

Some Highlights of Percolation

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Abstract

We describe the percolation model and some of the principal results and open problems in percolation theory. We also discuss briefly the spectacular recent progress by Lawler, Schramm, Smirnov and Werner towards understanding the phase transition of percolation (on the triangular lattice).

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1. Introduction and description of the percolation model

Percolation was introduced by Broadbent and Hammersley (see [14],[15]) as a probabilistic model for the flow of fluid or a gas through a random medium. It is one of the simplest models which has a phase transition, and is therefore a valuable tool for probabilists and statistical physicists in the study of phase transitions. For many mathematicians percolation on general graphs may be of interest because it exhibits relations between probabilistic and topological properties of graphs. On the applied side, percolation has been used to model the spread of a disease or fire, the spread of rumors or messages, to model the displacement of oil by water, to estimate whether one can build nondefective integrated circuits with certain wiring restrictions.

We shall give a brief survey of some of the important results obtained for this model and list some open problems. The present article is only a very restricted survey and its references (in particular to the physics literature) are far from complete. We apologize to the authors of relevant articles which we have not cited. Earlier surveys are in [42], [21], [22], [37], [38], and the reader can find more elaborate treatments in the books [20], [63] and [55].

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The oldest (indirect) reference to percolation that I know of is a problem submitted to the Amer. Math. Monthly (vol 1, 1894, pp. 211-212) in 1894 by De Volson Wood, Professor of Mechanical Engineering at the Stevens Inst. of Technology in Hoboken NJ. Here is the text of the problem.

“An actual case suggested the following:

An equal number of white and black balls of equal size are thrown into a rectangular box, what is the probability that there will be contiguous contact of white balls from one end of the box to the opposite end? As a special example, suppose there are 30 balls in the length of the box, 10 in the width and 5 (or 10) layers deep.”

Even though percolation theory was not invented to answer this problem, it naturally came to study problems of this kind. By the way, we still have no answer to De Volson Wood’s problem. Percolation as a mathematical theory was invented by Broadbent and Hammersley ([14],[15]). Broadbent wanted to model the spread of a gas or fluid through a random medium of small channels which might or might not let gas or fluid pass. To model these channels he took the edges between nearest neighbors on \mathbb{Z}^d and made all edges independently *open* (or passable) with probability p or *closed* (or blocked) with probability $1 - p$. Write P_p for the corresponding probability measure on the configurations of open and closed edges (with the obvious σ -algebra generated by the sets determined by the states of finitely many edges). A *path* on \mathbb{Z}^d will be a sequence (finite or infinite) v_1, v_2, \dots of vertices of \mathbb{Z}^d such that for all $i \geq 1$, v_i and v_{i+1} are adjacent on \mathbb{Z}^d . The edges of such a path are the edges $\{v_i, v_{i+1}\}$ between successive vertices and a path is called *open* if all its edges are open. Broadbent’s original question amounted to asking for

$$P_p\{\exists \text{ an open path on } \mathbb{Z}^d \text{ from } \mathbf{0} \text{ to } \infty\}. \quad (1.1)$$

This question has an obvious analogue on any infinite connected graph \mathcal{G} with edge set \mathcal{E} and vertex set \mathcal{V} . Again one makes all edges independently open or closed with probability p and $1 - p$, respectively, and one denotes the corresponding measure on the edge configurations by P_p . E_p is expectation with respect to P_p . An open path is defined as before with \mathcal{G} taking the role of \mathbb{Z}^d . A path (v_1, v_2, \dots) is called *self-avoiding* if $v_i \neq v_j$ for $i \neq j$. (1.1) now is replaced by

$$P_p\{\exists \text{ an infinite self-avoiding open path starting at } v\}, \quad (1.2)$$

with v any vertex in \mathcal{V} .

The preceding model is called *bond-percolation*. There is also an analogous model, called *site-percolation*. In the latter model all edges are assumed passable, but the vertices are independently open or closed with probability p or $1 - p$, respectively. An open path is now a path all of whose vertices are open. One is still interested in (1.2). Site percolation is more general than bond percolation in the sense that the positivity of (1.2) for some v in bond-percolation on a graph \mathcal{G} is equivalent to the positivity of (1.2) for some v in site-percolation on the covering graph or line graph of \mathcal{G} . However, site percolation on a graph may not be equivalent to bond percolation on another graph (see [40], Section 2.5 and Proposition 3.1).

Unless otherwise stated we restrict ourselves in the remaining sections to site percolation. We shall often use \mathcal{V} and \mathcal{E} to denote the vertex and edge set of whatever graph we are discussing at that moment, without formally introducing the graph as $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. It should be clear from the context what \mathcal{V} and \mathcal{E} stand for in such cases. For $A \subset \mathcal{V}$, we shall use $|A|$ to denote the number of vertices in A . Further if A, B and C are sets of vertices, then $A \leftrightarrow B$ means that there exists an open path from some vertex in A to some vertex in B , while $A \overset{\mathcal{C}}{\leftrightarrow} B$ means that there exists an open path with all its vertices in C , from some vertex in A to some vertex in B . In particular, with some abuse of notation, we have

$$\{|\mathcal{C}(v)| = \infty\} = \{v \leftrightarrow \infty\}.$$

Definition 1 We call a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ quasi-transitive if there is a finite set of vertices V_0 , such that for each vertex v there is a graph automorphism of \mathcal{G} which maps v to one of the vertices in V_0 .

All vertices which can be mapped by a graph automorphism to a fixed $v_0 \in V_0$ are equivalent for our purposes. In a quasi-transitive graph each vertex is equivalent to one of finitely many vertices. A special subclass is formed by the *transitive graphs*, which have $|V_0| = 1$, so that all vertices are equivalent for our purposes. (For example, the Cayley graph of a finitely generated group is transitive.)

We shall restrict ourselves here to graphs which are

$$\text{connected, infinite but locally finite, and quasi-transitive.} \quad (1.3)$$

Graphs which satisfy (1.3) automatically have countable vertex sets and edgesets. We define, for $v \in \mathcal{V}$,

$$\begin{aligned} \theta^v(p) &= P_p\{v \leftrightarrow \infty\} \\ &= P_p\{\exists \text{ an infinite self-avoiding open path starting at } v\}. \end{aligned} \quad (1.4)$$

For a quasi-transitive graph $\theta^v(p) = \theta^{v_0}(p)$ for some $v_0 \in V_0$. It is an easy consequence of the FKG inequality that either $\theta^v(p) > 0$ for all v or $\theta^v(p) = 0$ for all v (see [40], Section 4.1). We call $\theta^v(p)$ the *percolation probability (from v)*. Much of the earlier work on percolation theory deals with properties of the function $p \mapsto \theta^v(p)$, or more generally with the full distribution of the so-called cluster sizes. The *cluster* $\mathcal{C}(v)$ of the vertex v is the set of all points which are connected to the origin by an open path. By convention, this always contains the vertex v itself (even if v itself is closed in the case of site percolation). The clusters are the maximal components of the graph with vertex set \mathcal{V} and with an edge between two sites only if they are adjacent on \mathcal{G} and are both open. $\theta^v(p)$ is just the P_p -probability that $|\mathcal{C}(v)| = \infty$.

2. Existence of phase transition and related properties of the critical probability

The most important property of the percolation model is that it exhibits a *phase transition*, that is, there exists a threshold value p_c such that the global behavior of the system is quite different in the two regions $p < p_c$ and $p > p_c$. To make this more precise let us consider the percolation probability as a function of p . It is non-decreasing. This is easiest seen from Hammersley's ([31]) joint construction of percolation systems for all $p \in [0, 1]$ on \mathcal{G} . Let $\{U(v), v \in \mathcal{V}\}$ be independent uniform $[0, 1]$ random variables. Declare v to be p -open if $U(v) \leq p$. Then the configuration of p -open vertices has distribution P_p for each $p \in [0, 1]$. Clearly the collection of p -open vertices is nondecreasing in p and hence also $\theta(\cdot)$ is nondecreasing. Clearly $\theta^v(0) = 0$ and $\theta^v(1) = 1$. Roughly speaking the graph of $\theta^v(\cdot)$ (for a fixed v) therefore looks as in figure 1, but not all the features exhibited in this figure have been proven.

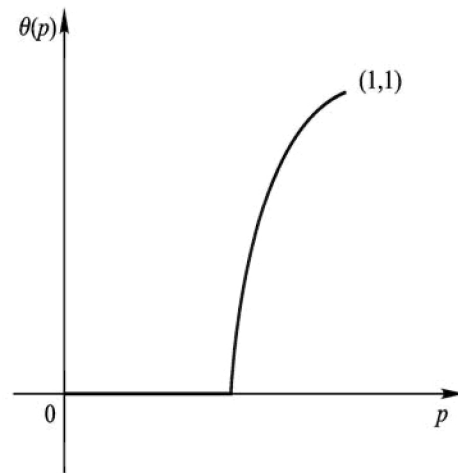


Figure 1: Graph of θ . Many aspects of this graph are still conjectural.

The *critical probability* is defined as

$$p_c = p_c(\mathcal{G}) = \sup\{p : \theta^v(p) = 0\}. \quad (2.1)$$

As remarked after (1.4) this is independent of v . By definition we then have

$$P_p\{|\mathcal{C}(v)| = \infty\} = 0 \text{ for } p < p_c, v \in \mathcal{V},$$

so that

$$\text{all clusters are finite a.s. } [P_p] \text{ when } p < p_c. \quad (2.2)$$

On the other hand, for $p > p_c$ there is a strictly positive P_p -probability that $|\mathcal{C}(v)|$ is infinite. It then follows from Kolmogorov's zero-one law that

$$P_p\{\text{some } |\mathcal{C}(v)| = \infty\} = 1, \quad p > p_c. \quad (2.3)$$

Thus the global behavior of the system is quite different for $0 \leq p < p_c$ and for $p_c < p < 1$. We therefore can say that there is a phase transition at p_c , provided

the intervals $[0, p_c)$ and $(p_c, 1]$ are both nonempty. It is easy to see from a so-called Peierls argument (just as in [29]) that $p_c(\mathcal{G}) > 0$ for any graph \mathcal{G} of bounded degree (and hence certainly if (1.3) holds). It is much harder to do show that $p_c(\mathcal{G}) < 1$ holds for certain \mathcal{G} . Hammersley [30] proved this for bond-percolation on \mathbb{Z}^d , but a similar argument works for site-percolation and various other periodic graphs. (Basically, we say that \mathcal{G} is *periodic* or can be *periodically imbedded in \mathbb{R}^d* if \mathcal{V} can be imbedded in \mathbb{R}^d (with $d \geq 2$) such that \mathcal{V} as well as the edges of \mathcal{G} , as represented by the straight line segments between the pairs of vertices adjacent in \mathcal{G} , form a subset of \mathbb{R}^d which is invariant under translations by d linearly independent vectors. If this is the case we call d the dimension of \mathcal{G} . We refer the reader to [40], Section 2.1 for details.) Thus

Theorem 1

$$0 < p_c(\mathbb{Z}^d) < 1.$$

Thus, at least on \mathbb{Z}^d , there really is a phase transition. On any graph one says that the system is in the *subcritical (supercritical) phase* if $p < p_c$ (respectively, $p > p_c$). Because percolation is such a simple model with a phase transition, percolation has received a great deal of attention from physicists. Percolation is one of the Potts models, corresponding to the parameter q in the Potts model equal to 1; the famous Ising model for magnetism is essentially the same as the Potts model with $q = 2$. One hopes that understanding of the percolation model will help understand all the Potts models and even the more general Fortuin-Kasteleyn or random cluster models (see [23], which also explains the relation, due to Fortuin and Kasteleyn, between random cluster models and Potts models).

The exact value of $p_c(\mathcal{G})$ is known only for a handful of graphs, and all of these are periodic two-dimensional graphs. This leads to

Open problem 1: Find $p_c(\mathcal{G})$ for a wide class of graphs.

However, it is generally agreed that the solution to this problem would not have any explanatory value. The critical probabilities which have been determined so far depend heavily on special symmetry properties of the underlying graph, and the values of these critical probabilities vary with the graph. One has therefore moved on to properties which are believed to be shared by large classes of graphs; see Section 4 below. The rigorously known critical probabilities can be found in [38], Chapter 3. Here we merely mention the one case which will be important later on:

$$p_c(\text{site percolation on triangular lattice}) = \frac{1}{2}. \quad (2.4)$$

Also known is the following asymptotic result, both for the site and for the bond version:

$$p_c(\mathbb{Z}^d) \sim \frac{1}{2d} \text{ as } d \rightarrow \infty. \quad (2.5)$$

This has been proven by several people; [35] gives the best higher order terms in (2.5).

One can define another critical probability as the threshold value for the finiteness of the clustersize of a fixed vertex. Thus,

$$p_T(\mathcal{G}) = \sup \{p : E_p\{|\mathcal{C}(v)|\} = \infty\}. \quad (2.6)$$

Since $P_p\{|C(v)| = \infty\} > 0$ for $p > p_c$, it is obvious that $E_p\{|C(v)|\} = \infty$ for all $p > p_c$, so that $p_T(\mathcal{G}) \leq p_c(\mathcal{G})$. It was a crucial step in establishing the known values for p_c to show that $p_T(\mathcal{G}) = p_c(\mathcal{G})$. The original proof of this fact was only for bond percolation on \mathbb{Z}^2 ([39]; this proof made strong use of crossing probabilities similar to those appearing in De Volson Wood’s problem in Section 1). Proofs of $p_T = p_c$ for some other special lattices are in [65] and [40]). Later Menshikov ([51]) and Aizenman and Barsky ([1]) gave independent and different proofs of exponential decay of the distribution of $|C(v)|$ for $p < p_c$. This is a cornerstone of the subject and is of course a much stronger statement than $p_T = p_c$.

Theorem 2 (*Menshikov and Aizenman and Barsky*) *Assume that \mathcal{G} is periodic. Then for $p < p_c(\mathcal{G})$ there exists constants $0 < C_1, C_2 < \infty$ such that*

$$P_p\{|C(v)| \geq n\} \leq C_1 e^{-C_2 n}, \quad n \geq 0. \tag{2.7}$$

(2.7) gives a basic estimate for the subcritical phase. By an earlier “subadditivity” argument of [45] (2.7) can be sharpened to a “local limit theorem” (see [20], Theorem 6.78): for each $p < p_c$ there exists a $0 < C_3(p) < \infty$ such that

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log P_p\{|C(v)| = n\} = C_3(p). \tag{2.8}$$

These results give us a measure of control over the subcritical phase. In the supercritical phase many estimates rely on another fundamental result of percolation theory, which was proven by Grimmett and Marstrand [24]. The simplest form of the result is as follows:

Theorem 3

$$p_c(\mathbb{Z}^d) = \lim_{k \rightarrow \infty} p_c(\mathbb{Z}_+^2 \times \{1, 2, \dots, k\}^{d-2}). \tag{2.9}$$

One may replace \mathbb{Z}_+^2 by \mathbb{Z}^2 here.

The graph appearing in the right hand side here consists of a finite number of copies of the first quadrant in \mathbb{Z}^2 or of the whole \mathbb{Z}^2 . Thus (before the limit is taken) this graph looks very much like \mathbb{Z}^2 and many of the special tools for percolation on \mathbb{Z}^2 can be applied to this graph. Because of this one could prove a number of results on \mathbb{Z}^d for $p > \lim_{k \rightarrow \infty} p_c(\mathbb{Z}_+^2 \times \{1, 2, \dots, k\}^{d-2})$. Theorem 3 now shows that these results hold throughout the supercritical regime (at least when $\mathcal{G} = \mathbb{Z}^d$ or a similar graph). As an example of this situation we mention a result of [43], namely the right hand inequality in (2.10) (the left hand inequality is due to [3]): For site percolation on \mathbb{Z}^d with $p > p_c(\mathbb{Z}^d)$ there exist $0 < C_4(p), C_5(p) < \infty$ such that

$$C_4(p) \leq -\frac{1}{n^{(d-1)/d}} \log P_p\{|C(v)| = n\} \leq C_5 \tag{2.10}$$

for all large n .

Open problem 2: Does

$$\lim_{n \rightarrow \infty} -n^{-(d-1)/d} \log P_p\{|C(v)| = n\} \text{ exist ?}$$

(under the conditions for (2.10)). The reader should notice the contrast between (2.6), (2.7) — which give exponential decay for the clustersize distribution in the subcritical case — and (2.10) which corresponds to a “stretched exponential” for the tail of the clustersize in the supercritical case. The tail of this distribution at criticality, i.e., for $p = p_c$ will be discussed in Section 4.

3. Uniqueness of infinite clusters and properties of the percolation probability

It is natural to ask “how many infinite clusters can there be ?” In [52] it is shown that for periodic graphs for each p , exactly one of the following three situations prevails:

$$\begin{aligned} P_p\{\text{there is no infinite open cluster}\} &= 1, \\ P_p\{\text{there is exactly one infinite open cluster}\} &= 1 \text{ or} \\ P_p\{\text{there are infinitely many infinite open clusters}\} &= 1. \end{aligned}$$

As pointed out in [58], the proof of [52] carries over to any quasi-transitive graph by a zero-one law for events which are invariant under graph automorphisms. Of course, the first alternative here holds for $p < p_c$, but can the last situation occur for some $p \geq p_c$? The first proof that this is impossible on \mathbb{Z}^d is in [4]. This proof was improved and generalized a few times, but the most elegant, and by now standard, proof is due to Burton and Keane [16]. Their method works for any amenable graph. To make this precise we define for any set $W \subset \mathcal{V}$,

$$\partial W = \{w \in \mathcal{V} : w \notin W \text{ but } w \text{ is adjacent to some } v \in W\}.$$

We call the graph \mathcal{G} *amenable* if there exists a sequence $\{W_n\} \subset \mathcal{V}$ for which $|\partial W_n|/|W_n| \rightarrow 0$.

Theorem 4 (*Burton and Keane*) *If \mathcal{G} satisfies (1.3), and if \mathcal{G} is amenable, then for all $p \in [0, 1]$*

$$P_p\{\text{there exist more than one infinite open cluster}\} = 0. \quad (3.1)$$

The proof of this result is the same as in [16], except that one should argue on the *expected number of encounter points* where Burton and Keane use the ergodic theorem to make the number of encounter points itself large. (We owe this observation to O. Häggström.) Simple examples (such as a regular tree) show that (3.1) does not have to hold for nonamenable graphs. This is one example of a relation between percolation properties and algebraic/topological properties of the underlying graph (see [10], [50] and [8] and some of their references for other examples). What can be said about uniqueness/nonuniqueness in the nonamenable case ? Benjamini and Schramm [10] introduced a further critical probability:

$$p_u = p_u(\mathcal{G}) := \inf\{p : \text{a.s. } [P_p] \text{ there is a unique infinite cluster}\}. \quad (3.2)$$

By definition $p_u \geq p_c$. We have $p_c < p_u = 1$ on a regular b -ary tree (in which all vertices have degree $b + 1$) with $b \geq 2$. The first example of a graph

with $p_c < p_u < 1$ was given in [25]. Note that there is no a priori reason why uniqueness should be monotone in p , that is why uniqueness a.s. $[P_{p'}]$ should imply uniqueness a.s. $[P_{p''}]$ whenever $p'' \geq p'$. This has been proven to be the case for graphs satisfying (1.3). More precisely, the following theorem (and somewhat more) is proven in [57] (see also [26] and [27]):

Theorem 5 *Let \mathcal{G} satisfy (1.3) and let the percolation configurations on \mathcal{G} be constructed simultaneously for all $p \in [0, 1]$ by Hammersley's method described in the beginning of Section 2. Let $N(p)$ be the number of p -open infinite clusters. Then a.s.,*

$$N(p) = \begin{cases} 0 & \text{for } p \in [0, p_c) \\ \infty & \text{for } p \in (p_c, p_u) \\ 1 & \text{for } p \in (p_u, 1]. \end{cases}$$

Note that this theorem does not give the value of $N(p)$ at $p = p_c$ or p_u (see also the lines after Theorem 1.2 in [26] and Open problem 3 below).

Other obvious questions concern the smoothness of the function $\theta^v(\cdot)$, and in particular whether this function is continuous. Clearly $p \mapsto \theta^v(p)$ is always continuous for $p < p_c$, since $\theta^v(p) = 0$ for all such p . Russo [53] noted that $\theta^v(\cdot)$ is everywhere right continuous and [11] proved that (under (1.3)) if for some fixed $p_0 > p_c$ there is a.s. $[P_{p_0}]$ a unique infinite cluster, then $\theta^v(\cdot)$ is also left continuous at p_0 . Thus, under (1.3), the remaining problem is

Open problem 3: Is $p \mapsto \theta^v(p)$ (left) continuous on $[p_c, p_u]$?

On \mathbb{Z}^d which has $p_c(\mathbb{Z}^d) = p_u(\mathbb{Z}^d)$, continuity is equivalent to $\theta(p_c) = 0$. It has long been conjectured that this is the case. It is known that this holds for $d \geq 19$ by the theory of Hara and Slade [34]; actually this deals with bond percolation, but should go through also for site percolation on \mathbb{Z}^d . It also follows from work of Harris [36] and the author [40], Theorem 3.1, that continuity holds when $d = 2$ (both for bond and site percolation). [8] and [9] prove that on a Cayley graph of a non-amenable group there is no percolation at p_c .

4. Behavior at and near p_c

From now on we shall restrict ourselves to transitive graphs which are periodically imbedded in \mathbb{R}^d , so that the origin is a vertex of the graph. Since all vertices are equivalent in a transitive graph, we drop the superscript v from various quantities such as $\theta(p)$; we further write \mathcal{C} for the open cluster of the origin.

We saw in (2.7) and (2.10) that the probability of a cluster of size $n < \infty$ decays exponentially or as a stretched exponential in the subcritical and supercritical regime, respectively. The behavior at criticality is quite different. In fact, it is believed that there exists constants $0 < C_i < \infty$ such that

$$C_6 n^{-(d-1)/2} \leq P_{p_c}\{|C(v)| \geq n\} \leq C_7 n^{-C_8}. \quad (4.1)$$

Indeed, for periodic graphs in dimension $d = 2$ the left hand inequality is proven in [12], but the argument remains valid in any dimension. In an Abelian sense

one knows even more (see the proof of Proposition 10.29 in [20]). The right hand inequality of (4.1) for certain two-dimensional graphs can be found in [40], Theorem 8.2, while for $d \geq 19$ with $\delta = 2$ it follows from [6] and [33]. It is natural to conjecture that

$$P_{p_c}\{|C(v)| \geq n\} \approx n^{-1/\delta} \tag{4.2}$$

for some $\delta = \delta(\mathcal{G}) > 0$, where $a(n) \approx b(n)$ means $\log(a(n))/\log b(n) \rightarrow 1$ as $n \rightarrow \infty$. (One may conjecture that (4.2) and similar relations below hold with an even stronger interpretation of \approx but we shall not pursue this here.)

(4.2) is one example of a so-called *power law*. Another (conjectured) power law is for $P_{p_c}\{v' \leftrightarrow v''\}$. It is believed that for some constant η

$$P_{p_c}\{v' \leftrightarrow v''\} \approx |v' - v''|^{2-d-\eta} \text{ as } |v' - v''| \rightarrow \infty. \tag{4.3}$$

Here $|v|$ denotes the ℓ^1 norm of the image of v under the imbedding into \mathbb{R}^d . Again this is supported by the following partial result for periodic graphs whose image under the periodic imbedding into \mathbb{R}^d is invariant under permutations of the coordinates. For such graphs (4.1) implies that there exist constants $0 < C_i < \infty$ such that

$$C_9|v' - v''|^{-C_{10}} \leq P_{p_c}\{v' \leftrightarrow v''\} \leq C_{11}|v' - v''|^{-C_{12}}. \tag{4.4}$$

Physicists also conjectured that various quantities behave like powers of $|p - p_c|$ as $p \rightarrow p_c, p \neq p_c$. Such conjectures are analogues of results which were known (often on a nonrigorous basis) or conjectured for related models. They were also rigorously known for quite some time for percolation on regular trees. In addition, Hara and Slade in a series of important papers (see in particular [33], [34]) have developed the so-called lace expansion technique to give us a good understanding of percolation in high dimensions. Roughly speaking, they prove many of the physicists conjectures for bond percolation on \mathbb{Z}^d with $d \geq 19$, by showing that most quantities show mean field behavior near p_c , that is, they have the same singularity on \mathbb{Z}^d with $d \geq 19$ as on a regular tree. It is believed that this will remain true for $d > 6$. In fact Hara and Slade can prove their results for percolation in any dimension > 6 for what they call “spread out” models. (These have \mathbb{Z}^d as vertex set but there may be some open bonds between points which are not nearest neighbors on \mathbb{Z}^d .)

The most common of the conjectured power laws (with the traditional names for the exponents) are as follows. Here $A(p) \approx B(p)$ means $\log A(p)/\log B(p) \rightarrow 1$ as $p \rightarrow p_c$ (with $p \neq p_c$).

$$\left(\frac{d}{dp}\right)^3 \left[\sum_{n=1}^{\infty} \frac{1}{n} P_p\{|C| = n\} \right] = \left(\frac{d}{dp}\right)^3 E_p\{|C|^{-1}\} \approx |p - p_c|^{-1-\alpha}, \tag{4.5}$$

$$\theta(p) \approx (p - p_c)^\beta, \quad p \downarrow p_c, \tag{4.6}$$

$$\chi(p) := E\{|C(v)|; |C(v)| < \infty\} \approx |p - p_c|^{-\gamma}. \tag{4.7}$$

Another power law is supposed to hold for the so-called *correlation length*, $\xi(p)$. Intuitively speaking, if $p \neq p_c$, the correlation length is the minimal size a cube should have so that one can detect from a typical percolation configuration in such

a cube that p is not equal to p_c . On scales which are small with respect to the correlation length, the system is expected to behave as if it is critical. On the other hand, on scales which are large with respect to the correlation length, one should be able to partition the system into cubes of edglength equal to a large multiple of the correlation length and regard these cubes as “supersites”; for $p > p_c$ (respectively $p < p_c$) these supersites should behave as sites in site percolation with a p value close to 1 (respectively close to 0). On such scales the details of the lattice other than its dimension should play little role, if any. Several possible formal definitions are in use for the correlation length. Here we define the correlation length $\xi(p)$ by

$$[\xi(p)]^{-1} = \lim_{n \rightarrow \infty} -\frac{1}{n} \log P_p\{\mathbf{0} \leftrightarrow ne_1, \mathbf{0} \nleftrightarrow \infty\}, \quad (4.8)$$

where e_1 is the first coordinate vector. Strictly speaking, [19] only proves that this is a good definition for (bond or site) percolation on \mathbb{Z}^d , but this definition should make sense with minor changes for percolation on general periodic graphs. The conjectured powerlaw then takes the form

$$\xi(p) \approx |p - p_c|^{-\nu}. \quad (4.9)$$

Other power laws have been conjectured for electrical conductance and for the graph-theoretical length of an open crossing between opposite faces of a cube.

In all these cases proofs of power *bounds* instead of actual power laws are known for many graphs which are periodically imbedded in \mathbb{R}^d with $d = 2$ or d large (see [40], Chapter 8, [20], Chapter 10, [33], [34]). For instance,

$$C_{13}|p - p_c|^{-1} \leq \chi(p) \leq C_{14}|p - p_c|^{-C_{15}} \text{ for } p < p_c. \quad (4.10)$$

In fact, the left hand inequality holds for all d (see [5], [20], Theorem 10.28).

Most remarkable is the conjecture of “universality”. That is, it is generally believed that each of the so-called critical exponents $\alpha, \beta, \gamma, \delta, \eta, \nu$ depends for periodic graphs on the dimension d only, and not on the details of the graph \mathcal{G} . For instance, they should have the same value for bond and for site percolation on \mathbb{Z}^2 and on the triangular lattice. This is in contrast to the critical probability p_c , which definitely *does* depend on the details of \mathcal{G} . For this reason the principal concern these days is to establish power laws and universality, and little attention is being paid to open problem 1. (See next section for more on what is now known.)

There are also nonrigorous arguments to derive simple relations between various of these exponents. These are the so-called scaling laws:

$$\alpha + \beta(\delta + 1) = 2 \quad (4.11)$$

$$\gamma + 2\beta = \beta(\delta + 1) \quad (4.12)$$

$$\gamma = \nu(2 - \eta) \quad (4.13)$$

$$d\nu = \gamma + 2\beta \text{ for } 2 \leq d \leq 6. \quad (4.14)$$

The last relation, which involves the dimension d is called a hyper-scaling law. These scaling relations, except (4.11), have been established for many graphs with

$d = 2$ (assuming that the exponents exist; see [41]) and are also known for high d . There is also a “conditional proof” of the hyperscaling relation . That is, (4.14) (or rather the relation $(2 - \eta) = d(\delta - 1)/(\delta + 1)$) has been shown to be implied by other (not yet established) laws for percolation ([13]). It is widely believed that these other laws hold for $3 \leq d \leq 6$.

There are predictions by physicists of the values of these exponents when $d = 2$ or when d is large. In fact, as we already pointed out, it is believed that all these exponents are even independent of d for $d > 6$. The existence of these exponents and their predicted values have now been proven to be correct when \mathcal{G} is the triangular lattice ([60], [61], [62], [48]). It is also known that these exponents (except for α and perhaps η) exist and take the same values as on a regular tree for $d \geq 19$ ([33], [34], [32]).

This section raises the obvious and very extensive

Open problem 4: Prove power laws, universality and scaling relations.

In the next section we shall describe some of the progress made on this problem in dimension 2. We already mentioned the work of Hara and Slade in high dimensions. No progress has been made in dimensions 3, 4 and 5. So we may pose a more modest problem for these dimensions.

Open problem 5: Find upper and lower power bounds for $\theta(p), \chi(p), \xi(p)$ and $P_{p_c}\{|\mathcal{C}| \geq n\}$ when $3 \leq d \leq 5$.

As we pointed out above, bounds on one side are already known for most of these quantities, but as far as we know no bound of the form

$$\xi(p) \leq C_{16}|p - p_c|^{-C_{17}} \tag{4.15}$$

has been proven for $3 \leq d \leq 5$, not even for $p < p_c$. This is probably the most fundamental bound to prove, from which several other bounds might follow. Note that it is not hard to see that on \mathbb{Z}^d

$$\xi(p) \geq C_{18}|\log(p_c - p)|^{-(d-1)/d}(p_c - p)^{-1/d} \text{ for } p < p_c. \tag{4.16}$$

Indeed, the proof of the left hand inequality in (4.4) actually gives

$$P_{p_c}\{0 \overset{[0,n]^d}{\leftrightarrow} ne_1\} \geq C_9 n^{-3(d-1)}. \tag{4.17}$$

From this one trivially has for $p < p_c$

$$\begin{aligned} P_p\{\mathbf{0} \leftrightarrow kn e_1\} &\geq [P_p\{\mathbf{0} \overset{[0,n]^d}{\leftrightarrow} ne_1\}]^k \\ &\geq \left[\left(\frac{p}{p_c}\right)^{(n+1)^d} P_{p_c}\{\mathbf{0} \overset{[0,n]^d}{\leftrightarrow} ne_1\}\right]^k \\ &\geq \left[\left(\frac{p}{p_c}\right)^{(n+1)^d} C_9 n^{-3(d-1)}\right]^k. \end{aligned} \tag{4.18}$$

Now take $n = \lceil |\log(p_c - p)| \rceil^{1/d} (p_c - p)^{-1/d}$ and estimate $\xi(p)$ from

$$\begin{aligned} [\xi(p)]^{-1} &= \lim_{k \rightarrow \infty} -\frac{1}{kn} \log P_p\{\mathbf{0} \leftrightarrow kn e_1, \mathbf{0} \leftrightarrow \infty\} \\ &= \lim_{k \rightarrow \infty} -\frac{1}{kn} \log P_p\{\mathbf{0} \leftrightarrow kn e_1\} \text{ for } p < p_c. \end{aligned}$$

A somewhat different aspect of the behavior of critical percolation concerns the random variable

$$N(v) := \inf\{\text{number of closed vertices in any path from } \mathbf{0} \text{ to } v\}. \quad (4.19)$$

If no percolation occurs for $p = p_c$, then $N(v) \rightarrow \infty$ as $v \rightarrow \infty$, a.s. $[P_{p_c}]$. For bond percolation on \mathbb{Z}^2 it is known that $[\sigma_{p_c}(N(v))]^{-1}[N(v) - E_{p_c}N(v)]$ satisfies a central limit theorem with $E_{p_c}N(v) \asymp \log|v|$ and $\sigma_{p_c}(N(v))$, the standard deviation of $N(v)$, of order $[\log|v|]^{1/2}$ (see [44]). On \mathbb{Z}^d with $d \geq 3$ it is only known ([18]) that $N(v) = O(|v|^\varepsilon)$ a.s. $[P_{p_c}]$, for every $\varepsilon > 0$.

Open problem 6: Improve the bound for $N(v)$ and find a limit theorem for $N(v)$ in dimension ≥ 3 .

5. Conformal invariance and SLE

In this section we only consider graphs which are periodically imbedded in \mathbb{R}^2 . Special attention will be paid to site percolation on the triangular lattice.

We already briefly discussed the interpretation of the correlation length in the preceding section. In view of (4.4), the definition (4.8) assigns the value ∞ to the correlation length when $p = p_c$, at least if $P_{p_c}\{|\mathcal{C}| = \infty\} = 0$, as is widely believed (and is known for $d = 2$ or $d \geq 19$). Thus the correlation length is not a useful length scale for critical percolation. Other than the spacing between vertices, there seems to be no lengthscale which plays a role for critical percolation. In this case one may hope to take some sort of limit without normalization of a critical percolation system in a larger and larger region. It is not clear in what topology one should take a limit. Matters look somewhat friendlier if one fixes a region and considers a limit as the spacing between vertices tends to zero. Even then it is not clear what topology will be most useful for taking a limit. A discussion of these issues can be found in [2] and the beginning of [59]. Putting this problem aside, let us first ask for limits of simple quantities such as crossing probabilities. Let D be a Jordan domain in \mathbb{R}^2 with a smooth boundary and let A_1 and A_2 be two disjoint arcs of ∂D . Identify \mathcal{G} with its periodic imbedding in \mathbb{R}^2 . (This imbedding is not unique, but for the present purposes we can just fix some imbedding.) We can then define $\delta\mathcal{G}$ as the result of multiplying the image of \mathcal{G} under the imbedding by a factor $\delta > 0$. This image has vertices located at $\{\delta v : v \in \mathcal{V}\}$ and edges between two points $\delta v', \delta v''$ if and only if v' and v'' are adjacent in \mathcal{G} . For any percolation configuration on \mathcal{G} we say that there exists an open path on $\delta\mathcal{G}$ from A_1 to A_2 in D if there is an open path v_1, \dots, v_m on \mathcal{G} such that $\delta v_i \in D$ for $2 \leq i \leq m-1$ and the edge between δv_1 and δv_2 intersects A_1 and the edge between δv_{m-1} and δv_m intersects A_2 . We then define

$$h(D, A_1, A_2, \delta) := P_{p_c}\{\exists \text{ open path on } \delta\mathcal{G} \text{ from } A_1 \text{ to } A_2 \text{ in } D\}, \quad (5.1)$$

and ask whether this has a limit as $\delta \downarrow 0$. (Here is where contact is made with De Volson Wood's problem in the Amer. Math. Monthly.) It is conjectured that this limit, call it $h(D, A_1, A_2)$, exists, and moreover that it is conformally invariant. By

this we mean that if ϕ is a conformal map from D onto $D' = \phi(D)$ which extends to a homeomorphism between $\bar{D} := \text{closure of } D$ and \bar{D}' , then

$$h(D, A_1, A_2) = h(\phi(D), \phi(A_1), \phi(A_2)). \tag{5.2}$$

Conformal invariance of a limit of critical percolation had been conjectured by physicists (see [17] and its references) on the grounds that this had been found in related models. The stress on studying this for crossing probabilities is due to [46], which also credits Aizenman with the formulation of conformal invariance for crossing probabilities (actually in a slightly more general form than (5.2)). Cardy used conformal invariance and the Riemann mapping theorem to equate $h(D, A_1, A_2)$ to $h(\mathbb{H}, [z, 0], [1, \infty))$, where \mathbb{H} is the upper half plane and $z \in (-\infty, 0)$ a suitable point on the boundary of \mathbb{H} , i.e., the real axis. He then derived (nonrigorously) a differential equation for $h(\mathbb{H}, [z, 0], [1, \infty))$ as a function of z . From this he obtained an explicit formula for $h(\mathbb{H}, [z, 0], [1, \infty))$, and hence for $h(D, A_1, A_2)$ in special cases, such as when D is a rectangle and A_1, A_2 two opposite sides of D . In an astonishing paper Smirnov [60] succeeded in showing that for site percolation on the triangular lattice, the limit $h(D, A_1, A_2)$ indeed exists and is conformally invariant. To do this Smirnov introduces an extra variable $z \in \bar{D}$, and considers

$$f(z, D, A_1, A_2, \delta) := P_{p_c} \{ \exists \text{ self-avoiding open path on } \delta\mathcal{G} \text{ from } A_1 \text{ to } B_1 \cup A_2 \text{ in } D \text{ which separates } z \text{ from } B_2 \},$$

where B_1, B_2 are the arcs on ∂D between A_1 and A_2 (i.e., the boundary of D consists of the four arcs A_1, B_1, A_2, B_2 and one successively traverses these arcs as one goes around the boundary of D in one direction). He now shows that any limit of $f(z, D, A_1, A_2, \delta)$ along a subsequence $\delta_n \downarrow 0$ is a harmonic function of $z \in D$ which has to satisfy certain boundary conditions which uniquely determine the limit. Therefore $\lim_{\delta \downarrow 0} f(z, D, A_1, A_2, \delta)$ exists. Moreover, the limit is conformally invariant, because it is characterized as the harmonic function which satisfies a certain boundary condition. The original problem for the crossing probabilities $h(D, A_1, A_2, \delta)$ can be treated as a special case, by letting z approach the single point in $\bar{A}_2 \cap \bar{B}_1$. One can find the limit function $h(D, A_1, A_2)$ explicitly if D is a rectangle, and A_1, B_1, A_2, B_2 its sides, and thereby one can recover Cardy's formula.

Somewhat before Smirnov, Schramm [59] had introduced *stochastic Loewner evolutions* (SLE) in order to describe a scaling limit of growing random sets (and in particular the scaling limit of loop erased random walk in dimension two, and related processes). For percolation, the simplest version of SLE is probably the so-called chordal SLE (see [54]), described as follows. Let \mathbb{H} and $\bar{\mathbb{H}}$ be the open upper and closed upper half plane, respectively, and let $\{B(t)\}_{t \geq 0}$ be a standard Brownian motion starting at 0. Let $g_t(z)$ be the solution of the Loewner equation

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - \xi(t)}, \quad g_0(z) = z \tag{5.3}$$

with $\xi(t) = \sqrt{\kappa}B(t)$ for some parameter $\kappa > 0$. The solution to (5.3) exists for $t < \tau(z) := \inf \{s : 0 \text{ is a limit point of the set } \{g_u(z) - \xi(u), u < s\}\}$. Define

$H_t := \{z \in \mathbb{H} : \tau(z) > t\}$, $K_t = \{z \in \overline{\mathbb{H}} : \tau(z) \leq t\}$. Chordal SLE_κ is the collection of maps $\{g_t : t \geq 0\}$. It turns out that g_t is the unique conformal homeomorphism from H_t onto \mathbb{H} for which $\lim_{z \rightarrow \infty} [g_t(z) - z] = 0$. It is shown in [54] that for $\kappa \neq 8$ there exists a continuous path $\gamma : [0, \infty) \rightarrow \overline{\mathbb{H}}$ such that K_t is the hull of $\gamma[0, t]$, that is, K_t is the closure of the union of the bounded components of $\overline{\mathbb{H}} \setminus \gamma[0, t]$. In many situations one can also start with the path γ and then define g_t as the conformal homeomorphism from its hull K_t onto \mathbb{H} . This must then satisfy a Loewner equation (5.3). γ is called the *trace* of the corresponding SLE process.

Schramm ([59]) showed that the scaling limit of loop erased random walk can be described by an analogue of SLE_2 in the unit disc. ([59] still had to assume that this scaling limit exists and is conformally invariant, but this has since been proven in [49]). In [59] Schramm expresses the belief that SLE_6 is appropriate for the description of the scaling limit of the boundary of percolation clusters. This has been proven to be correct for percolation on the triangular lattice.

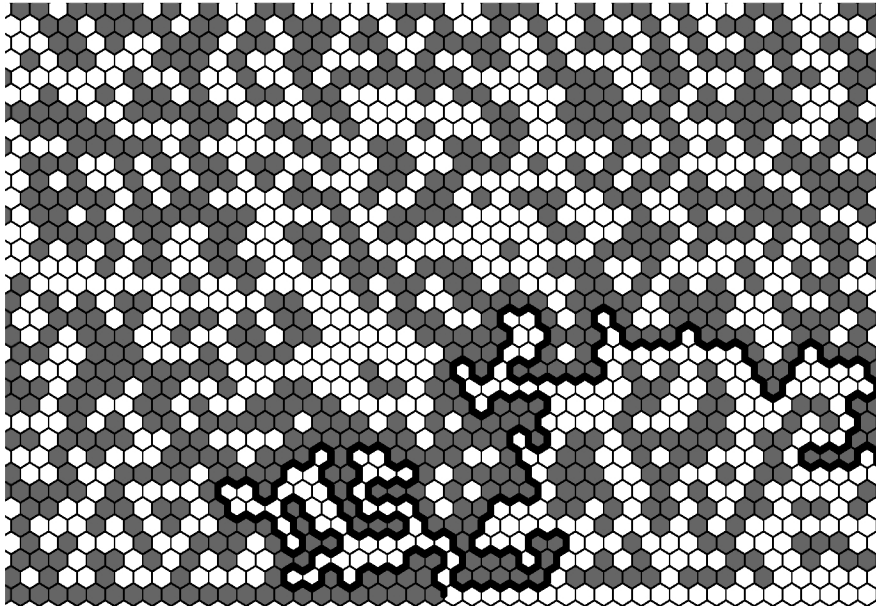


Figure 2: The exploration process, which separates the open (white) hexagons from the closed (black) ones. We thank Oded Schramm for providing us with this figure.

To give a specific example, consider the hexagonal lattice, imbedded in \mathbb{R}^2 in such a way that the hexagonal faces which intersect the x-axis have their centers on this axis and that the origin lies on the common boundary of two such faces. Make the hexagonal faces in the upper half plane independently open or closed with probability 1/2. If one thinks of the centers of the hexagonal faces as vertices of the triangular lattice then one sees that this is equivalent to critical site percolation

on the triangular lattice on \mathbb{H} (recall that its critical probability equals $1/2$). Now impose the boundary condition that all faces with center on the positive (negative) x -axis are open (closed, respectively). There is then a curve, γ_δ , in the upper half plane and running on the boundaries of some of the hexagonal faces, which starts at 0 and traverses the boundary between the open cluster of the positive x -axis and the closed cluster of the negative x -axis. This curve is called the *exploration process*; see Figure 2. The distribution of this curve γ_δ converges to the distribution of the trace of SLE_6 (as the mesh size goes to zero, and using the Hausdorff metric on the space of curves, determined up to parametrization) (see [60], [61]). Actually these references discuss the analogous situation on an equilateral triangle instead of \mathbb{H} and concentrate on showing the existence of the limit. The identification of the limit as SLE_6 is based on the work of Lawler, Schramm and Werner ([47], [64]).

To prove this result Smirnov ([60], [61]) first uses a compactness argument to show that any sequence $\delta_n \downarrow 0$ has a subsequence along which the distribution of γ_{δ_n} converges to some distribution $\mu^{e,p}$ on Hölder continuous curves. Then he proves that $\mu^{e,p}$ is independent of the subsequence $\{\delta_n\}$ by showing that $\mu^{e,p}$ has certain properties which characterize SLE_6 . This of course also shows that $\mu^{e,p}$ is the distribution of the trace of SLE_6 . The second step relies on a reduction of various $\mu^{e,p}$ -probabilities to crossing probabilities of the form (5.1) and on the existence and conformal invariance of the limit of (5.1). In addition it relies on a “locality property.” Note that one can construct γ_δ from “local” information only; at any step γ_δ turns to the right (left) if its tip has a closed (respectively, open) hexagon in front of it.

SLE turns out to be the perfect tool for calculating critical exponents. Lawler, Schramm, Smirnov and Werner in [48] and [62] were able to use the correspondence with SLE_6 to prove for percolation on the triangular lattice, not only Cardy’s formula, but also the power laws (4.2), (4.3), (4.6), (4.7) and (4.9) with the values for $\nu, \beta, \gamma, \delta$ and η which were predicted by physicists (see [62] for relevant references). It was further shown in [7] by Beffara that the Hausdorff dimension of the trace of SLE_6 is $7/4$. Thus, this is also the Hausdorff dimension of the exploration process “in the scaling limit.” This dimension had already been predicted by Saleur and Duplantier [56].

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