\infty$ is unramified takes the form

$$L(s,\pi_\infty) = \prod_{j=1}^{m} \left(\pi^{-s/2}\,\Gamma\left(\frac{s-\mu_{j,\pi}(\infty)}{2}\right)\right) \tag{3}$$

As with $\zeta(s)$ and $L(s,\chi)$ the key analytic properties of $L(s,\pi)$ are known [3]. These being the analytic continuation and functional equation:

$$L(s,\pi_\infty)\,L(s,\pi) = \epsilon_\pi q_\pi^{s-1/2}\,L(1-s,\tilde{\pi}_\infty)\,L(1-s,\tilde{\pi}) \tag{4}$$

where $q_\pi \in N$ is the conductor of $\pi$, $\epsilon_\pi$ is of modulus 1 and is the "sign" of the functional equation and $\tilde{\pi}$ is the contragredient of $\pi$ [4]. We let $\lambda_\pi$ be the quantity $(\sum_{j=1}^{m} |\mu_{j,\pi}(\infty)|^2)^{1/2}$ and call it the archimedean size of $\pi$.

General philosophies and conjectures [5] (which among other things encompass the Artin conjectures) assert that any $L$-function (from automorphic forms on more general groups over number fields or from varieties defined over number fields) are products of these $L(s,\pi)$'s. These are therefore the primitive objects in the theory of $L$-functions. Undoubtedly the two central analytic problems in the theory are:

I. The Grand Riemann Hypothesis (GRH), which asserts that the zeroes of the completed $L$-function $\xi(s,\pi) = L(s,\pi_\infty)\, L(s,\pi)$ all lie on $\mathrm{Re}(s) = 1/2$.

II. The (generalized) Ramanujan conjectures [100]: if $\pi_p$ is unramified then

$$|\alpha_{j,\pi}(p)| = 1$$

while if $\pi_\infty$ is unramified

$$\mathrm{Re}\,(\mu_{j,\pi}(\infty)) = 0.$$

There are no known direct relations between Conjectures I for these different primitive $L$-functions and it is of course possible that the original RH [6] is true for $\zeta(s)$ but that it fails for some general $L(s,\pi)$. This however seems unlikely and the theme of this report is the role played by *families* of $L$-functions which may often be employed to analyze a given $L(s,\pi)$.

Conjectures I and II have many far reaching implications. The most interesting applications of Conjecture I follow from its use for a *family* of $L$-functions rather than for a single function such as $\zeta(s)$. While these Conjectures remain out of reach at present, the approximations to them, some of which are described below, lead in many cases to the resolution of the problem at hand. Conjecture II for $m = 1$ is trivial. For $m = 2$ there are some important special cases (including Ramanujan's original one) known [7,96,8] (interestingly, the proof in these cases involves reducing Conjecture II to function field generalizations of Conjecture I). The case when $m = 2$ for the place at $\infty$ is equivalent to the conjecture that the first eigenvalue of the Laplacian on the hyperbolic quotient $\Gamma(N)\backslash\mathbb{H}$, $\Gamma(N)$ being the congruence subgroup $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv I(N); a,b,c,d \in \mathbb{Z},\ ad - bc = 1 \right\}$, is at least 1/4 [9]. The local bounds towards II which use only that $\pi_p$ and $\pi_\infty$ are generic [10] assert that

$$\text{and} \qquad \begin{aligned} p^{-1/2} < |\alpha_{j,\pi}(p)| &< p^{1/2} \\ \left|\mathrm{Re}\,(\mu_{j,\pi}(\infty))\right| &< \tfrac{1}{2} \end{aligned} \tag{5}$$

To go beyond this basic bound one uses global methods. In particular, the use of *families* of $L$-functions as described below lead to the best known results.

Associated with $\pi$ as above are other $L$-functions which have conjectured analytic continuations and functional equations. First and foremost is $L(s, \pi \otimes \tilde{\pi})$ [11], [12], [13] whose analytic properties are completely understood [14], [15]. The local factor $L(s, \pi_p \otimes \tilde{\pi}_p)$ at a prime $p$ at which $\pi$ is unramified is given by

$$L(s, \pi_p \otimes \tilde{\pi}_p) = \prod_{j,k} \left(1 - \alpha_{j,\pi}(p)\, \overline{\alpha_{k,\pi}(p)}\, p^{-s}\right)^{-1} \tag{6}$$

There are other important cases which are partially understood such as $L(s, sym^2\pi)$ where $sym^2\pi$ is the symmetric square representation [16], [17]. In fact, for $\pi$ on $GL_2$ the analytic theory of the symmetric square $L$-function is complete [18], [19] and recently the same has been achieved for the symmetric cube [20]. For a survey of these techniques, results and their limitations see [21], [22],[23]. We note that establishing the expected analytic properties of $L(s, sym^k\pi)$ for all $k$ would lead to a proof of Conjecture II, as well as the conjecture about the distribution of the $\{\alpha_{j,\pi}(p)\}_{j=1}^m$ as $p \to \infty$.

The basic result towards I, which in the case of $\zeta(s)$ is the key ingredient in the proof of the prime number theorem and is based on the non-negativity of the coefficients of associated Dirichlet series, is that $L(s, \pi) \neq 0$ for $\mathrm{Re}(s) = 1$. This general result may be proven by this 100 year old technique together with the analytic properties of $L(s, \pi \otimes \tilde{\pi})$ (or one may use the Eisenstein series directly [24] which yields the *same* zero-free region). The quality of the lower bound for $L(1+it, \pi)$ (or equivalently a zero-free region) in terms of the parameters $t$, $\lambda_\pi$, $q_\pi$ is more or less the same in all cases except for one major (and tantalizing) lacuna - the possible "Landau-Siegel Zero." That is in the case that $\chi$ is a quadratic ($\chi^2 = 1$) Dirichlet character, then instead of an *effective* lower bound for $L(1, \chi)$ of the form $\gg (\log q_\chi)^{-1}$ which is established for the other $\chi$'s, only the lower bounds of $(\log q_\chi)/\sqrt{q_\chi}$ when $\chi(-1) = 1$ [25] and of $\frac{\log q_\chi}{\sqrt{q_\chi}} \prod_{\substack{p \backslash q_\chi \\ p \neq q_\chi}} \left(1 - \frac{[2\sqrt{p}]}{p+1}\right)$ when $\chi(-1) = -1$ [26], [27], are known (the latter has striking applications to class numbers of imaginary quadratic fields and is a prime example of an application of $GL_2$ theory to $GL_1$). Put another way, there may be an $L(s, \chi)$ with a real zero very close to 1 (in terms of the conductor), which we call a Landau-Siegel Zero [28], [29]. Interestingly, it appears that only such a $\chi$ ($\chi^2 = 1$) can have such an extreme violation of I. In [30] and [32] it is shown (using the positivity of the coefficients of an appropriate Dirichlet series) that for any $GL_2$ form $\pi$ as well as its symmetric square (if it is not of "CM" type) there are no Landau-Siegel zeroes. The last is technically very useful especially when applying the Petersson formula [31] and its generalization [33], see for example [34].

## Section 2. Sub-convexity

A consequence of Conjectures I and II which is used in many of their applications is the "Lindelof Hypothesis" which asserts that for any $\pi$ on $GL_m$ ($m$ fixed) and $\epsilon > 0$ there is $C_\epsilon < \infty$ such that

$$\left| L\left(\frac{1}{2} + it, \pi\right) \right| \leq C_\epsilon \left( (|t|+1)^m (\lambda_\pi + 1)\, q_\pi \right)^\epsilon \tag{7}$$

The functional equation (4) together with II and a standard convexity argument in complex analysis imply that

$$\left| L\left(\frac{1}{2}+it,\pi\right)\right| \le C_\epsilon \left(\left(|t|+1\right)^m \left(\lambda_\pi+1\right) q_\pi\right)^{1/4+\epsilon} \tag{8}$$

Some of the most interesting applications of (7), (for example, to estimation of Fourier coefficients of 1/2-integral weight modular forms [35] or to problems in Quantum Chaos [36]) require only a sub-convexity bound in (8) - that is, in one of the $t, \lambda$ or $q$ aspects an exponent $\delta < 1/4$ in (8). In $GL_1$ the first such bound is essentially due to Weyl [37] in the $t$-aspect while [38] is still the best known in the $q$-aspect. For $GL_2$, the series of papers [39], [40], [41], [42], [43] establish sub-convexity bounds in each of the $\lambda, q$ and $t$-aspects. An application of this in the $t$-aspect to quantum unique ergodicity is given in [34] while when applied in the $q$-aspect to $L(\frac{1}{2}, \pi\otimes\chi_q)$, $\chi_q^2=1$ ($\pi$ fixed), it yields a solution (albeit ineffective due to the possible Landau-Siegel zero) of the long standing problem of determining which large integers are represented by a positive definite integral ternary quadratic form [44]. The novel technique leading to the sub-convexity estimate is "amplification" which proceeds by embedding $L(s,\pi)$ in a suitable family $\mathcal{F}$ of $L$-functions. See [45] for a description of the method and [46] and [47] for some other instances of its use. An interesting and basic problem is to develop sub-convexity bounds in the various aspects for $\pi$'s on $GL_m$, $m \ge 3$.

## SECTION 3. LOCAL DISTRIBUTION OF ZEROES

The asymptotics of the number of zeroes $\rho_\pi$ of $\xi(s,\pi)$ is well known. As $T\to\infty$

$$\#\{\rho_\pi | 0 \le \text{Im}\,(\rho_\pi) \le T\} \sim \frac{mT\log T}{2\pi} \tag{9}$$

For $GL_1$ $L$-functions, it is shown in [48] that a positive proportion of these zeroes are on the line, $\text{Re}(s)=1/2$. The proof is based on a technique called "mollification" and it has been used to establish a similar result for $GL_2$ $L$-functions [49]. Another approach to this type of result was introduced in [50]. It has the advantage of producing simple zeroes and in [51] this method was developed further to show that at least 40% of the zeroes of $\zeta(s)$ are on $\text{Re}(s)=1/2$ and are simple.

For the rest of this section we will assume Conjecture I and discuss the fine structure of the distribution of the zeroes. This is of interest both in arithmetic applications as well as giving insight into the nature (eg spectral) of the zeroes. Write the zeroes $\rho_\pi$ as $\frac{1}{2}+i\gamma_\pi$ and order them:

$$\ldots \le \gamma_\pi^{(-2)} \le \gamma_\pi^{(-1)} \le 0 \le \gamma_\pi^{(1)} \le \gamma_\pi^{(2)} \ldots \tag{10}$$

In view of (9), in order to examine the distribution of the *local* spacings between the zeroes we re-normalize and consider the numbers $\hat{\gamma}_\pi^{(j)} = (m\gamma_\pi^{(j)} \log \gamma_\pi^{(j)})/2\pi$, $j \ge 1$. Their consecutive spacings are the numbers $\hat{\gamma}_\pi^{(j+1)} - \hat{\gamma}_\pi^{(j)}$. The pair correlation is the local density of the numbers $\hat{\gamma}_\pi^{(j)} - \hat{\gamma}_\pi^{(k)}$, $j \ne k \le N$ (as $N\to\infty$). The $k$-th ($k \ge 2$) consecutive spacings and $n \ge 3$ correlations are defined similarly

[58]. For the zeroes of $\zeta(s)$ it was shown in [52] that for a restricted class of test functions the pair-correlation density approaches the density $\left(1 - \left(\frac{\sin \pi x}{\pi x}\right)^2\right) dx$, as $N \to \infty$. It was further noted there that this density is the same as the known [53] pair-correlation density for the eigenvalues of a typical (for Haar measure) large unitary matrix [54]. This ensemble of random matrices has been much studied by Physicists [55] (for example in connection with models for the spectral lines of heavy nuclii) and goes by the name the Circular Unitary Ensemble, (CUE). All the local spacing statistics for the eigenvalues of a random matrix in this ensemble are the same as for the related Gaussian Unitary Ensemble (GUE) [54]. In [56] a detailed numerical investigation of the hypothesis that the local spacing distributions of the high zeroes of $\zeta(s)$ follow CUE laws, has been carried out. In particular, the local spacing distributions for the 70 million zeroes near the $10^{20}-th$ zero follow the CUE predictions (almost perfectly!). In [57], the $n = 3$ and in [58] all the $n$-level correlations are determined analytically (again in restricted ranges). The results being precisely the CUE $n$-level correlation densities. At the phenomenological level, this CUE feature is perhaps the most interesting discovery about $\zeta(s)$ since Riemann's Conjecture I and it points to the spectral nature of the zeroes. In [58] the $n \geq 2$ correlations are determined for *any* $L(s, \pi)$ and are found to be *universally* CUE. Numerical experiments for various $\pi$'s in $GL_1$ [59] and $GL_2$ [60] strongly confirm this CUE phenomenon. Thus, unlike the distributions of the $\{\alpha_{j,\pi}(p)\}_{j=1}^m$ as $p \to \infty$, which depend on the symmetry type of $\pi$, the local distributions of the high zeroes of any $L(s, \pi)$ appear to be universally CUE.

The function field analogues of $\zeta(s)$ offer much insight into the above. Replacing the rational numbers $\mathbb{Q}$ by a finite extension $k$ of $\mathbb{F}_q(t)$, $\mathbb{F}_q$ being a finite field with $q$-elements, one obtains an analogue of $\zeta(s)$ due to Artin [61]. If $C$ is a curve over $\mathbb{F}_q$ with function field $k$ then the associated zeta function $\zeta(T, C/\mathbb{F}_q)$ is a rational function with $2g$ zeroes, where $g$ is the genus of $C$. The analogue of Conjecture I in this setting has been known for over 50 years [97]. The Frobenius morphism on $C$ is intimately related to $\zeta(T, C/\mathbb{F}_q)$ and is crucial in the proofs of I. In [62] the local spacings between the zeroes of $\zeta(T, C/\mathbb{F}_q)$ is examined. It is shown that as $q$ and $g(C)$ go to infinity the zeroes of the typical (but not every!) $\zeta(T, C/\mathbb{F}_q)$ obey the CUE spacing laws. The sources of this law are clearly identified as: (A) The monodromy of the representation of $\pi_1$ of the family of curves of genus $g$ on $H^1$ of a given curve is "big," it being $Sp(2g)$. (B) The equidistribution of the Frobenius conjugacy classes in the monodromy [8]. (C) The (universal) law for the eigenvalue spacings for the typical matrix in *any* large compact classical group being CUE [62].

In this function field setting, one can also determine the distributions of the zeroes near the point of symmetry (for the functional equation), for a family of zeta or $L$-functions. Again, this follows from the calculation of these distributions for the scaling limits of the monodromy groups of the family and unlike the universality above, these are found to be sensitive to the symmetry of the family [62]. The analogous questions in the rational number case, for various families $\mathcal{F}$ of $L(s, \pi)$'s, has been investigated recently [63]. Ordering the $\pi \in \mathcal{F}$ by their conductors $q_\pi$ one examines the distribution of the (scaled) *low-lying* zeroes. That is, for $j \geq 1$ fixed, the distribution in $[0,\infty)$ of the numbers $(\gamma_\pi^{(j)} \log q_\pi)/2\pi$, as $\pi$ varies over $\mathcal{F}$ [1], and the densities of the numbers $(\gamma_\pi \log q_\pi)/2\pi$ again as $\pi$ varies over $\mathcal{F}$. It is found [63], [64] that these follow the distributions predicted by the symmetry of the family $\mathcal{F}$, when the latter can be determined from the function field analogue. For example, for the family $\mathcal{F}_I$ of $L(s,\chi)$'s where $\chi$ is a quadratic ($\chi^2 = 1$) Dirichlet character, the distribution of the low-lying zeroes follows the *symplectic* $Sp(\infty)$ scaling distributions [62]. This is convincingly confirmed by numerical experiments [60] for $q_\chi$'s of size $10^{12}$. Further confirmation is given by the analytic determination (in restricted ranges) of the densities of the low-lying zeroes for this family [65], [63], [60]. Another example is the family $\mathcal{F}_{II}$ of holomorphic cusp forms $\pi$ of weight 2 for the congruence subgroups $\Gamma_0(N)$ of the modular group. The symmetry type of $\mathcal{F}_{II}$ is *orthogonal* ie $O(\infty)$, at least as far as the analytic computations of the densities of the low-lying zeroes [64].

The above densities of the low-lying zeroes in a family determine in particular the percentages of $\pi \in \mathcal{F}$ for which $L(1/2,\pi) = 0$ (or it's derivative if $L(1/2,\pi) = 0$ for the trivial reason of the sign of the functional equation). For certain families such as $\mathcal{F}_{II}$ above this together with the Birch and Swinnerton-Dyer Conjectures [94] give information about the ranks of the group of rational points on elliptic curves and abelian varieties over $\mathbb{Q}$. In particular, for $\mathcal{F}_{II}$ above one obtains from the analytic results on the densities [66], [67], [64], sharp estimations for the ranks of the Jacobian $J_0(N)/\mathbb{Q}$ of the curves $X_0(N)/\mathbb{Q}$ (which analytically is $\Gamma_0(N)\backslash\mathbb{H}$) as well as for the dimension of largest quotient $M_0(N)/\mathbb{Q}$ ([68], [69]) of $J_0(N)$ which is of rank zero.

While for the above families $\mathcal{F}$ as well as for numerous others [63], [64] the proposed symmetry "$G(\mathcal{F})$" is compelling, it is premature to guess whether it is appropriate for all families. The reason being, that numerical experiments (for moderate size conductors) with certain families of elliptic curves [70], [71] indicate that their ranks are persistently larger than the symmetry (as well as the function field) predicts. Whether this "excess rank" is a consequence of too small a range of computation or whether it is truly there, is a fascinating question whose understanding will no doubt be very instructive.

## SECTION 4. NON-VANISHING FOR FAMILIES

The question of the number of $\pi$'s in $\mathcal{F}$ (ordered by conductor) for which $L(s,\pi)$ is non-zero at a special point arises in a number of contexts. In the basic problem of existence of *cusp* forms for general subgroups of $SL_2(\mathbb{R})$ [72], in the correspondence between forms of 1/2-integral weight and integral weight [73], [74] and in connection with the Birch and Swinnerton-Dyer Conjecture. There are many results asserting that infinitely many $\pi \in \mathcal{F}$ have their $L$-function not zero at a specific point and in some cases even good lower bounds for the number of such $\pi$'s. For example, for the family $\pi_1 \otimes \pi$ with $\pi_1$ fixed on $GL_2$ and $\pi$ varying (with fixed conductor) by increasing $\lambda_\pi$, non-vanishing at special points on the critical line are established in [75], [76]. These have applications to the problem of existence of cusp forms mentioned above. For the family of quadratic twists $\chi$ of

[1] We apologize for the bad notation and hope the reader does not get too confused between $\pi$ the number and $\pi$ the representation.

a given (modular) elliptic curve $E/\mathbb{Q}$ it is shown in [77] and [78] that infinitely many of the values $L(\frac{1}{2}, E \otimes \chi)$ are not zero and also infinitely many of the values $L'(\frac{1}{2}, E \otimes \chi)$ are not zero for $\chi$'s with $\epsilon_{E\otimes\chi} = -1$. This, when combined with [79] has applications to the B-S conjecture for elliptic curves. A challenging unsolved problem which as yet is at the limit of the analytic methods [80] is to show that a positive proportion of the values $L(\frac{1}{2}, E \otimes \chi)$ are not zero. The results on the densities of low-lying zeroes for this family (of Section 3) imply this, however, they appeal to Conjecture I. For special $E$'s a positive proportion of non-vanishing has been established by algebraic methods [81], [82]. In the "vertical" case of twisting such $L$-functions by $\chi$'s of high order, non-vanishing results are proven in [91].

For the family $\mathcal{F}_{II}$ of Section 3, it is shown in [83] that at least 50% of the $L(\frac{1}{2}, \pi)$'s are non-zero, where $\pi$ varies over cusp forms of (say) weight 2 for $\Gamma_0(N)$ and with $\epsilon_\pi = 1$, as $N \to \infty$. (Based on numerical calculations [66] it is conjectured that 100% of these should be non-zero). This result when combined with [79] implies that the dimension of $M_0(N)$ is at least 1/4 of the dimension of $J_0(N)$, as $N \to \infty$. The number 50% above is of fundamental significance (for this as well as for a number of other families [83]) since any improvement of the percentage (in the quantitative form in which the 50% is established) would lead to a proof that there are no Landau-Siegel zeroes! This type of relation, that the distribution of the low-lying zeroes of a family are controlled by the zeroes of other $L$-functions, is not surprising from the function field analysis mentioned in Section 3, see [62]. The proof of this 50% result uses amongst many things an appropriate method of mollification. The proof of the implication to Landau-Siegel zeroes makes use of the following result which is proven either using forms of 1/2-integral weight or the relative trace formula [74], [84]: Let $\pi$ be a (self-dual) cusp form with trivial central character for $GL_2/K, K$ a number field, then $L(\frac{1}{2}, \pi) \geq 0$. Note that since $L(s, \pi)$ is real for $s \in \mathbb{R}$ this inequality is an immediate consequence of Conjecture I for $L(s, \pi)$. That it can be proven unconditionally is quite striking especially since the $GL_1$ analogue - that is $L(\frac{1}{2}, \chi) \geq 0$, $\chi$ quadratic, is not known. Returning $J_0(N)$, in [85] and [86] non-vanishing results are established which together with [27] imply that the rank of $J_0(N)$ is at least 7/16 of $\dim J_0(N)$.

The non-vanishing in a family is also a very powerful tool in attacking Conjecture II. The approach via the family of $L$-functions, $L(s, \pi \otimes \tilde{\pi} \otimes \chi)$ as $\chi$ varies over Dirichlet characters was initiated in [87]. It was convincingly applied in [88] to give estimates for $\alpha_\pi(p)$, $p$ finite, where $\pi$ is a Maass cusp form on $GL_2/\mathbb{Q}$. In [89] a general approach via *non-vanishing* of partial $L$-functions at special points in such a family, was introduced. It leads to the best known bounds towards Conjecture II [90]. If $\pi$ is an automorphic cusp form for $GL_m(K)$ and $\pi$ is unramified at a place $v$ of $K$, then

$$\left| \log_{N(v)} |\alpha_{j,\pi}(v)| \right| \leq \frac{1}{2} - \frac{1}{m^2+1}, \quad \text{if } v \text{ is finite and } N(v) \text{ its norm} \tag{11}$$

$$\left| \mathrm{Re}(\mu_{j,\pi}(v)) \right| \leq \frac{1}{2} - \frac{1}{m^2+1} \quad \text{for } v \text{ archimedean} \tag{12}$$

This result for $GL_3$ combined with the symmetric square correspondence from

$GL_2 \to GL_3$ [19] leads to the bounds on $GL_2$:

$$\left| \log_{N(v)} |\alpha_{j,\pi}(v)| \right| \leq \frac{1}{5}, \quad \text{if } v \text{ is finite} \tag{13}$$

$$\left| \mathrm{Re}(\mu_{j,\pi}(v)) \right| \leq \frac{1}{5}, \quad v \text{ archimedean} \tag{14}$$

Interestingly (13) was derived earlier in [92] by special use of the exceptional group $F_4$. (14) implies a lower bound of 21/100 for the first eigenvalue of the Laplacian on $\Gamma_0(N)\backslash\mathbb{H}$. This goes beyond the 3/16 bound [9] which was based on estimating sums of Kloosterman sums using [93]. Thus (14) provides for the first time cancellations in sums of Kloosterman sums on arithmetic progressions [89].

## SECTION 5. FINAL COMMENTS

We note that numerical experimentation played a key role in the discoveries and (or) confirmations of Conjecture I by Riemann, of Conjecture II by Ramanujan, of the Conjecture of Artin [61] and that of Birch and Swinnerton-Dyer [94].

While we may still have to wait for some time for the complete resolutions of Conjectures I and II, these like other fundamental problems have generated marvellous mathematics. Various things are falling into place. The function field analogues are very suggestive and the evidence for there being a natural spectral interpretation of the zeroes [2] as well as a symmetry group for families is rather convincing. The last bodes well since in the function field the proof of the general cases of Conjecture I make essential use of monodromy of families [8]. Similarly at the present time the most powerful techniques (in the number field case) have emerged from considerations of families. Averaging over families in $GL_2$ theory is usually achieved by the trace formula [95] but often and more profitably, it can be gotten from the older Petersson formula [31]. The approximations to Conjectures I and II that have been established are good enough in many instances to resolve completely some classical problems.

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