Free Probability Theory: Random Matrices and von Neumann Algebras

DAN VOICULESCU*

Department of Mathematics, University of California
Berkeley, CA 94720, USA

0 Introduction

Independence in usual noncommutative probability theory (or in quantum physics) is based on tensor products. This lecture is about what happens if tensor products are replaced by free products. The theory one obtains is highly noncommutative: freely independent random variables do not commute in general. Also, at the level of groups, this means instead of $\mathbb{Z}^n$ we will consider the noncommutative free group $F(n) = \mathbb{Z} * \cdots * \mathbb{Z}$ or, looking at the Cayley graphs, a lattice is replaced by a homogeneous tree.

Three different models of free probability theory are provided by convolution operators on free groups, creation and annihilation operators on the Fock space of Boltzmann statistics, and random matrices in the large N limit.

Important problems on the von Neumann algebras of free groups have been solved using free probability techniques, and surprisingly the random matrix model has played a major role in this. In another direction there is a free entropy quantity that goes with free independence.

Concerning connections with other fields we should signal that combinatorial objects (noncrossing partitions, random permutations) have appeared in free probability theory and that random matrices are used in physics.

We have divided our survey into five sections:

(1) Free random variables
(2) Free harmonic analysis
(3) Asymptotic models
(4) Applications to operator algebras
(5) Free entropy.

At the end, an Appendix explains a few basic notions in operator algebras for the reader not conversant in C^*- and W^*-algebras.

1 Free Random Variables

For noncommutative probability spaces, the usual prescription applies: replace the functions on a space by elements of a (possibly noncommutative) algebra. Thus:

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1.1 DEFINITION. A noncommutative probability space is a unital algebra $\mathcal{A}$ over $\mathbb{C}$ endowed with a linear functional $\phi : \mathcal{A} \to \mathbb{C}$ such that $\phi(1) = 1$. The elements of $\mathcal{A}$ are called random variables and the distribution of a random variable $a \in \mathcal{A}$ is the map $\mu_a : \mathbb{C}[X] \to \mathbb{C}$ given by $\mu_a(P(X)) = \phi(P(a))$.

The above definition is only an algebraic caricature, sufficient for discussing questions such as independence. (Positivity and almost everywhere convergence require additional structure: $\mathcal{A}$ a C^*-algebra and ϕ a state or, even more, a von Neumann algebra with a normal state. In the C^*-case, if $a = a^*$, the distribution functional μ_a extends to a compactly supported probability measure on $\mathbb{R}$.)

Usually independence is modeled on tensor products. The idea of free probability theory is to replace tensor products by free products.

1.2 DEFINITION. A family of subalgebras $1 \in \mathcal{A}_i \subset \mathcal{A}$ $(i \in I)$ in a noncommutative probability space $(\mathcal{A}, \phi)$ is called a *free family of subalgebras* if

$$\phi(a_1 \dots a_n) = 0$$

whenever $a_j \in \mathcal{A}_{i(j)}$ with $i(j) \neq i(j+1)$ $(1 \le j < n)$ and $\phi(a_j) = 0$ $(1 \le j \le n)$. Families of subsets or of random variables in (A, ϕ) are free if the generated unital subalgebras are free.

As for usual independence, if the free family of subalgebras $\mathcal{A}_i$ $(i \in I)$ generates $\mathcal{A}$, then ϕ is completely determined by the restrictions $\phi|\mathcal{A}_i$ $(i \in I)$. What distinguishes freeness and independence is that free random variables are highly noncommuting.

1.3 EXAMPLES. (a) Let $G = \mathop{*}\limits_{i \in I} G_i$ be a free product of groups and let λ be the left regular representation of G on $\ell^2(G)$. Let further W and W_i $(i \in I)$ be the weakly closed subalgebras generated by $\lambda(G)$ and $\lambda(G_i)$ respectively. The von Neumann trace $\tau : W \to \mathbb{C}$ is given by $\tau(T) = \langle T\delta_e, \delta_e \rangle$ where δ_g $(g \in G)$ is the canonical basis of $\ell^2(G)$. Then the W_i $(i \in I)$ are free in (W, τ).

(b) Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{TH} = \oplus_{k \ge 0} \mathcal{H}^{\otimes k}$ where $\mathcal{H}^{\otimes 0} = \mathbb{C}1$. Let further $\ell(h)\xi = h \otimes \xi$ be the creation operators on the full Fock space $\mathcal{TH}$ and let $\varepsilon(X) = \langle X1, 1 \rangle$ be the vacuum expectation. If $\mathcal{H}_i$ $(i \in I)$ are mutually orthogonal subspaces of $\mathcal{H}$, then the generated subalgebras $C^*(\ell(\mathcal{H}_i))$ $(i \in I)$ are free in $(C^*(\ell(\mathcal{H})), \varepsilon)$.

1.4 The analogue of the Gaussian law in the free context is the semicircle law, i.e. probability measures on $\mathbb{R}$ with densities having a semiellipse graph: 0 if $|t-a| > R$ and equal to $2\pi^{-1}R^{-2}(R^2 - (t-a)^2)^{\frac{1}{2}}$ if $|t-a| \le R$. Indeed, we have the following

FREE CENTRAL LIMIT THEOREM [32]. *If $(f_n)_{n \in \mathbb{N}}$ is a free family of random variables in (A, ϕ) so that $\phi(f_n) = 0$ $(n \in \mathbb{N})$,*

$$\lim_{N \to \infty} N^{-1} \sum_{1 \le n \le N} \phi(f_n^2) = 4^{-1} R^2 > 0$$

$$\sup_{n \in \mathbb{N}} |\phi(f_n^k)| < \infty \quad \textit{for all } \; k \in \mathbb{N}$$

then, if

$$S_N = N^{-\frac{1}{2}} \sum_{1 \le k \le N} f_k$$

the distributions μ_{S_N} converge pointwise on $\mathbb{C}[X]$ to the semicircle law with density $2\pi^{-1}R^{-2}\,\mathrm{Re}(R^2 - t^2)^{\frac{1}{2}}$.

Convergence in the previous theorem is in a very weak sense. Actually, in the free context convergence to the central limit is much stronger than in usual probability theory (see [1] concerning this superconvergence).

1.5 Roughly speaking, the Gaussian process over a real Hilbert $\mathcal{H}$ space is the process indexed by $\mathcal{H}$, the random variable corresponding to h being $\langle\,\cdot\,, h\rangle : \mathcal{H} \to \mathbb{R}$, when $\mathcal{H}$ is endowed with the Gaussian measure. This is part of the Gaussian functor of second quantization, which takes real Hilbert spaces and contractions to commutative von Neumann algebras with specified trace state and trace- and unit-preserving completely positive maps. The canonical anticommutation relations provide a fermionic analogue. We have found a third such functor, which is the free analogue of these.

THE FREE ANALOGUE OF THE GAUSSIAN FUNCTOR [32]. *If $\mathcal{H}$ is a real Hilbert space, let $\mathcal{H}_{\mathbb{C}}$ be its complexification and let $\mathcal{T}\mathcal{H}_{\mathbb{C}}$ and $\ell(h)$ be as in 1.3(b). Let further $s(h) = 1/2(\ell(h) + \ell(h)^*)$.*

(i) *The von Neumann algebra $\Phi(\mathcal{H})$ generated by $s(\mathcal{H})$ is isomorphic to the II_1 factor of a free group on* dim $\mathcal{H}$ *generators (if* dim $H > 1$*) and the trace state is given by the vacuum expectation $\langle \cdot 1, 1\rangle$.*

(ii) *If $T : \mathcal{H}_1 \to \mathcal{H}_2$ is a contraction there is a unique completely positive map $\Phi(T) : \Phi(\mathcal{H}_1) \to \Phi(\mathcal{H}_2)$ such that*

$$(\Phi(T))(X)1 = \mathcal{T}(T_{\mathbb{C}})(X1)\ .$$

The map $\Phi(T)$ is trace and unit preserving.

(iii) *If $(\mathcal{H}_i)_{i\in I}$ is a family of pairwise orthogonal subspaces in $\mathcal{H}$ and $v(i)$ are the corresponding inclusions, then $(\Phi(v(i)))(\Phi(\mathcal{H}_i))_{i\in I}$ is free in $\Phi(\mathcal{H})$.*

(iv) *Orthogonal vectors correspond to free variables via the map $s : \mathcal{H} \to \Phi(\mathcal{H})$ and the distribution of $s(h)$ is a centered semicircle law.*

Gaussian processes are obtained by mapping the index set of the process into a Hilbert space and then composing with the Gaussian process over the Hilbert space. Composing with the free analogue (i.e. with $s : \mathcal{H} \to \Phi(\mathcal{H})$) one gets the free analogue of Gaussian processes. Free increments correspond to the requirement of orthogonal increments for the map into the Hilbert space. For instance, Brownian motion corresponds to $\mathcal{H} = L^2(0,\infty)$ and the Hilbert space curve $[0,\infty) \ni t \to \chi_{[0,t)} \in L^2(0,\infty)$. The free analogue of Brownian motion is then obtained by taking $[0,\infty) \ni t \to s(\chi_{[0,t)})$, a possibility used in [28].

1.6 Generalizations of various parts of the free probability context have been studied. We would like to mention here the following two.

(a) Free products with amalgamation over an algebra B [32], [36]. One replaces the complex field $\mathbb{C}$ by an algebra B over $\mathbb{C}$. The noncommutative probability space A is then an algebra containing B as a subalgebra and the expectation ϕ is a $B-B$-bimodule projection $\phi : A \to B$. There is a corresponding definition of B-freeness and the corresponding operator-algebra context has also been studied.

(b) Deformed Cuntz relations [6]. A natural model in which free random variables arise is provided by the creation operators $\ell(h)$ (Example 1.3(b)). They satisfy the Cuntz relations $\ell(h)^*\ell(k) = \langle h, k\rangle I$. A deformation of these relations is

$$\ell(h)^*\ell(k) - \mu\ell(k)\ell(h)^* = \langle h, k\rangle I$$

$\mu \in [-1, 1]$. This provides an interpolation between the three cases $\mu = -1, 0, 1$, which correspond, respectively, to the fermionic, free, and bosonic creation operators.

1.7 Free stochastic integrations. Stochastic integration in the free case has been studied in papers by R. Speicher, K. R. Parthasarathy, B. K. Sinha, F. Fagnola, L. Accardi, and B. Kummerer.

2 Free Harmonic Analysis

2.1 Free convolution. The distribution of the sum of two independent random variables is the (additive) convolution of their distributions. By analogy on $\Sigma = \{f : \mathbb{C}[X] \to \mathbb{C} | f \text{ linear}, f(1) = 1\}$, there are operations $\boxplus$ and $\boxtimes$ called, respectively, additive and multiplicative free convolution so that if a, b are free random variables in some noncommutative probability space then $\mu_{a+b} = \mu_a \boxplus \mu_b$ and $\mu_{ab} = \mu_a \boxtimes \mu_b$ [32]. Because this does not depend on the concrete realizations of the variables with distribution μ_a, μ_b and because the sum of self-adjoint operators is self-adjoint, the product of the unitaries unitary, etc., we have that $\boxplus$ extends to an operation on compactly supported probability measures on $\mathbb{R}$, while $\boxtimes$ defines operations on the compactly supported probability measures on $\mathbb{R}^\times$, $\mathbb{R}_+$, and $\mathbb{T}$. Clearly $\boxplus$ is commutative and actually $\boxtimes$ is also commutative. Moreover, [2] $\boxplus$ extends to an operation on all probability measures on $\mathbb{R}$, while $\boxtimes$ extends to an operation on probability measures on $\mathbb{R}_+$ — which correspond to operations on "unbounded" random variables.

2.2 The linearizing transforms. The computation of free convolution can be done using a linearizing transform. This is like computing the usual convolution of two probability measures using the logarithm of the Fourier transform (which linearizes convolution).

THEOREM [33]. *If $\mu \in \Sigma$ let $G_\mu(z) = z^{-1} + \sum_{n\geq 1} \mu(X^n)z^{-n-1}$ and let $K_\mu(z) \in z^{-1} + \mathbb{C}[[z]]$ be such that $G_\mu(K_\mu(z)) = z$. Then $R_\mu(z) = K_\mu(z) - z^{-1}$ has the property that $R_{\mu_3} = R_{\mu_1} + R_{\mu_2}$ if $\mu_3 = \mu_1 \boxplus \mu_2$.*

If μ is a compactly supported probability measure then G_μ is its Cauchy transform and R_μ is analytic near 0. The linearization result also extends to the case of unbounded supports using analytic functions in angular domains ([2], an intermediate generalization is given in [16]).

A similar linearization result holds for the multiplicative free convolution $\boxtimes$ [34] and also has an analytic function extension to the case of unbounded supports [2].

2.3 ***F*-Infinitely divisible laws.** A probability measure μ on $\mathbb{R}$ is called F-infinitely divisible if for every $n \in \mathbb{N}$ there is $\mu_{1/n}$ so that $\underbrace{\mu_{1/n} \boxplus \cdots \boxplus \mu_{1/n}}_{n \text{ times}} = \mu$. A family of probability measures $(\mu_t)_{t \geq 0}$ on $\mathbb{R}$ is an F-convolution semigroup if $\mu_{t+s} = \mu_t \boxplus \mu_s$ and μ_t depends continuously on t. There is bijection between F-infinitely divisible measures and F-convolution semigroups. Stationary processes with free increments naturally lead to these definitions.

If $(\mu_t)_{t \geq 0}$ is an F-convolution semigroup, then the Cauchy transforms $G(t, z) = G_{\mu_t \boxplus \gamma}(z)$ for some probability measure γ on $\mathbb{R}$ satisfy the complex quasilinear equation

$$\frac{\partial G}{\partial t} + \frac{\partial G}{\partial z}\ \phi(G) = 0$$

where $\phi(z) = R_{\mu_1}(z)$. In particular, the complex Burger equation

$$\frac{\partial G}{\partial t} + \alpha G\ \frac{\partial G}{\partial z} = 0$$

is the analogue of the heat equation, as $R_\mu(z) = \alpha z$ if μ is a centered semicircle law (which is the free analogue of the Gauss law).

THEOREM. *μ is F-infinitely divisible iff R_μ has an analytic extension to $\{z \in \mathbb{C} | \mathrm{Im}\ z < 0\}$ with values in $\{z \in \mathbb{C} | \mathrm{Im}\ z \leq 0\}$.*

(The case of compactly supported measures is given in [33], the intermediate case of measures with finite variance in [16], and the result in full generality in [2].)

The condition on the imaginary part of $R_\mu(z)$ implies the existence of an integral representation, which makes the above theorem an analogue of the Levy-Khintchine theorem. The analogy goes even further when we remark that the free Poisson distribution defined by

$$\lim_{n \to \infty} \left((1 - \frac{a}{n})\delta_0 + \frac{a}{n}\ \delta_b\right)^{\boxplus n}$$

has the R-function $R(z) = ab(1 - bz)^{-1}$. The free Poisson measure is given by

$$\mu = \begin{cases} (1-a)\delta_0 + \nu & \text{if } 0 \leq a \leq 1 \\ \nu & \text{if } a > 1 \end{cases}$$

where ν has support in $[b(1 - \sqrt{a})^2, b(1 + \sqrt{a})^2]$ and density $(2\pi bt)^{-1}(4ab^2 - (t - b(1 + a))^2)^{\frac{1}{2}}$.

2.4 F-Stable laws [2]. Replacing usual convolution by free convolution in the definition of stable laws one defines F-stable laws. F-stable laws were classified in [2], up to taking certain linear combinations, the main types are given by

(i) $R(z) = a, \ \ a \in \mathbb{C}, \ \ \mathrm{Im}\ a \le 0$
(ii) $R(z) = z^{\alpha-1}\mathrm{sign}(\alpha - 1), \quad \alpha \in (0,1) \cup (1,2)$
(iii) $R(z) = \log z$.

As for infinitely divisible laws this runs essentially parallel to the classical case.

The usual Cauchy distribution and the free Cauchy distribution, given by $R(z) = -i$ coincide.

2.5 Multiplicative F-infinite divisibility. Infinitely divisible probability measures with respect to the operation $\boxtimes$ on $\mathbb{T}$ were classified in [1] and on $\mathbb{R}_+$ in [2].

Note that the generating function for the measure μ, which is the free analogue of the multiplication Gaussian distribution (i.e. of the log-normal distribution) $\psi(z) = \sum_{n\ge 1} \mu(X^n)z^n$, can be expressed using the generating series for rooted labelled trees

$$\sum_{n\ge 1} \frac{n^{n-1}}{n!} z^n .$$

2.6 Noncrossing partitions. Because the map $\mu \to R_\mu$ linearizes the free convolution it follows that if $R_\mu(z) = \sum_{n\ge 0} R_{n+1}(\mu)z^n$ the coefficients $R_{n+1}(\mu)$ are polynomials in the moments of μ and $R_{n+1}(\mu_1 \boxplus \mu_2) = R_{n+1}(\mu_1) + R_{n+1}(\mu_2)$. The $R_{n+1}(\mu)$ are the free analogues of the cumulants of μ. In [29] it was shown that the formulae giving the free cumulants are entirely analogous to those for the usual cumulants if we replace the lattice of all partitions of $\{1,\ldots,n\}$ by the lattice of noncrossing partitions (i.e. partitions with crossing pairs $\{a,c\},\{b,c\}$ where $a < b < c < d$ do not lie in different sets of the partition). There are more general such formulae based on noncrossing partitions [29], [30], [20], [21] which characterize freeness of sets of random variables. It seems that the passage from all partitions to the noncrossing partitions is the combinatorial aspect of going from usual independence to free independence.

2.7 Generalizations of the free harmonic analysis. (a) **B-free convolution.** Free convolution and its linearization were extended to the context of B-freeness in [36]. A combinatorial approach based on noncrossing partitions to linearization and to the classification of infinitely divisible distributions (with moments of all orders) in the B-free context was developed in [30].

Multiplicative free convolution is no longer commutative for general B and there are nonlinear systems of differential equations that replace linearization [36].

(b) Deformed linearization maps. The linearization map involves certain canonical forms of random variables in creation and annihilation operators on the full Fock-space. Passage to the deformed Cuntz-relation was used to construct deformed free convolution [6] and its linearization map [21].

3 Asymptotic Models

3.1 Gaussian random matrices. The semicircle law that appears in the free central limit theorem also occurs in Wigner's work on the asymptotic distribution of eigenvalues of large Gaussian random matrices [41], [42]. The explanation we found [38] for this coincidence is that large Gaussian random matrices with independent entries give rise asymptotically to free random variables. Moreover, this asymptotic model is the bridge connecting classical and free probability theory. Indeed, independence of matrix-valued random variables is transformed into free independence of the corresponding noncommutative random variables (asymptotically).

The precise statements are as follows.

Let (A_n, ϕ_n) and (A_∞, ϕ_∞) be noncommutative probability spaces and let $(X_{n,i})_{i\in I}$, $n \in \mathbb{N} \cup \{\infty\}$ be in A_n. Then $(X_{n,i})_{i\in I}$ *converges in distribution to* $(X_{\infty,i})_{i\in I}$ if

$$\lim_{n\to\infty} \phi_n(P((X_{n,i})_{i\in I})) \longrightarrow \phi_\infty(P(X_{\infty,i})_{i\in I})$$

for every noncommutative polynomial P in indeterminates indexed by I. In particular the $(X_{n,i})_{i\in I}$ are *asymptotically free* if they converge in distribution to a free family.

A family $(x_i)_{i\in I}$ is called *semicircular* if the x_i have equal centered semicircle distributions and are free. In a C^*-probability space we require in addition that $x_i = x_i^*$.

For asymptotics of random matrices the appropriate (A_n, ϕ_n) are $A_n = \bigcap_{1\le p<\infty} L^p(\Omega, M_n)$ where $(\Omega, d\sigma)$ is some standard probability measure space and

$$\phi_n(X) = \frac{1}{n}\int_\Omega \mathrm{Tr}\; X(\omega)\, d\sigma(\omega)\;.$$

THEOREM [38]. *Let $Y(\iota, n) = (a(i,j;n,\iota)_{1\le i,j\le n}) \in A_n$ be real random matrices $(\iota \in I)$. Assume $a(i,j;n,\iota) = a(j,i;n,\iota)$ and $\{a(i,j;n,\iota) | 1 \le i \le j \le n,\ \iota \in I\}$ is a family of independent Gaussian $(0, 1/n)$ random variables. Let further $D_n \in A_n$ be a constant diagonal random matrix having a limit distribution as $n \to \infty$. Then $\{Y(\iota,n)|\iota \in I\} \cup \{D_n\}$ is asymptotically free as $n \to \infty$ and $\{Y(\iota,n)|\iota\in I\}$ converges in distribution to a semicircular family.*

3.2 Unitary random matrices. Using polar decomposition (i.e. noncommutative functional calculus) and results of Gromov-Milman on isoperimetric inequalities yields stronger versions of the preceding result for unitary random matrices.

THEOREM [38]. *Given $\varepsilon > 0$ and a nontrivial element*

$$g = g_{i_1}^{k_1} g_{i_2}^{k_2} \cdots g_{i_m}^{k_m}$$

$(m \ge 1,\ k_j \ne 0,\ i_s \ne i_{s+1})$ of the free group on p generators, let

$$\Omega_n(g) = \{(u_1, \ldots, u_p) \in (U(n))^p | \tau_n(u_{i_1}^{k_1} \cdots u_{i_m}^{k_m})| < \varepsilon\}$$

where $\tau_n = n^{-1}\mathrm{Tr}$ *is the normalized trace on* $(n \times n)$*-matrices. Then we have*

$$\lim_{n\to\infty} \mu_n(\Omega_n(g)) = 1$$

where μ_n *is the normalized Haar measure on* $(U(n))^p$.

The preceding theorem has a more general form where a constant diagonal unitary also appears. This implies asymptotic freeness results for random matrices, which as matrix-valued variables are independent and are distributed according to the invariant measures of unitary orbits of self-adjoint matrices (this includes random projections, etc.).

3.3 Further results. Further extensions of the preceding theorems include results for real symmetric and antisymmetric Gaussian random matrices [38], for matrices with fermionic entries [38], and matrices with independent non-Gaussian entries together with a finite-dimensional constant algebra [10]. A generalization of the random matrix result involving representations has been obtained in [4].

In a different direction in [18] freeness results were obtained for independent uniformly distributed random permutation matrices. (Further combinatorial results for words in independent random permutations related to this are given in [19].)

3.4 Applications. Many of the known asymptotic distribution of eigenvalue results for random matrices can be recovered from the asymptotic freeness results. Indeed, many of these are obtained via noncommutative functional calculus from random matrices like those in the preceding theorems. Hence the limit distribution of eigenvalues in the large n limit is the same as the distribution of an element in a certain algebra generated by free random variables, the distribution of which can be computed, in certain cases via free convolution operations.

Related to the asymptotic freeness results for random matrices, it was recently discovered in [5] that free convolution occurs asymptotically in the decomposition into irreducible representations of tensor products of representations of $U(n)$.

Last but not least there are applications to the II_1 factor of free groups, which we shall survey in the next section.

4 Applications to Operator Algebras

Free probability theory and especially asymptotic random matrix realization have led to a surge of new results on the von Neumann algebras of free groups. These recent results will be surveyed here, preceded by some background on II_1-factors.

4.1 II_1-Factors of discrete groups. A factor is a von Neumann algebra M with trivial center $Z(M) = \mathbb{C}I$. The factor M is type II_1 if it has a trace-state $\tau : M \to \mathbb{C}$ (which is then unique) and is infinite dimensional. As P ranges over projections in M, $\tau(P)$ takes all values in $[0,1]$, which corresponds to a geometry with subspaces having dimensions in $[0,1]$.

$L(G)$, the von Neumann algebra of the left regular representation $\lambda(G)$, is a II_1-factor iff G has infinite conjugacy classes (i.c.c.). The $L(G)$'s are a rich source of II_1-factors (G will be assumed countable in what follows).

By a deep theorem of Connes [7] all $L(G)$ with amenable G are isomorphic — the hyperfinite II_1-factor. This is the "best" among all II_1-factors; it has a large automorphism group and good finite-dimensional approximation properties, and there are approximately central elements (property Γ of von Neumann). The remarkable properties of the hyperfinite II_1-factor made an in-depth study of its subfactors possible.

At the other extreme are the $L(G)$'s for G with property T of Kazhdan [8], [9]. These II_1-factors have rigidity properties, few automorphisms, no approximation properties, and no approximate center (non-Γ). It is conjectured (by Connes) that isomorphisms among these $L(G)$'s imply isomorphisms of the corresponding groups.

The free group factors $L(F_n)$ ($n = 2, 3, \ldots, \infty$) have intermediate properties: some approximation properties (compact instead of finite-rank) and some properties towards rigidity (non-Γ). Like the hyperfinite II_1-factor, which is related to the fermionic context of the canonical anticommutation relations, the free group factors are related to the free analogue of the Gaussian functor. This could mean that the free group factors are the "best" among the "bad" (i.e. non-Γ) II_1-factors.

4.2 The free probability technique [37] Semicircular and circular systems are the free analogues of, respectively, independent real and complex Gaussian random variables. They provide convenient sets of generators for free group factors. The asymptotic random matrix models based on Gaussian random matrices are the source for many of the properties of circular and semicircular systems.

A system of self-adjoint random variables $(s_i)_{i\in I}$ is semicircular if the s_i's are free and have identical centered semicircle distributions. Similarly, $(c_i)_{i\in I}$ is circular if $(\mathrm{Re}\ c_i)_{i\in I} \cup (\mathrm{Im}\ c_i)_{i\in I}$ is semicircular.

A block of a Gaussian random matrix, being a matrix of the same kind, implies that if $p = p^* = p^2$ is free with respect to a semicircular system $(s_i)_{i\in I}$ then the compression $(ps_ip)_{i\in I}$ is semicircular in $(pAp, \phi(p)^{-1}\phi(\cdot))$. In the polar decomposition $c = u|c|$ of a circular element, u and $|c|$ are free. Cutting and pasting blocks of Gaussian random matrices have analogues for circular and semicircular systems. For instance, if $(c_{i,j;s})_{1\le i,j\le n, s\in S}$ is circular, then the matrices $X_s = \sum_{1\le i,j\le n} c_{i,j;s} \otimes e_{ij}$, $s \in S$, form a circular system.

I introduced this free probability technique and used it to obtain results on free group factors in [37]; the applications to free group factors were subsequently carried much further by Radulescu and Dykema.

4.3 The fundamental group $\mathcal{F}(L(F(\infty)))$. If M is a II_1-factor and $p = p^2 \in M$ the isomorphism class of pMp depends only on $\lambda = \tau(p)$ and is denoted M_λ. The fundamental group $\mathcal{F}(M)$ [17] consists of those $\lambda \in (0, 1]$ such that $M_\lambda \sim M$ and their inverses. For the hyperfinite II_1-factor R, $\mathcal{F}(R) = (0, \infty)$. By a result of Connes $\mathcal{F}(L(G))$ is countable if G is an i.c.c. group with property T.

THEOREM. $\mathcal{F}(L(F(\infty))) = (0, \infty)$.

That $\mathcal{F}(L(F(\infty))) \supset \mathbb{Q} \cap (0, \infty)$ was proved in [37] by me, the complete result was then obtained by Radulescu [23].

4.4 The compressions $(L(F(n)))_\lambda$

THEOREM [37]. $(L(F(n)))_{1/N} \sim L(F(N^2(n-1)+1)), \ N \in \mathbb{N}, \ n = 2, 3, \ldots, \infty.$

The preceding result, a first application of the free probability technique, was extended in several directions.

THEOREM [24], [25]. *If $p, q \in \mathbb{N}$, $2 \leq p < q$, and $\lambda = (p-1)^{\frac{1}{2}}(q-1)^{-\frac{1}{2}}$, then*

$$(L(F(p)))_\lambda = L(F(q)) \ .$$

Building on this, Dykema [12] and Radulescu [25] (independently) defined *interpolated free group factors* $L(F(s))$, $s > 1$, $s \in \mathbb{R}$, satisfying the formula in the preceding theorem for arbitrary real $q > p > 1$. Moreover for arbitrary real $p > 1$, $q > 1$,

$$L(F(p)) * L(F(q)) \sim L(F(p+q)) \ .$$

4.5 Free products. A few preliminary results [37], [10] identifying certain free product von Neumann algebras with free group factors were greatly extended by Dykema [11]. If A, B are injective separable von Neumann algebras with specified faithful normal trace-states and if $A * B$ is a factor, then it is isomorphic to one of the interpolated free group factors $L(F(s))$. Moreover, formulae for the parameter s are given in [11]. A further generalization is given in [13].

4.6 Subfactors. Radulescu has shown in [25] that $L(F(\infty))$ has subfactors of all allowable Jones indices < 4, i.e. the numbers $4\cos^2\frac{\pi}{n}$ of [15]. The proof involves random matrices and results of [22] on constructing subfactors via amalgamated free products. Note that the fundamental group of $L(F(\infty))$ being $(0, \infty)$ implies the existence of subfactors of indices ≥ 4.

4.7 The isomorphism problem. The question of whether the free group factors $L(F(m))$ are isomorphic or not for different values of m is still unresolved (this problem appears on Kadison's Baton Rouge problem list).

4.8 Type III factors. In [26] Radulescu showed that the free product of $L(\mathbb{Z})$ with the (2×2) matrix algebra endowed with a nontracial state is a type III factor and that its core is isomorphic to $L(F_\infty) \otimes B(H)$. Further results on free product type III factors were obtained by Barnett and Dykema.

4.9 Quasitraces. Uses of semicircular systems have not been confined to W^*-algebra questions. A surprising application of semicircular systems appears in Haagerup's solution of the quasitraces problem for exact C^*-algebras [14].

5 Free Entropy [39]

5.1 The definition of free entropy. In classical probability theory, the entropy of an n-tuple $f = (f_1, \dots, f_n)$ of random variables is given by

$$S(f) = -\int_{\mathbb{R}^n} p(t) \log p(t)\, dt$$

where p is the density of the distribution of $(f_1, \dots, f_n)$. To define a free entropy $\chi(X_1, \dots, X_n)$ for an n-tuple of self-adjoint random variables in a tracial W^*-probability space, we had to go back to Boltzmann's $S = k \log W$ (i.e., roughly, the entropy is proportional to the logarithm of the measure of a set of microstates) and take into account that independence of random matrices gives rise asymptotically to freeness. This means we will choose approximating microstates $\Gamma_R(X_1, \dots, X_n; m, k, \varepsilon)$ to be the sets of $(A_1, \dots, A_n) \in (\mathcal{M}_k^{sa})^n$ so that

$$|\tau(X_{i_1} \dots X_{i_p}) - k^{-1}\mathrm{Tr}(A_{i_1} \dots A_{i_p})| < \varepsilon$$

for all $1 \le p \le m$, $(i_1, \dots, i_p) \in \{1, \dots, n\}^p$ and $\|A_j\| \le R$, $1 \le j \le n$. With vol denoting the volume on $(\mathcal{M}_k^{sa})^n$ for the scalar product defined by the trace Tr, we take

$$\limsup_{k \to \infty} (k^{-2} \log \mathrm{vol}\, \Gamma_R(X_1, \dots, X_n; m, k, \varepsilon) + \frac{n}{2} \log k)$$

and then define $\chi(X_1, \dots, X_n)$ to be

$$\sup_{R>0}\ \inf_{m \in \mathbb{N}}\ \inf_{\varepsilon > 0}$$

of that quantity.

Note that a similar definition for the classical entropy is possible, taking instead of all matrices $\mathcal{M}_k$ only the diagonal ones.

5.2 Properties of free entropy

(1) For one variable X with distribution μ,

$$\chi(X) = \iint \log|s-t|\, d\mu(s)\, d\mu(t) + \tfrac{3}{4} + \tfrac{1}{2}\log 2\pi$$

(2) $\chi(X_1, \dots, X_n) \le \frac{n}{2} \log(2\pi e n^{-1} C)$ where $C^2 = \tau(X_1^2 + \cdots + X_n^2)$.

(3) If $(X_1^{(p)}, \dots, X_n^{(p)})$ converge strongly to $(X_1, \dots, X_n)$ then

$$\limsup_{p \to \infty} \chi(X_1^{(p)}, \dots, X_n^{(p)}) \le \chi(X_1, \dots, X_n)$$

(4) $\chi(X_1, \dots, X_{m+n}) \le \chi(X_1, \dots, X_m) + \chi(X_{m+1}, \dots, X_{m+n})$

(5) If $X_1, \dots, X_n$ are free, then $\chi(X_1, \dots, X_n) = \chi(X_1) + \cdots + \chi(X_n)$.

(6) Let $F = (F_1, \dots, F_n)$ where F_j are noncommutative power series in n indeterminates. Under suitable convergence assumptions and the existence of an inverse (with respect to composition) of the same kind,

$$\chi(F(X_1, \dots, X_n)) = \chi(X_1, \dots, X_n) + \log|\mathcal{J}|$$

where $|\mathcal{J}|$ (the "positive Jacobian") is the Kadison-Fuglede positive determinant of the differential $DF(X_1, \dots, X_n)$ viewed as an element of $M \otimes M^{\mathrm{op}} \otimes \mathcal{M}_n$.

5.3 The free analogue of Fisher's information measure. By analogy with the classical case, the free analogue of the Fisher information measure is

$$\Phi(X) = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1}(\chi(X + \sqrt{\varepsilon}S) - \chi(X))$$

where S is $(0,1)$-semicircular and X, S are free. If $d\mu(t) = v(t)dt$ (μ the distribution of X) then

$$\Phi(X) = \tfrac{2}{3} \int v^3(t)\, dt\ .$$

The free analogue of the Cramér-Rao inequality is

$$\left(\int v^3(t)\, dt\right)\left(\int t^2 v(t)\, dt\right) \geq \frac{3}{4\pi^2}$$

where $v \in L^1 \cap L^3$ is a probability density. Equality holds iff v is a centered semicircle law.

5.4 The free entropy dimension. The entropy being a kind of normalized (logarithm of) volume, one may imitate the idea of the Minkowski content and define a dimension quantity from the asymptotic of volumes of ε-neighborhoods. This is realized via a free semicircular perturbation. The free entropy dimension is

$$\delta(X_1, \ldots, X_n) = n + \limsup_{\varepsilon \to 0} \frac{\chi(X_1 + \varepsilon S_1, \ldots, X_n + \varepsilon S_n)}{|\log \varepsilon|}$$

where $(S_1, \ldots, S_n)$ and $(X_1, \ldots, X_n)$ are free and $(S_1, \ldots, S_n)$ is a semicircular system.

(1) $\delta(X_1, \ldots, X_n) \leq n$ and it is ≥ 0 if $X_1, \ldots, X_n$ can be realized in a free group factor $L(F_m)$.

(2) $\delta(X_1, \ldots, X_{p+q}) \leq \delta(X_1, \ldots, X_p) + \delta(X_{p+1}, \ldots, X_{p+q})$.

(3) If $X_1, \ldots, X_n$ are free then $\delta(X_1, \ldots, X_n) = \delta(X_1) + \cdots + \delta(X_n)$.

(4) If μ is the distribution of X, $\delta(X) = 1 - \sum_{t \in \mathbb{R}} (\mu(\{t\}))^2$.

5.5 Free entropy dimension and smooth changes of generators

THEOREM. *If $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ are semicircular generators of the same W^*-algebra M and if $Y_1, \ldots, Y_m$ are "smooth noncommutative functions of $X_1, \ldots,$ X_n" then $n \geq m$.*

Here Y_j is a smooth noncommutative function of $(X_1, \ldots, X_n)$ if

$$d_2(Y_j, W^*(X_1 + \varepsilon S_1, \ldots, X_n + \varepsilon S_n)) = O(\varepsilon^s) \quad \text{for all } \ s < 1$$

where d_2 is the 2-norm distance defined by the trace $(S_1, \ldots, S_n)$ semicircular and free with respect to $(X_1, \ldots, X_n)$. For instance, elements obtained via suitably convergent noncommutative power series are smooth.

Note that if "smooth" could be replaced by "Borel" the corresponding result would imply $m \neq n \Rightarrow L(F_m)$ nonisomorphic to $L(F_n)$. In particular, the same conclusion, concerning the isomorphism problem of free group factors, would be reached, from an affirmative answer to the

SEMICONTINUITY PROBLEM. *If* $(X_1^{(p)}, \dots, X_n^{(p)})$ *converges strongly to* $(X_1, \dots, X_n)$ *does it follow that* $\liminf_{p\to\infty} \delta(X_1^{(p)}, \dots, X_n^{(p)}) \geq \delta(X_1, \dots, X_n)$ *?*

The explicit formula for δ in case $n = 1$ implies an affirmative answer.

Note also that if $X_1, \dots, X_n$ are free and generate a factor, then comparing $\delta(X_1, \dots, X_n)$ with the results of [11] we have

$$W^*(X_1, \dots, X_n) \simeq L(F(\delta(X_1, \dots, X_n))) .$$

Appendix: Operator Algebra Glossary

C^*-algebras are involutive Banach algebras isomorphic to norm-closed sub-algebras of the algebra of all bounded operators on some complex Hilbert space $B(\mathcal{H})$ and which together with an operator T contain its adjoint T^*.

A functional $\phi : A \to \mathbb{C}$ (A a C^*-algebra) is a *state* if $\|\phi\| = 1$ and ϕ is positive, i.e. $\phi(a^*a) \geq 0$ for all $a \in A$. By a theorem of Gel'fand-Naimark *commutative* ***C^*-algebras*** are precisely the algebras of continuous functions $C_0(X)$ vanishing at infinity on some locally compact space — states are Radon probability measures on X.

A *von Neumann algebra* M (or *W^*-algebra*) is a $*$-subalgebra of $B(\mathcal{H})$ that contains the identity and is closed in the weak operator topology, i.e. if x_i is a net of operators in M and $\langle x_i, h, k\rangle \to \langle xh, k\rangle$ for some $x \in B(\mathcal{H})$ and all $h, k \in \mathcal{H}$, then x is in M.

A functional $\tau : A \to \mathbb{C}$ on an algebra is a ***trace*** if $\tau(ab) = \tau(ba)$ for all $a, b \in A$.

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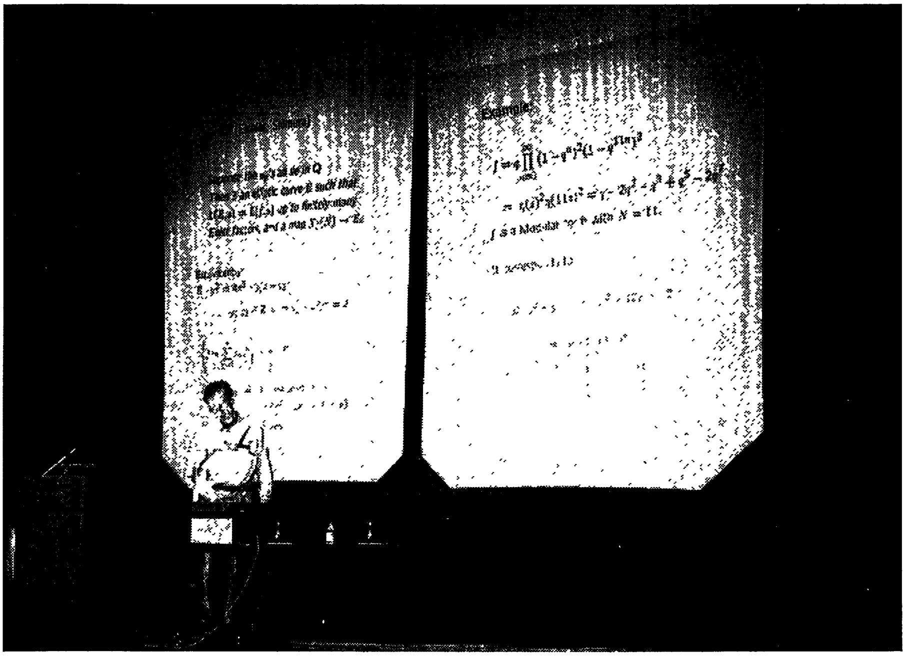

Andrew Wiles, last plenary speaker