Geometric Model Theory

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"Contemporary symbolic logic can produce useful tools – though by no means omnipotent ones – for the development of actual mathematics, more particularly for the development of algebra, and it would appear, of algebraic geometry." This statement (with a reference to still older roots) was made by Abraham Robinson in his 1950 address to the ICM. Instances of such uses of logic include the correction and proof by Ax-Kochen of a p-adic conjecture of Artin's ([1]), and the Denef - Van den Dries proof of a p-adic analytic conjecture of Serre ([13]). The internal development of model theory since the 70's has led to entirely new techniques, that, combined with the older ones, have begun to find applications to diophantine geometry. It is the purpose of this talk to explain these methods and connections.

The present applications use only the finite-dimensional part of model theory (in a sense to be explained). Shelah and his followers created a theory of much greater generality (superstability, supersimplicity) incorporating many of the features of the finite dimensional case. One hopes that future applications will use this power. This exposition will limit itself to the finite-dimensional heartland (finite Morley rank, S1-rank).

Instead of defining the abstract context for the theory, I will present some of its results in a number of special, and hopefully more familiar, guises: compact complex manifolds, ordinary differential equations, difference equations, highly homogeneous finite structures. Each of these has features of its own, and the transcription of the general results is not routine; they are nonetheless readily recognizable as instances of a single theory. The current applications to dipohantine geometry arise by way of the difference and differential "examples". §2 and §6 will describe the model theory behind these results, and the prospects and difficulties lying ahead.

1 Example 1: compact analytic spaces

A complex manifold is a space obtained by gluing open discs in $\mathbb{C}^n$, using complex analytic gluing maps. A *closed analytic subset* of a complex manifold M is a closed subset, cut out locally by the vanishing of finitely many analytic functions. This defines a topology on M. An *analytic subvariety* is an irreducible closed analytic set, i.e. one that is not the union of closed proper subsets. Every closed subset in this topology has dimension strictly less than $\dim(M)$, and is the union of finitely many analytic subvarieties. By a (complex analytic) *space* we will mean, in this section, the complement of a closed analytic U' in a closed analytic subvariety V of a compact complex manifold. (Let $\mathcal{C}$ denote the class of such spaces.) We do not however wish to remember the *construction* of V, nor the sheaf of analytic functions or even the topology on V. Instead we are interested in describing the family $\mathcal{Z}(V)$, $\mathcal{Z}(V^n)$ of analytic subvarieties of M and of its Cartesian powers; and the interaction of V with other spaces W by means of $\mathcal{Z}(V \times W)$.

We would like to map out the category of analytic spaces X, with a view to the internal geometry of the subvarieties of X and of $X \times Z$ for other Z. We will that this category is not at all homogeneous: some spaces have a very rich internal geometry, others a very poor one; some interact with each other, some do not. The different features can be well differentiated by a close look at products of *minimal* varieties X, those that have no proper infinite subvarieties. This is the case though it is very far from being true that every variety can be decomposed as such a product.

Among the minimal varieties, we will find very sharp dividing lines. The *algebraic curves* lie in a class of their own. The non-algebraic complex tori fall into another distinct class; their geometry is essentially *linear*. The third class, about which model theory says least, consists of the minimal varieties whose geometry is trivial (at least generically) from our "subvarieties of Cartesian powers" point of view. These three classes exemplify a deep and general trichotomy, and in the present category has decisive influence on the geometry of all varieties (not just on products of minimal ones.)

Algebraic varieties Among the analytic spaces are those with the structure of algebraic varieties. These have a very rich geometry of subvarieties. In particular, in dimension > 1, they have algebraic *families* of subvarieties, having arbitrarily large dimension.

A general model theoretic principle, to be discussed later, shows that this richness *characterizes* algebraic varieties.

The complex projective space $\mathbb{P}^n$, for example, contains the family $\mathcal{F}_d$ of hypersurfaces cut out by homogeneous polynomials of degree d in $n+1$ variables; this family is parameterized by $\binom{n+d}{d-1}$-dimensional projective space.

Intersecting the elements of $\mathcal{F}_d$ with a projective variety - a subvariety V of $\mathbb{P}^n$ - yields large families of subvarieties on V. We thus see in passing that any projective variety is "rich" (V or V^2 have many subvarieties.) By the model theoretic characterization alluded to above, it follows that projective varieties are algebraic. This indeed fits in with a classical theorem of Chow's, asserting in more detail that projective varieties are automatically defined by finitely many homogeneous polynomials.

1.1 Minimal spaces and the semi-minimal analysis M is called *minimal* if it has no proper analytic subvarieties, other than points.

Every one-dimensional (connected) complex manifold is minimal, but there are also many others. For example, if Λ is the subgroup of $\mathbb{C}^n$ generated by a sufficiently general $\mathbb{R}$-basis, the *torus* $T = \mathbb{C}/\Lambda$ is minimal.

Given a minimal M, a subgroup G of $Sym(n)$, one can form the space M^n/G. Such spaces, as well as subspaces of their finite products, will be called *semi-minimal*. We will later (1.6) obtain a good description of semi-minimal spaces (in terms of minimal ones.)

The following theorem is an instance of Shelah's theory of "regular types" (adapted to minimal types using a contribution of Lascar's.)

Theorem 1.1 *Let $V \in \mathcal{C}$. There exists a minimal space $Y \in \mathcal{C}$ and a $F \in \mathcal{Z}(V \times Y)$, inducing a morphism from the complement of an analytic subset V' in V, onto a subspace of $Y^{[k]}$.*

The theorem provides a proper closed subvariety V_0 of V, and a map $f : (V \setminus V_0) \to L_1$ with L_1 semi-minimal. (f is defined by: $f(a) = \{b : (a,b) \in F\}$.) Once f is found, the theorem can be re-applied to V_0 and to each fiber of f. This process, "the semi-minimal analysis", terminates after a finite number of steps.

Remark 1.1 *There is a largest semi-minimal image V_{sm} of V (in the sense of 1.1); it is unique at least up to "birational isomorphism" (or even a constructible bijection).*

1.2 Orthogonality Let X,Y be a variety. We say that X *dominates* Y if there exists a subvariety Z of $X \times Y$, such that the projection of Z to X has finite fibers, while the projection to Y is surjective (or it may miss a proper closed subset.) For *algebraic* varieties, X dominates Y iff $\dim(X) \geq \dim(Y)$. However this is far from being true in general.

Two varieties X,Y are called *orthogonal* if every proper subvariety $T \subset X^m \times Y^n$ is contained in $U \times Y^n$ or in $X^m \times V$ for some closed analytic U, V of smaller dimension. When X,Y are minimal, this implies that every closed subvariety of $X^m \times Y^n$ is a rectangle $U \times V$.

Theorem 1.2 *[Shelah]*

1. *For minimal X,Y, X dominates Y iff they are not orthogonal. Domination is an equivalence relation on minimal spaces.*

2. *Each X dominates a finite number of minimal Y (up to domination equivalence.) For each such Y, there exists a maximal integer m such that X dominates Y^m.*

3. *Two varieties are not orthogonal iff they dominate a common minimal.*

1.3 Non-orthogonality and liaison groups If a minimal, occurring beyond the first level of the semi-minimal analysis is non-orthogonal to an earlier one, their interaction must be mediated by a group action. For example:

Theorem 1.3 *Let X be a space, $f : X \to X_{sm}$ the maximal semi-minimal quotient, and $a \in X_{sm}$, $X(a) = f^{-1}(a)$, and let $g : X(a) \to X(a)_{sm}$ be the semi-minimal quotient of $X(a)$. If $X(a)_{sm}$ is an algebraic variety, it is a homogeneous space for an algebraic group.*

1.4 Dimensions Each compact complex manifold has a complex analytic dimension, the number of complex parameters needed locally to determine a point. A more intrinsic dimension from our point of view assigns each minimal space dimension 1. More generally, we say inductively that X has (Morley) dimension $d+1$ if it does not have dimension $\leq d$, and contains an infinite collection of subvarieties X_i of dimension d, with $\dim(X_i \cap X_j) < d$ for $i \neq j$.

It can be shown that for minimal X, for any $Y \subset X^n$, $\dim_{Morley}(Y) = e \dim_{\mathbb{C}}(Y)$ where $e = \dim_{\mathbb{C}}(X)$ does not depend on Y. (This resembles the relation between complex and real dimension, with $e = 2$.) When working systematically with the geometry of X^n and its subvarieties, the intrinsic dimension is helpful even if one is already aware of the complex analytic dimension. For instance, subspace of dimension one are treated as *curves*; it is useful to know in advance that the intersection of two such curves must be finite (as does not follow directly from the analytic dimension.)

1.5 Classification of minimal spaces: ampleness vs. modularity

Families of varieties Given $X \in \mathcal{Z}(M \times P)$, and $a \in P$, let

$$X(a) = \{b \in M : (b,a) \in X\}$$

Then $X(a) \in \mathcal{Z}(M)$. As a varies through P, (or perhaps through the complement in P of a proper closed analytic subvariety), we will say that the varieties $X(a)$ form a uniform family of subvarieties of M. Without changing the family of sets $X(a)$, it is possible to replace X and P in such a way that the sets $X(a)$ are distinct for distinct elements $a \in P$. The *dimension* of the family is then $\dim(P)$.

A space is called *geometrically modular* if, for each k, there exists an absolute bound to the dimension of any uniform family of subvarieties of V^k. The significance of this condition will be explained later; for now we view it as an expression of a sharp difference between algebraic curves and the other minimal varieties. For minimal V, it can be shown that the bound is $k-l$, where $l = \dim X(a)$.

The terms "locally modular" and "1-based" are also used in the literature. The first refers to a condition on the lattice of algebraically closed subsets, that we will not enter into here. The latter refers to the following:

Definition 1.2 *A space V is* 1-based *if for any k, through a sufficiently general point $a \in V^k$, there pass only countably many subvarieties of V^k.*

Equivalently, no uniformly definable family of subvarieties intersects in that point. A dimension - counting argument shows that geometric modularity is equivalent to 1-basedness.

Example: non-algebraic tori Some complex tori can be embedded in projective space; the embedding is then an algebraic subvariety of projective space (defined by the vanishing of finitely many homogeneous polynomials.) These are called Abelian varieties, and have a rich structure of subvarieties; they are not geometrically modular. We will see later however that any minimal complex torus that is not an Abelian variety is geometrically modular. For a sufficiently general torus, the subvarieties of T^n passing through a point $a = (a_1, ..., a_n)$ are not only countable in number but completely explicit: they are defined by equations of the form $\sum n_i(x_i - a_i) = 0$.

Theorem 1.4 *Let $V \in \mathcal{C}$ be minimal, and not algebraic. Then V is geometrically modular.*

1.6 Classification of minimal spaces: geometric triviality If V is a geometrically modular minimal space, through a typical point of V^k there pass at most countably many curves. There are always at least k curves through $a = (a_1, \ldots, a_k)$, namely those "parallel to the axes": $(a_1, \ldots, a_{k-1}) \times V, \ldots, (a_1) \times V \times (a_3, \ldots, a_k), V \times (a_2, \ldots, a_k)$.

Call V *geometrically trivial* if for every $a \in V^k$,(except perhaps for a finite union of proper subvarieties), these k curves are the only ones passing through a. (This condition implies equally strong constraints on subvarieties of higher dimension passing through a general point.)

A complex torus T can never be geometrically trivial. For example, for each rational $\frac{a}{b}$ ($(a, b) = 1$) and any point $c = (c_1, c_2) \in T^2$, one has the subvariety

$$\{(y_1, y_2) : ay_1 + by_2 = ac_1 + bc_2\}$$

passing through c.

It can be shown more generally that a subvariety of a group variety can never be geometrically trivial.

Theorem 1.5 ([15]) *Let V be minimal, modular, and not geometrically trivial. Then there exists a minimal U equivalent to V and admitting a group structure, whose graph is a subvariety of U^3.*

Putting together Theorems 1.4, 1.5, we obtain

Corollary 1.3 (Trichotomy) *Every minimal variety X is geometrically trivial, or equivalent to a geometrically modular group variety, or is algebraic.*

It can be shown, from modularity, that a geometrically modular group variety U must be commutative ([19]). It is very likely that U must be a complex torus; this requires proof, and provides an example of the kind of work needed to adapt the general theory to a special context.

1.7 INTERNAL STRUCTURE OF SEMI-MINIMAL SETS

THEOREM 1.6 *Let X be a minimal variety. Let Y be a subvariety of X^n.*

1. *If X is algebraic, then Y is algebraic.*
2. *If X is a geometrically modular group, then Y is defined by linear equations $\sum a_{ij} X_i = b_j$, with respect to the group structure, and certain analytic endomorphisms a_{ij}.*
3. *If X is geometrically trivial, then Y is a direct product of minimal varieties Y_j*

Item (1) (with $X = \mathbb{P}^1$ or $\mathbb{P}^n$) is a classical theorem of Chow's. In model theoretic language, the *induced structure* on the complex analytic X is precisely that given by *algebraic* geometry. Here the result is derived from a general model-theoretic *recognition theorem* for algebraic geometry, ([23]). Having recognized algebraic geometry, the model theory hands the variety over to methods best suited to it.

Item (2) (taken from [19]) shows that the induced structure on complex tori is given by *linear algebra* (over the endomorphism ring.) The linearity is relative to the group structure; it is not comparable within the category we work in to the additive group of $\mathbb{C}$.

In (3), each Y_j is a subvariety of $X^{l(j)}$, a certain product of $l(j)$ of the n factors of X^n. The statement is a fairly direct consequence of the definition of geometric triviality. Note that (3) gives no information in the case $\dim(Y) = 1$. In this respect the information concerning geometrically trivial varieties is less decisive than in the other cases.

COROLLARY 1.4 (TO (3)) *Let A be a geometrically modular group variety, minimal as a group variety. Then A is a minimal variety.*

Thus if a non-algebraic torus has no proper nontrivial sub-tori (a condition easily verified), then it has no proper analytic subvarieties of any kind (other than points.)

Combining Theorem 1.6 with the theory of orthogonality, we see that a subvariety of a product of geometrically modular group varieties, geometrically trivial varieties, and algebraic varieties, is itself a product of the same form. Any semi-minimal variety is domination-equivalent to such a product.

1.8 LOCAL-GLOBAL PRINCIPLES The above theory of minimal and semi-minimal varieties is useful to the extent that global properties of arbitrary varieties can be reduced to properties of their minimal components. This happens often; we give just one example here.

THEOREM 1.7 ([5]) *Let $V \in \mathcal{C}$, and assume every minimal variety occurring in the semi-minimal decomposition of V is geometrically modular. Then so is V.*

In view of 1.4, this expresses the idea that $\mathcal{Z}(V)$ can be "large" *only* as an effect of algebraic varieties within V. As a corollary, one can globalize also 1.4(2):

Theorem 1.8 *Let X be a complex torus. Assume X has a maximal chain $(0) = V_0 \subset V_1 \subset \ldots \subset V_n = X$ of sub-tori, and no quotient V_{i+1}/V_i is an Abelian variety. Then the conclusion of 1.6 (3) holds for X.*

Sometimes just one layer in the semi-minimal analysis controls the situation. Shelah's Theorem 1.2 (3) is an example of this, using the first layer alone. Here is an example where only the *last* layer matters. It is a local-global principle for the notion of geometric triviality.

Theorem 1.9 *Let $g : X \to Y$ be the last stage of the canonical semi-minimal analysis. Assume the minimal varieties associated with the semi-minimal fibers X_b $(b \in Y)$ are all geometrically trivial, of dimension $\leq n$ say. Then through any $a \in X^m$ (outside some proper subvariety) there pass at most mn distinct curves (one dimensional spaces.)*

2 Model theoretic inputs: finite Morley rank theory

The theory described in the last section was in reality developed in a more general context. We stated it for compact complex manifolds essentially as a device of exposition, hoping to illuminate the general theory without plunging immediately into abstraction. We will now make some comments on the model theoretic setting.

2.1 Quantifier elimination A *first-order structure* in the sense of model theory has many "universes", called *sorts*. The sorts are assumed to be closed under finite Cartesian products; if a structure with a single universe M is presented, the other sorts will be the Cartesian powers M^n; it is there that the model theory will take place. One is given a family of subsets of the various sorts, the *basic relations*. One considers not only the given subsets, but also others formed from them using the "first-order operations": pullbacks and pushforwards under projections and diagonal maps, finite unions and intersections, and complements. *Any hope for a useful model theory depends on some control over the outcome of the first-order operations*. The strongest form of this control is:

Quantifier-elimination: Every projection of a Boolean combination of basic relations, is itself a basic relation.

(cf.[7]). This must be achieved separately in each application, and is rarely trivial.

In the example presented in §1, the sorts are the complex manifolds; the basic relations are the complex analytic subvarieties. Quantifier elimination was proved by Boris Zil'ber; the main ingredient is the theorem (Remmert, Grauert) that images of analytic subvarieties under proper maps are analytic.

Zil'ber also proved that the structure consisting of compact complex manifolds satisfies the appropriate axioms of dimension theory, so that the general results on structures of finite Morley rank, and on Zariski geometries [23], apply.

2.2 Structures with dimension §1 is a simple transcription of a part of the theory of structures of finite Morley dimension. These are first order structures, with a non-negative integer-valued function on the definable sets, satisfying the condition in §1.3. The same dimension theory will work for differential algebra. For our difference and quasi-finite examples, we will use a modification, S1-dimension, defined in the same way but with $X_i = X(a_i)$ assumed to be taken from a uniform family.

The theory of semi-minimal reduction, and the theory of orthogonality, are due to Shelah ([37]). They are instances of his much more general theory of regular types in superstable theories. A part of the theory, in the finite dimensional case, appeared in the work of Morley and of Baldwin-Lachlan on categoricity. The books [34], [4],[35] are general references for this section, and contain further references.

Modularity is the most important concept of geometric model theory. It appeared first in work of Lachlan's [28] on the $\aleph_0$-categorical theories, and of Zilber's in the $\aleph_1$-categorical and totally categorical theories ([40]). There are many equivalent definitions of modularity; Lachlan's original definition involved the absence of *pseudo-planes*, structures modeled roughly on plane geometry. The idea is the existence of a sharp dividing line between the combinatorial and linear worlds (modularity), and between nonlinear, geometric complexity, as found in algebraic geometry. This was successfully generalized from the categorical cases to the superstable and general stable frameworks, and beyond that (perhaps not yet in full) to simple theories. It is clear that the idea continues to be meaningful and important in much wider domains, not yet technically developed.

Theorem 1.4 follows from the main theorem of [23]. It states that structures with a dimension theory having the basic properties of the dimension theory of algebraic varieties, and with *large* uniform families of subvarieties, must arise from algebraic geometry. It is not assumed there that the structure arises from analytic geometry or from any other specific geometry. The "basic properties" are here understood to include the "dimension theorem": intersection with a codimension - one variety lowers dimension by at most one, in every component. This is the only general result used in §1 that requires assumptions beyond that of finite Morley rank. This was originally conceived as a foundational result, showing that algebraic geometry is sui generis.

The proof of [23] involves geometric constructions in powers X^n, using the intrinsic dimension. One-dimensional sets are viewed as curves, and one constructs tangent spaces to them synthetically. (Note that this is applied, in §1, to complex analytic spaces, where Morley dimension one translates to higher complex dimension!)

The analogous theorem is now known ([33]) for structures with a dimension theory analogous to that of the reals (called "O-minimal" to recall the ordering; cf. [38].) A similar result may well be true for much more general types of geometries, including in particular p-adic geometries, and it would be valuable to develop it. The rest of the theory in §1 has not been developed even for the O-minimal context (where "semi-minimality" is in effect built into the assumptions.)

3 Differential equations

(General reference: [31]) A theory fully parallel to that of §1 exists for algebraic ordinary differential equations. The most interesting difference is the identification of the nontrivial, geometrically modular objects; the non-algebraic tori of §1 are replaced with certain equations, discovered by Manin and deeply studied by Buium, associated to any algebraic family of Abelian varieties. It is at first surprising that such a preliminary model-theoretic investigation of the basic geography of algebraic differential equations should discover Abelian varieties in a special role.

The results apply more generally to systems of (nonlinear) algebraic partial differential equations whose set of solutions is finite-dimensional in an appropriate sense. (In classical language, "the general solution involves only finitely many arbitrary constants".) Technically, we fix a field k, and let $k\{X\}$ be the ring of differential polynomials over k in variables $X = (X_1, \dots, X_m)$. We use ODE's or PDE's; in positive characteristic, we use Hasse- Witt derivatives. We assume the equations generate a differential ideal J such that for every prime $p \supset J$, $k\{X\}/p$ has finite transcendence degree over k. This condition is automatic for a nontrivial ODE in one variable. In characteristic $p > 0$, on the other hand, infinitely many equations are required.

An important open problem is the extension of the theory to less constrained systems of PDE's; Shelah's theory of superstability is available, but not the required generalization of the trichotomy theorem [23] (analog of 1.6(1)).

The necessary *quantifier elimination* was achieved by A. Robinson in characteristic 0, Delon, Ershov, Wood in positive characteristic; (cf. [12]). Certain verifications concerning the dimension theory, and the identification of the geometrically nontrivial minimal modular sets, are from [20]. (The approach we take here will make both of these essentially immediate, for finite dimensional systems.)

It is here that applications to diophantine geometry first arose, using a connection discovered by Buium, [6]. The model theory handles all characteristics with equal ease. It provides the only known proof of the Mordell-Lang conjecture in characteristic $p > 0$; cf. [17], [18] [2]. We will not go into details here, but will discuss a related result in §4.

There are several possible ways to describe the first order structure associated with such differential equations.

1) The standard model theoretic approach defines a *universal domains* for differential algebra. These are differential fields, in which every consistent, countable set of differential equations has a solution. The *sorts* can be taken to be the solution sets in this universal domain, to given equations; the basic relations, called Kolchin-closed sets, are defined by further equations.

2) One can define the category using the differential equations themselves, disregarding the sets of solutions.

3) The variant we will use will is a purely geometric representation of the differential equations. (It uses points again, but these are related to the points of the sorts of (1) only indirectly, via (2)). We will restrict attention to characteristic 0, and to ODE's, and work over an algebraically closed base field k with a trivial derivation.

The sorts will be smooth algebraic varieties endowed with algebraic vector fields; i.e. of of pairs (V, s) where V is a smooth variety over k, and $s : V \to TV$ is a section of the tangent bundle. The product of two sorts (V, s) and (V', s') is naturally defined. The basic relations are now the integral subvarieties, i.e. the algebraic subvarieties U of V such that s restricts to a section of TU. (Formally or analytically, we can define a flow corresponding to s; the integral subvarieties are then those fixed by the flow, and it is not surprising that their Boolean combinations are closed under projections.)

We will be interested in algebraic families $\{U\}$ of algebraic subvarieties V, that are left invariant by the flow corresponding to s. Such a family can be obtained by first taking the product of (V, s) with another object (P, t), fixing an integral subvariety R of $(V \times P, (s, t))$, and then letting

$$\{U\} = \{R(p) : p \in P\}$$

with

$$R(p) = \{a \in V : (a, p) \in R\}$$

Any element of an invariant family will be called s-coherent. $\mathcal{Z}(\underline{V})$ is the set of s-coherent subvarieties of V. Thus every point is s-coherent, as well as every integral subvariety of s. We will refer to refer to these as differential -algebraic varieties.

As in §1, we are interested in criteria for the abundance or scarcity of subvarieties of a given flow; the geometry of such subvarieties; and of the reducibility of one vector field to another by algebraic or algebraic differential transformations. The theory of §1 has a perfect analog here. Here, $\underline{V}$ is *minimal* iff V has no s-coherent subvarieties, except for points and all of V.

In particular, the trichotomy is true in this context. We must however identify the analogs of algebraic varieties, and the geometrically modular groups.

If the vector field is trivial, $s = 0$, every subvariety is an integral subvariety, and the geometry on V is ordinary algebraic geometry. It can be shown conversely (Ph.D. theses of Mesmer, Sokolovic; cf. [2]) that a minimal set, abstractly bi-interpretable with an algebraically closed field, must be isomorphic to a curve C endowed with the zero vector field. Let us call such minimal differential varieties *algebraic*. The corresponding semi-minimal sets are closely connected to the algebraically integrable flows. Part of the theory will thus take the form, in the present context, of recognition results for algebraically integrable vector fields.

The analog of non-algebraic complex tori is interesting. We are looking for the minimal coherent sets, possessing a group structure, and satisfying the conclusion of 1.4. The right equations were discovered by Manin, [30], and by Buium in a role closer to their status here. (A quick description, essentially following Buium: Let $A \to U$ be a family of Abelian varieties. For $v \in U$, let M_v be the maximal extension of A_v by a vector group. We have $M \to A \to U$, and now any vector field t on V canonically lifts to a vector field s on M: we have $TM \to TV$; the group structure on M_v can be prolonged to one on $N_v = (TM)_{(v,t(v))}$, so that N_v becomes an extension of M_v by the vector group TM_v; since M_v is the universal vector extension of A, there exists a unique section of $N_v \to M_v$. This gives s.)

Theorem 3.1 *There is a 1-1 correspondence between non-isotrivial families of Abelian varieties over k, up to isogeny, and families of geometrically modular minimal differential varieties. up to equivalence*

"Non-isotrivial" means essentially that the different Abelian varieties in the family are not isomorphic to each other. The equivalence of minimal sets is that of non-orthogonality, §1.2. This recognition theorem ([20]) allows us to state the trichotomy of [23] thus:

Theorem 3.2 *Every minimal differential algebraic variety is either geometrically trivial, algebraic, or equivalent to a Manin-Buium variety*

We also obtain a theorem on the internal structure of Manin-Buium varieties similar to 1.6, in particular 1.6 (3). This result was reproved by Buium and Pillay by analytic methods. The trichotomy has no analytic proof at present.

Geometrically trivial equations Geometric triviality severely limits the possible complexity of the internal geometry on a minimal differential variety V, but leaves open the question of its precise structure. For ODE's of differential order one, we have a complete answer. It is essentially the simplest possible one, of no structure at all. A differential variety V *has trivial internal structure* if the only subvarieties of V^m are the coordinate subvarieties V^l (defined by equations $X_i = a_i$.) Equations defining such varieties can have only a finite number of algebraic solutions; indeed over a differential field of transcendence degree k, they can have at most k solutions. Conversely the condition of finitely many algebraic solutions over a finitely generated field, characterizes geometrically trivial equations, up to equivalence.

Theorem 3.3

- *Let X be a geometrically trivial ODE of order* 1. *There exists a finite map $g : X \to Y$, Y another ODE of order* 1, *such that Y has trivial internal structure.*
- *$X = \{X_a : a \in T\}$ be a family of geometrically trivial ODE's of order* 1, *and assume the generic X_a is geometrically trivial. Then there exist differential rational maps $b : T \to T'$, another family Y of order* 1 *ODE'S, AND $g : X \to Y$, such that (for generic a, with $b = b(a)$) X_a is equivalent to Y_b, Y_b is trivial, and such that Y_b, $Y_{b'}$ are equivalent only if $b = b'$.*

This kind of control over the internal structure and the variation of arbitrary *minimal* ODE's would make for a much more powerful theory (about arbitrary algebraic ODE's).

(1) is proved ([26]) by a slight modification of [25], while (2) is proved by a combination of model-theoretic and geometric methods (see [22] for the case of positive genus.) Further results would presumably be proved geometrically, perhaps by extensions of the method of [25]; the model theory may be helpful in suggesting the correct higher dimensional version (the Manin-Buium equations must be taken into account.)

We note that Jouanolou's theorem [25] was used directly by Vojta, to bound the number of rational points on curves over function fields. The model theoretic method uses Buium-Manin equations for similar results applying to subvarieties of Abelian varieties. These results use only one part of the trichotomy, the gap between geometrically modular and algebraic. This state of affairs suggests that the gap between geometrically trivial minimal varieties, and between geometrically modular groups (1.5, here 3.2) may be used for results on rational points on varieties of general type. A higher-dimensional version of 3.3 would be one of the missing ingredients for such an attempt.

Here is a statement of the trichotomy that does not mention minimality. The proof combines the trichotomy and the analogs of 1.9 and 1.3. (The statement of this theorem in the abstract contained an inaccurate mixture of the languages of approaches (2) and (3).)

Theorem 3.4 *Assume $\underline{V} = (V, s)$ is not geometrically trivial. After possibly removing from V a finite number of lower dimensional integral subvarieties, and possibly pushing forward by an s - equivariant map with finite fibers, one of the following occurs:*

a. *There exists a map $f : V \to W$, W an algebraic variety of dimension ≥ 1, such that the vector field s is parallel to the fibers of f.*

b. *There exists an equivariant map $f : V \to V'$, V' an algebraic variety of smaller dimension carrying a vector field s', such that the fibers of f are principal homogeneous spaces for algebraic groups; and the action respects the vector field.*

c. *There exists a map $f : V \to V'$ as in (3) such that s is the pullback over V' of a Buium-Manin family.*

4 Difference equations

A *difference equation* is analogous to a differential equation, but involves a discrete difference operator σ in place of a differential operator. Classically one thinks of the field of rational or meromorphic functions, and defines $f^\sigma(z) = f(z+1)$, or $f^\sigma(z) = f(qz)$. The Leibnitz rule is replaced by the fact that σ is an automorphism : $\sigma(fg) = \sigma(f)\sigma(g)$. Thus a *difference domain* is defined to be an integral domain with a distinguished field endomorphism. (See [11]).

There are also arithmetic sources of difference equations: the Galois group of $\mathbb{Q}$, and the Frobenius endomorphisms $x \mapsto x^{p^m}$ in characteristic $p > 0$. The latter play a fundamental role among all difference domains; for instance it can be shown that a simple, finitely generated difference domain (L, σ) always has $\sigma(x) = x^{p^m}$ for some p and m. We will not enter here into this story.

The theory described in §1, §2 is available in full, though a great deal more work is needed to access the model theoretic inputs or reprove them in suitable form ([8]). In particular a semi-minimal analysis and a trichotomy theorem exist. Here we will just highlight two of the places where the theory complements rather than merely parallels the differential case.

4.1 Fixed fields It can be shown that the equation $x^\sigma = x$, defining the fixed field, is one-dimensional for an appropriate dimension theory; it is an analog of the minimal varieties encountered before. It corresponds to $Dx = 0$ in the differential case and in characteristic 0, it is the only non-geometrically modular minimal difference variety. (In characteristic $p > 0$, one must add equations such as $x^{\sigma^2} = x^p$.) The situation is more interesting however in that the fixed field is not algebraically closed, even in a universal domain for difference fields.

For example, in the differential case, it can be shown either by means of differential Lie theory (Phyllis Cassidy) or of model theory (Sokolovic) that every *simple group* defined by differential equations, and finite-dimensional in our sense, is isomorphic to an algebraic group over the field of constants. In the difference case, twisted groups arise. Let G be a simple algebraic group, and let $h : G \to G$ be a graph isomorphism of G. Then one can use difference equations to define a subgroup of G:

$$G(h;\sigma) = \{a \in G : h(a) = \sigma(a)\}$$

For instance, if $G = GL_n$, and $h(M) = M^{t^{-1}}$ for a matrix M, then $G(h;\sigma)$ is the unitary group U_n over the fixed field of σ^2, with respect to the conjugation σ of that field.

While the classification up to isomorphism is possible, we will only discuss the classification up to virtual isogeny (G_1,G_2 are virtually isogenous if there exists G and homomorphisms $h_i : G \to G_i$ with finite kernel, and image of finite index.) It can be shown that $G(h;\sigma)$ defines (in the universal domain) a group virtually isogenous to a simple one.

Theorem 4.1 *A simple group definable by difference equations is virtually isogenous to some $G(h;\sigma)$*

This gives a connection to finite simple groups, more precisely to "horizontal" families of finite simple groups (e.g. $PSL(n,q)$ with fixed n and varying q.) One obtains an infinite family of (almost) simple groups from $G(h;\sigma)$ by letting $G(h,q)$ be the solutions to $G(h;\sigma)$ in the "Frobenius difference field", the difference field consisting of an algebraically closed field of characteristic $p > 0$, and the automorphism $\sigma(x) = x^q$. All the families occur (including the Ree and Suzuki groups) making the statement of the classification very natural in this context. See [HP 94], [21]

4.2 Geometrically modular, nontrivial equations. In the case of differential algebra, they corresponded to non-isotrivial simple Abelian varieties. In characteristic 0 difference algebra, they still lie on simple Abelian varieties, but precisely on those whose isogeny class is defined over a finite extension of the fixed field (as well as on the multiplicative group G_m). They correspond to non-cyclotomic irreducible equations over the endomorphism group. For example, let $f(T) = \sum a_i T^i$ be a polynomial over $\mathbb{Z}$. Let E_f be the subgroup of the multiplicative group defined by $X^f(\sigma) = 1$, or more precisely

$$\Pi_{a_i>0}\sigma^{a_i}(X) = \Pi_{a_j<0}\sigma^{-a_j}(X)$$

THEOREM 4.2 *E_f is minimal iff f is irreducible over $\mathbb{Z}$. Whether or not it is minimal, E_f is geometrically modular iff E_f has no cyclotomic factors. In this case, every subset of $(E_f)^n$ defined using difference equations is a Boolean combination of subgroups and their cosets. In particular this is true for the intersection of any algebraic variety with $(E_f)^n$.*

A similar result is true for Abelian varieties. For the multiplicative group, at least for the simple equation we will consider below, it is easy to prove directly. The proof in [16] uses the trichotomy, proved for difference equations in [8]: non-linearity inside a group implies non-modularity; this implies the presence of a field; one recognizes the field as a finite extension of the fixed field, thus involving the equation $\sigma^n(X) = X$, or E_{T^n-1}; the non-orthogonality of E_f to this equation implies that f is cyclotomic.

4.3 FINITENESS FOR TORSION POINTS In [16], the above was used to give a new proof of the Manin-Mumford conjecture on torsion points on semi-Abelian varieties, proved originally in (for curves on Abelian varieties) in [36]. The conjecture states that the number of torsion points on a curve of genus > 1 is finite; more generally, any variety intersects the torsion points in a finite union of translates of group varieties. The new proof gives effective and indeed explicit (though doubly exponential) bounds; this is automatic from the difference-algebra nature of the proof, more precisely from the fact that one bounds the number of points of a certain difference equation *in any difference field* and not only in number fields.

Here is the proof for the case of curves on powers of the multiplicative group (where the result goes back at least to Lang.) Let a be an even-order root of unity. Then a^3 is a root of unity of the same order. So there exists an automorphism σ of $\mathbb{Q}(a)$ with $\sigma(a) = a^3$. Similarly if $a^n = 1$, n odd, there exists an automorphism σ with $\sigma(a) = a^2$. Putting these together, and letting $f(T) = (T-3)(T-2)$, we can find an automorphism σ such that $E_f = E_f(\sigma)$ contains all roots of 1. Now by 4.2, the intersection of any curve with $(E_f)^n$, in any difference domain, is finite unless the curve is a multiplicative translate of a subgroup of $(G_m)^n$, i.e. it is defined by a purely multiplicative equation. A fortiori this holds for the smaller set consisting of the roots of unity.

4.4 TATE-VOLOCH CONJECTURE

CONJECTURE 4.3 (TATE-VOLOCH) *Let A be an Abelian variety over $\mathbb{C}_p$, the completion of the algebraic closure of $\mathbb{Q}_p$. Let $C \subset A$ be a curve of genus > 1, and let T be the group of torsion points of A. Then there exists a finite $F \subset T$ and a p-adic open neighborhood of $T \setminus F$, that meets C in a finite set.*

Certain cases were proved by Buium, Silverman, Tate-Voloch. When A is an Abelian variety over $\mathbb{Q}_p$ with good reduction, and one considers only torsion points T_p of order prime to p, the proof of the Manin-Mumford conjecture above – combined with a standard idea of nonstandard analysis – immediately yields a proof of Tate-Voloch. A sketch:

The assumptions are used to find a geometrically modular difference equation E_f, and an automorphism σ of $\bar{\mathbb{Q}}_p$, such that $T_p \subset E_f$ in the difference field $(\bar{\mathbb{Q}}_p, \sigma)$.

By 4.2, $F = C \cap E_f$ is finite in *any* difference field.

Assume $E_f \setminus F$ has points arbitrarily close to C. Then, using the compactness theorem of model theory, or nonstandard analysis, one can find a field L extending $\mathbb{Q}_p$ with a nonstandard p-adic valuation, and a point a on E_f whose distance to C is infinitesimal. Modifying the field by identifying sufficiently near elements, we obtain a residue difference field $\bar{L}$ and a point $\bar{a}$ on $E_f \setminus F$, whose distance to C is zero. Then $\bar{a} \in (C \cap E_F) = F$, a contradiction.

Note that this proof could not work directly with T or T_p in place of E_f; a "nonstandard torsion point" is just not torsion, nor has any other immediately obvious properties; whereas E_f is defined by an equation, so is respected by ultraproducts.

This proof was improved by Thomas Scanlon, both in the number theory part (obtaining the automorphism f under less restrictive conditions) and the model theory (using orthogonality as well as geometric modularity.) He proved:

THEOREM 4.4 *The Tate-Voloch conjecture is true when A is over a finite extension of $\mathbb{Q}_p$.*

5 QUASI-FINITE STRUCTURES

5.1 LIE-COORDINATIZED STRUCTURES

In the previous examples, a first-order structure was given; the existence of a dimension theory, a semi-minimal decomposition, and a structure theory for the minimal geometries was proved. Here we will go in the opposite direction. A certain class of linear geometries ("basic Lie geometries") is explicitly defined, and one considers structures having a semi-minimal analysis in terms of these geometries. ("Lie- coordinatizable structures".) One then proves the existence of a global dimension theory, global modularity, a structure theory for definable groups, existence of good finite approximations, axiomatizability, and other properties. The results of this section are from [10].

5.1.1 THE BASIC GEOMETRIES The full list includes all the "classical geometries" (Weyl): linear, unitary, orthogonal, symplectic; over an arbitrary *finite* field. (There are also some slightly less classical variants.) For definiteness, we take them to be $\aleph_0$-dimensional (later finite dimensional ones will be considered too.)

The simplest three examples:

1. A pure set X. (The only relations on X^n are the diagonals.)

2. A vector space V over $GF(3)$. (The basic relations: $\sum a_i X_i = 0$.)

3. A vector space V over $GF(3)$ with a symmetric bilinear form $V \times V \to GF(3)$.

4. A pair (V, V^*) of vector spaces over $GF(2)$. (Basic relations: addition on V and on V^*; a pairing $(\ ,\ ) : V \times V^* \to GF(2)$.)

We will be interested in these geometries when they are *embedded* in a structure M. This means (e.g. in case (2) above:) V coincides with a sort in M, or with a definable subset of a sort in M; and *a subset of V^n is definable in M if and only if it is definable in the vector space V.* (In case (2), iff it is a finite Boolean combination of relations $\sum a_i X_i = 0$.)

When more than one geometry is involved, say two geometries J_1, J_2, we will assume they are *jointly embedded*: the disjoint union of J_1,J_2 as structures, is embedded. This is equivalent to an *orthogonality* condition on J_1,J_2 as embedded in M.

This condition is more complicated when a family of geometries is involved, and we will omit it. If a geometry is embedded in M, it is automatically minimal in the sense that it has S1-dimension 1 (cf. §2.2)

5.1.2 DEFINITION OF LIE-COORDINATIZABLE STRUCTURES Let M be a first-order structure (§2). We assume a class of basic geometries is jointly embedded in M (for simplicity, consider a finite class.) We consider the class $\mathcal{M}$ of basic geometries, and principal homogeneous spaces over groups associated with the basic geometries. (Essentially, affine spaces corresponding to the vector spaces.) *We assume §1.1 - Theorem 1.1 and the remark following it - are true in M with respect to the class $\mathcal{M}$.* Thus for each definable $D \subset M$, there exist $J_1, ..., J_n \in \mathcal{M}$ and a nontrivial definable map $f : D \to (\cup_i J_i)^{[n]}$.

We also assume that M is $\aleph_0$-categorical, or that $Aut(M)$ has finitely many orbits on M^n, for any n. (Note that this is the case for each of the basic geometries.) It follows that the process of semi-minimal analysis - finding a function on each fiber of f above into other semi-basic geometries, and iterating - *terminates after finitely many steps.* (Cf. [10] for details.)

5.1.3 EXAMPLE Let M be a free Abelian group of exponent 4. M contains $V = 2M = \{x \in M : 2x = 0\}$. This can be shown to be an embedded geometry (of type (2) on our short list.) The map $f : M \to V$ is given by: $f(x) = 2x$. For $a \in V$, $f^{-1}(a)$ is a homogeneous space over V itself.

The following theorem lifts to a Lie coordinatizable structure, some easy but important properties of the basic geometries themselves.

THEOREM 5.1 *Let M be Lie-coordinatizable.*

1. *M has finite S1-dimension.*
2. *M is geometrically modular.*
3. *M has the finite model property: every finite set of first order sentences true in M, is true in a finite structure.*
4. *In fact M is the union of* finite homogeneous substructures: *finite substructures N, such every partial map from N to N extending to an automorphism of M , extends to an automorphism of N.*

5. M is relatively finitely axiomatizable*, over the Lie geometries in M.*

This type of theorem was first proved by Zilber: he showed that a totally categorical structure is Lie- coordinatizable, by a single basic geometry of type (1) or (2), and proceeded to conclude 5.1 (2) and (3). (The assumption of total categoricity was in effect: finite Morley dimension, and a single unknown minimal set, satisfying 5.1 (3). Zilber globalized this last assumption, but his proof went by way of a classification of the geometry involved; no direct proof of a local-global principle for 5.1(3) is known.) [9] extended this to the case of many geometries. It follows from (3), and this was Zilber's original motivation, that totally categorical structures are not finitely axiomatizable. (4) means that a single first order sentence, together with the isomorphism type of the basic geometries embedded in M, determines the isomorphism type of M. Now each of the basic geometries is itself determined by a single sentence together with their *dimension*. Thus (4) is equivalent to the statement that M is axiomatized by finitely many sentences, together with finitely many axiom schemes asserting that certain sets are infinite. It follows in particular that only countably many Lie coordinatizable structures exist.

5.2 Highly symmetric finite graphs

Our subject here is the class $C(\beta)$ of all finite graphs M, whose automorphism group has $\leq \beta$ orbits on four-tuples of vertices.

To say that a large graph has a bounded number of orbits on vertices already implies it has some symmetries; but an arbitrary finite graph is easily coded in a (not much larger) graph whose automorphism group is transitive on vertices, or even pairs or triples of vertices. At $k = 4$ something new happens; the symmetry condition permeates all parts of the graph, and becomes stable under the naming of boundedly many parameters.

The following remark shows the first connection between a single, infinite, Lie-coordinatizable structure, and a *class* of finite , highly homogeneous structures.

Remark 5.1 *Let M be a Lie - coordinatizable structure. Let Γ be a definable graph in M. Let β be the number of orbits of $Aut(M)$ on Γ^4. Let $C(M)$ be the class of finite homogeneous substructures of M, and*

$$C(M, \Gamma) = \{N \cap \Gamma : N \in C(M)\}$$

Then $C(M; \Gamma) \subset C(\beta)$.

The proof is immediate from the definition of homogeneous substructure.

If M has k Lie geometries (for simplicity), $J_1, ..., J_k$ then a homogeneous substructure N of M can be assigned k "dimensions": $\dim(J_i \cap N), \ldots, \dim(J_k \cap N)$. It can be shown that N is determined up to isomorphism by these dimensions. The remark thus provides some very orderly subfamilies of $C(\beta)$.

Example Let $V(n)$ be an n-dimensional vector space over a fixed finite field F, say $GF(5)$. As any eight elements of $V(n)$ are contained in a copy of $V(8)$, the automorphism group $GL(n,F)$ has no more orbits on $V(n)^8$ than $GL(8,5)$ has on $V(8)^8$; this number is bounded by 5^{64}). Let $\Gamma(n)$ be the graph whose vertices are 2-dimensional subspaces of $V(n)$, with an edge between two subspaces contained in the same 3-dimensional space. Then $GL(n,F)$ acts on $\Gamma(n)$ by automorphisms, and has $\leq 5^{64}$ orbits on $\Gamma(n)^4$. So $\Gamma(n) \in C(5^{64})$. Similarly, any class of graphs formed uniformly out of the $V(n)$ falls into a single $C(\beta)$.

We show that $C(\beta)$ consists entirely of such graphs:

Theorem 5.2 *There exist finitely many Lie-coordinatizable structures $M_1, \ldots, M_r$, such that $C(\beta) = \cup_{1\leq i\leq r} C(M_i)$.*

The entire theory applies to finite structures of any "signature", e.g. hypergraphs, and not only to graphs (and the "4" remains 4.) The theorem was proved by Lachlan for certain subclasses of $C(\beta)$: the graphs (or hypergraphs) that are *homogeneous* in the sense that every partial automorphism extends to an automorphism. In this case, only the trivial geometry (1) occurs in the Lie coordinatized structure.

We will not have time to bring out the power of 5.2, but will list some consequences that can be stated without further definitions, in the language of group theory, combinatorics and complexity, respectively.

Corollary 5.2 *There exists a bound $h = h(b)$ such that for any $M \in C(\beta)$, $Aut(M)$ has at most h distinct non-Abelian Jordan-Holder components. The isomorphism type of M is determined by the set of $\leq h$ simple components of $Aut(M)$, up to a bounded number of possibilities.*

Each of these simple components typically occurs unboundedly often in $Aut(M)$; in addition very large Abelian groups occur. The corollary hinges on a correspondence between the basic geometries embedded in a Lie-coordinatizable structure, and the simple components of the finite approximations to the structure.

The next corollary is a a version of the global modularity principle. Consider bipartite graphs $\Gamma = (P, L, I \subset (P \times L))$. Let $I(b) = \{a \in P : (a,b) \in I\}$. Let π, λ, l_b denote the sizes of $P, L, I(b)$ respectively. Let $l = min\{l_b : b \in L\}$.

Theorem 5.3 *Let Γ vary through a family of bipartite graphs in $C(\beta)$. Assume that for $b \neq b' \in L$, $|I(b) \cap I(b')| = o(l)$. Then*

$$\lambda l \leq O(p)$$

By contrast, if (P, L, I) is a projective plane, then $\pi = \lambda \sim l^2$, while $|I(b) \cap I(b')| = 1$. The theorem thus says that no bipartite graph in $C(\beta)$ is combinatorially similar to a projective plane; this is rather close to Lachlan's original formulation in the stable $\aleph_0$-categorical framework.

The theorem is obtained from a local-global principle for modularity; the modularity of the basic geometries themselves is a consequence of the classification of the finite simple groups. It would be interesting to know if the above combinatorial statement can be obtained without the heavy group theory. (In [10] a number of principles of a roughly similar nature are formulated; if all are assumed, one obtains a direct proof of the relevant part of the classification (classification of the large finite simple groups having highly symmetric permutation representations in the above sense, or occurring as components in groups that do.)

Finally,

Corollary 5.3 *Membership of a graph in $C(k)$ is decidable in polynomial time. So is the problem of deciding isomorphism between two graphs in $C(k)$*

This is analogous to a famous result of Luks (Proc. 21 FOCS), concerning graphs of bounded valency, but here the graphs are at the opposite extreme (and in particular have bounded diameter.)

5.3 Proof of 5.2 In [27], the primitive permutation groups with few orbits on 4-tuples are analyzed group-theoretically. The conclusion is an almost precise classification of the possibilities. The proof relies massively on the classification of the finite simple groups, and on related methods.

It follows from this result that to each $\Gamma \in C(\beta)$ one can associate a finite approximation M_Γ to a Lie coordinatized structure, such that Γ, M_Γ have the same automorphism group.

A very soft translation into model theory shows that M_Γ and Γ interpret each other; Γ can be viewed as a sort in a structure, built out of M_Γ. A formula ϕ_Γ describes the construction of Γ from M_Γ.

The difficulty is that the soft connection between automorphism groups and formulas says nothing of the length of the formula. It may be as large as the finite structure it describes. Take for instance the class $\{P_n = (V_n, V_n^*)\}$ of dual pairs (V_n is an n-dimensional $GF(2)$-vector space; V_n^* is the dual.) The pair $P_n = (V_n, V_n^*)$ (or a suitable graph formed from it) has the same automorphism group as V_n. So we may have $M_{P_n} = V_n$. Yet there is no formula of bounded length that constructs V^* from V. In this case, we were given the wrong basic geometry, and we have to find *another* that does have a construction of bounded length. (In this case, it is just P_n itself.)

We take an ultraproduct of the structures Γ, and M_Γ , obtaining infinite structures Γ^*, $M^* = (M_\Gamma)^*$. In a language with formulas of nonstandard size, M^* interprets Γ^*, so Γ^*, in this rich language, is Lie coordinatizable. *We now prove that the class of Lie coordinatizable structures is closed under interpretations.* This is nontrivial and lengthy; the interpreted structure will no longer have the original coordinatizing geometries, and one must go via more global properties (such as geometric modularity) that are inherited when the language is reduced. We apply this theorem to the reduct Γ^* *in the graph language*, obtaining a new Lie coordinatization. If done appropriately, it can now be shown that the original Γ are homogeneous substructures of Γ^*.

Robinson dreamed of rewriting number theory using nonstandard analysis. The hope is that ultrapowers will smooth out the finite irregularities and help to bring out the uniform behavior behind the undecidability. Some theorems of number theory (some treated by Robinson, and Robinson - Roquette) are very naturally stated in nonstandard language. The trouble is that when only one road leads from standard to nonstandard territory, a direct nonstandard proof is homotopic to a standard one. Only if two distinct paths lead to the same point can we get a truly new proof. In both uses of nonstandard ideas reported on here, the second road is provided by an axiomatization (difference fields, Lie-coordinatized structures) together with a method of analysis of abstract models of these axioms (In both these cases, finite S1-dimension and related concepts of definable groups.) To extend the scope of such results in number theoretic directions, one must develop both new quantifier-elimination results, beyond local fields, and corresponding generalizations of stability capable of dealing with them.

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