# Hyperbolic Systems of Conservation Laws in One Space Dimension

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## Abstract

Aim of this paper is to review some basic ideas and recent developments in the theory of strictly hyperbolic systems of conservation laws in one space dimension. The main focus will be on the uniqueness and stability of entropy weak solutions and on the convergence of vanishing viscosity approximations.

## 1. Introduction

By a *system of conservation laws* in $m$ space dimensions we mean a first order system of partial differential equations in divergence form:

$$\frac{\partial}{\partial t}U + \sum_{\alpha=1}^{m} \frac{\partial}{\partial x_\alpha} F_\alpha(U) = 0, \qquad U \in I\!\!R^n, \ \ (t,x) \in I\!\!R \times I\!\!R^m.$$

The components of the vector $U = (U_1, \ldots, U_n)$ are the *conserved quantities.* Systems of this type express the balance equations of continuum physics, when small dissipation effects are neglected. A basic example is provided by the equations of non-viscous gases, accounting for the conservation of mass, momentum and energy. The subject is thus very classical, having a long tradition which can be traced back to Euler (1755) and includes contributions by Stokes, Riemann, Weyl and Von Neumann, among several others. The continued attention of analysts and mathematical physicists during the span of over two centuries, however, has not accounted for a comprehensive mathematical theory. On the contrary, as remarked in [Lx2], [D2], [S2], the field is still replenished with challenging open problems. In several space dimensions, not even the global existence of solutions is presently known, in any

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significant degree of generality. Until now, most of the analysis has been concerned with the one-dimensional case, and it is only here that basic questions could be settled. In the remainder of this paper we shall thus consider systems in one space dimension, referring to the books of Majda [M], Serre [S1] or Dafermos [D3] for a discussion of the multidimensional case.

Toward a rigorous mathematical analysis of solutions, the main difficulty that one encounters is the lack of regularity. Due to the strong nonlinearity of the equations and the absence of diffusion terms with smoothing effect, solutions which are initially smooth may become discontinuous within finite time. In the presence of discontinuities, most of the classical tools of differential calculus do not apply. Moreover, for general $n \times n$ systems, the powerful techniques of functional analysis cannot be used. In particular, solutions cannot be represented as fixed points of a nonlinear transformation, or in variational form as critical points of a suitable functional. Dealing with vector valued functions, comparison arguments based on upper and lower solutions do not apply either. Up to now, the theory of conservation laws has progressed largely by *ad hoc* methods. A survey of these techniques is the object of the present paper.

The Cauchy problem for a system of conservation laws in one space dimension takes the form

$$u_t + f(u)_x = 0, \tag{1.1}$$

$$u(0, x) = \bar{u}(x). \tag{1.2}$$

Here $u = (u_1, \ldots, u_n)$ is the vector of *conserved quantities*, while the components of $f = (f_1, \ldots, f_n)$ are the *fluxes*. We shall always assume that the flux function $f : I\!R^n \mapsto I\!R^n$ is smooth and that the system is *strictly hyperbolic*, i. e., at each point $u$ the Jacobian matrix $A(u) = Df(u)$ has $n$ real, distinct eigenvalues

$$\lambda_1(u) < \cdots < \lambda_n(u). \tag{1.3}$$

As already mentioned, a distinguished feature of nonlinear hyperbolic systems is the possible loss of regularity. Even with smooth initial data, it is well known that the solution can develop shocks in finite time. Therefore, solutions defined globally in time can only be found within a space of discontinuous functions. The equation (1.1) must then be interpreted in distributional sense. A vector valued function $u = u(t, x)$ is a *weak solution* of (1.1) if

$$\iint \left[ u\, \phi_t + f(u)\, \phi_x \right] dx dt = 0 \tag{1.4}$$

for every test function $\phi \in \mathcal{C}_c^1$, continuously differentiable with compact support. In particular, the piecewise constant function

$$u(t, x) \doteq \begin{cases} u^- & \text{if} \quad x < \lambda t, \\ u^+ & \text{if} \quad x > \lambda t, \end{cases} \tag{1.5}$$

is a weak solution of (1.1) if and only if the left and right states $u^-, u^+$ and the speed $\lambda$ satisfy the famous Rankine-Hugoniot equations

$$f(u^+) - f(u^-) = \lambda \left(u^+ - u^-\right).\tag{1.6}$$

When discontinuities are present, the weak solution of a Cauchy problem may not be unique. To single out a unique "good" solution, additional *entropy conditions* are usually imposed along shocks [Lx1], [L3]. These conditions often have a physical motivation, characterizing those solutions which can be recovered from higher order models, letting the diffusion or dispersion coefficients approach zero (see [D3]).

In one space dimension, the mathematical theory of hyperbolic systems of conservation laws has developed along two main lines.

**1.** The *BV* setting, pioneered by Glimm (1965). Solutions are here constructed within a space of functions with bounded variation, controlling the *BV* norm by a wave interaction potential.

**2.** The $\mathbf{L}^\infty$ setting, introduced by DiPerna (1983), based on weak convergence and a compensated compactness argument.

Both approaches yield results on the global existence of weak solutions. However, it is only in the *BV* setting that the well posedness of the Cauchy problem could recently be proved, as well as the stability and convergence of vanishing viscosity approximations. On the other hand, a counterexample in [BS] indicates that similar results cannot be expected, in general, for solutions in $\mathbf{L}^\infty$. In the remainder of this paper we thus concentrate on the theory of *BV* solutions, referring to [DP2] or [S1] for the alternative approach based on compensated compactness.

We shall first review the main ideas involved in the construction of weak solutions, based on the Riemann problem and the wave interaction functional. We then present more recent results on stability, uniqueness and characterization of entropy weak solutions. All this material can be found in the monograph [B3]. The last section contains an outline of the latest work on stability and convergence of vanishing viscosity approximations.

## 2. Existence of weak solutions

Toward the construction of more general solutions of (1.1), the basic building block is the *Riemann problem*, i.e. the initial value problem where the data are piecewise constant, with a single jump at the origin:

$$u(0, x) = \begin{cases} u^- & \text{if } x < 0, \\ u^+ & \text{if } x > 0. \end{cases}\tag{2.1}$$

Assuming that the amplitude $|u^+ - u^-|$ of the jump is small, this problem was solved in a classical paper of Lax [Lx1], under the additional hypothesis

(H) For each $i = 1, \ldots, n$, the $i$-th field is either *genuinely nonlinear*, so that $D\lambda_i(u) \cdot r_i(u) > 0$ for all $u$, or *linearly degenerate*, with $D\lambda_i(u) \cdot r_i(u) = 0$ for all $u$.

The solution is self-similar: $u(t,x) = U(x/t)$. It consists of $n + 1$ constant states $\omega_0 = u^-$, $\omega_1, \ldots, \omega_n = u^+$ (see Fig. 1). Each couple of adjacent states $\omega_{i-1}$, $\omega_i$ is separated either by a *shock* (the thick lines in Fig. 1) satisfying the Rankine Hugoniot equations, or else by a *centered rarefaction*. In this second case, the solution $u$ varies continuously between $\omega_{i-1}$ and $\omega_i$ in a sector of the $t$-$x$-plane (the shaded region in Fig. 1) where the gradient $u_x$ coincides with an $i$-eigenvector of the matrix $A(u)$.
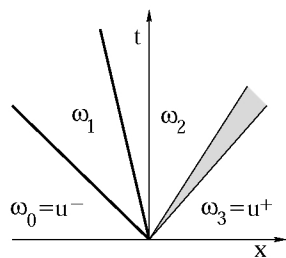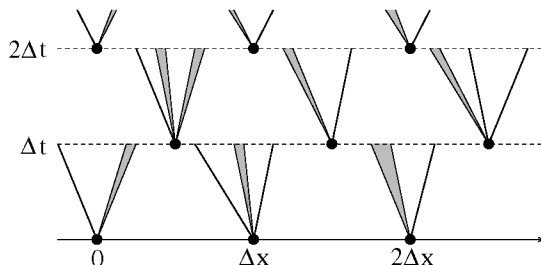


Figure 1                                    Figure 2

Approximate solutions to a more general Cauchy problem can be constructed by patching together several solutions of Riemann problems. In the Glimm scheme (Fig. 2), one works with a fixed grid in the $x$-$t$ plane, with mesh sizes $\Delta x$, $\Delta t$. At time $t = 0$ the initial data is approximated by a piecewise constant function, with jumps at grid points. Solving the corresponding Riemann problems, a solution is constructed up to a time $\Delta t$ sufficiently small so that waves generated by different Riemann problems do not interact. By a random sampling procedure, the solution $u(\Delta t, \cdot)$ is then approximated by a piecewise constant function having jumps only at grid points. Solving the new Riemann problems at every one of these points, one can prolong the solution to the next time interval $[\Delta t, \ 2\Delta t]$, etc...
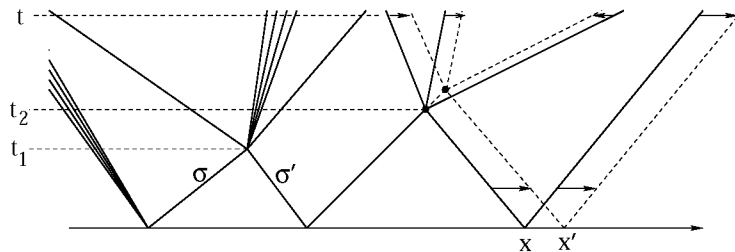


Figure 3

An alternative technique for contructing approximate solutions is by wave-

front tracking (Fig. 3). This method was introduced by Dafermos [D1] in the scalar case and later developed by various authors [DP1], [B1], [R], [BJ]. It now provides an efficient tool in the study of general $n \times n$ systems of conservation laws, both for theoretical and numerical purposes [B3], [HR].

The initial data is here approximated with a piecewise constant function, and each Riemann problem is solved approximately, within the class of piecewise constant functions. In particular, if the exact solution contains a centered rarefaction, this must be approximated by a *rarefaction fan*, containing several small jumps. At the first time $t_1$ where two fronts interact, the new Riemann problem is again approximately solved by a piecewise constant function. The solution is then prolonged up to the second interaction time $t_2$, where the new Riemann problem is solved, etc... The main difference is that in the Glimm scheme one specifies a priori the nodal points where the the Riemann problems are to be solved. On the other hand, in a solution constructed by wave-front tracking the locations of the jumps and of the interaction points depend on the solution itself, and no restarting procedure is needed.

In the end, both algorithms produce a sequence of approximate solutions, whose convergence relies on a compactness argument based on uniform bounds on the total variation. We sketch the main idea involved in these a priori BV bounds. Consider a piecewise constant function $u : \mathbb{R} \mapsto \mathbb{R}^n$, say with jumps at points $x_1 < x_2 < \cdots < x_N$. Call $\sigma_\alpha$ the amplitude of the jump at $x_\alpha$. The *total strength of waves* is then defined as

$$V(u) \doteq \sum_\alpha |\sigma_\alpha|. \tag{2.2}$$

Clearly, this is an equivalent way to measure the total variation. Along a solution $u = u(t,x)$ constructed by front tracking, the quantity $V(t) = V\big(u(t,\cdot)\big)$ may well increase at interaction times. To provide global a priori bounds, following [G] one introduces a *wave interaction potential*, defined as

$$Q(u) = \sum_{(\alpha,\beta)\in\mathcal{A}} |\sigma_\alpha \sigma_\beta|, \tag{2.3}$$

where the summation runs over the set $\mathcal{A}$ of all couples of approaching waves. Roughly speaking, we say that two wave-fronts located at $x_\alpha < x_\beta$ are *approaching* if the one at $x_\alpha$ has a faster speed than the one at $x_\beta$ (hence the two fronts are expected to collide at a future time). Now consider a time $\tau$ where two incoming wave-fronts interact, say with strengths $\sigma$, $\sigma'$ (for example, take $\tau = t_1$ in Fig. 3). The difference between the outgoing waves emerging from the interaction and the two incoming waves $\sigma, \sigma'$ is of magnitude $\mathcal{O}(1) \cdot |\sigma\sigma'|$. On the other hand, after time $\tau$ the two incoming waves are no longer approaching. This accounts for the decrease of the functional $Q$ in (2.3) by the amount $|\sigma\sigma'|$. Observing that the new waves generated by the interaction could approach all other fronts, the change in the functionals $V, Q$ across the interaction time $\tau$ is estimated as

$$\Delta V(\tau) = \mathcal{O}(1) \cdot |\sigma\sigma'|, \qquad \Delta Q(\tau) = -|\sigma\sigma'| + \mathcal{O}(1) \cdot |\sigma\sigma'| V(\tau-).$$

If the initial data has small total variation, for a suitable constant $C_0$ the quantity

$$\Upsilon(t) \doteq V\big(u(t,\cdot)\big) + C_0\, Q\big(u(t,\cdot)\big)$$

is monotone decreasing in time. This argument provides the uniform BV bounds on all approximate solutions. Using Helly's compactness theorem, one obtains the convergence of a subsequence of approximate solutions, and hence the global existence of a weak solution.

**Theorem 1.** *Let the system (1.1) be strictly hyperbolic and satisfy the assumptions (H). Then, for a sufficiently small $\delta > 0$ the following holds. For every initial condition $\bar{u}$ with*

$$\|\bar{u}\|_{\mathbf{L}^\infty} < \delta, \qquad\qquad Tot.\,Var.\{\bar{u}\} < \delta, \qquad\qquad (2.4)$$

*the Cauchy problem has a weak solution, defined for all times $t \geq 0$.*

This result is based on careful analysis of solutions of the Riemann problem and on the use of a quadratic interaction functional (2.3) to control the creation of new waves. These techniques also provided the basis for subsequent investigations of Glimm and Lax [GL] and Liu [L2] on the asymptotic behavior of weak solutions as $t \to \infty$.

# 3. Stability

The previous existence result relied on a compactness argument which, by itself, does not provide informations on the uniqueness of solutions. A first understanding of the dependence of weak solutions on the initial data was provided by the analysis of front tracking approximations. The idea is to perturb the initial data by shifting the position of one of the jumps, say from $x$ to a nearby point $x'$ (see Fig. 3). By carefully estimating the corresponding shifts in the positions of all wave-fronts at a later time $t$, one obtains a bound on the $\mathbf{L}^1$ distance between the original and the perturbed approximate solution. After much technical work, this approach yielded a proof of the Lipschitz continuous dependence of solutions on the initial data, first in [BC1] for $2 \times 2$ systems, then in [BCP] for general $n \times n$ systems.

**Theorem 2.** *Let the system (1.1) be strictly hyperbolic and satisfy the assumptions (H). Then, for every initial data $\bar{u}$ satisfying (2.4) the weak solution obtained as limit of Glimm or front tracking approximations is unique and depends Lipschitz continuously on the initial data, in the $\mathbf{L}^1$ distance.*

These weak solutions can thus be written in the form $u(t,\cdot) = S_t\bar{u}$, as trajecories of a semigroup $S : \mathcal{D} \times [0,\infty[\; \mapsto \mathcal{D}$ on some domain $\mathcal{D}$ containing all functions with sufficiently small total variation. For some Lipschitz constants $L, L'$ one has

$$\left\| S_t\bar{u} - S_s\bar{v} \right\|_{\mathbf{L}^1} \leq L\,\|\bar{u} - \bar{v}\|_{\mathbf{L}^1} + L'|t - s|, \qquad\qquad (3.1)$$

for all $t, s \geq 0$ and initial data $\bar{u}, \bar{v} \in \mathcal{D}$.

An alternative proof of Theorem 2 was later achieved by a technique introduced by Liu and Yang in [LY] and presented in [BLY] in its final form. The heart of the matter is to construct a nonlinear functional, equivalent to the $\mathbf{L}^1$ distance, which is decreasing in time along every pair of solutions. We thus seek $\Phi = \Phi(u, v)$ and a constant $C$ such that

$$\frac{1}{C} \cdot \|v - u\|_{\mathbf{L}^1} \leq \Phi(u, v) \leq C \cdot \|v - u\|_{\mathbf{L}^1}, \tag{3.2}$$

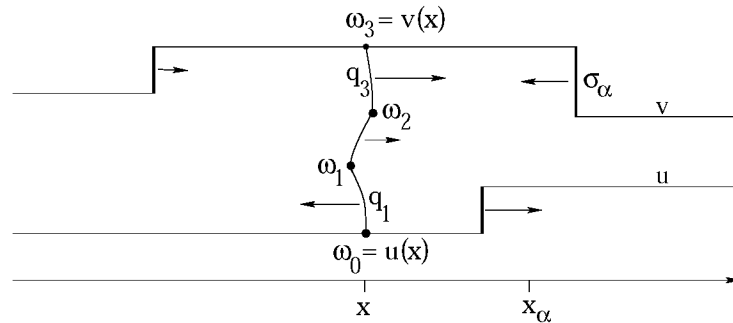$$\frac{d}{dt} \Phi\big(u(t),\ v(t)\big) \leq 0. \tag{3.3}$$



Figure 4

In connection with piecewise constant functions $u, v : \mathbb{R} \mapsto \mathbb{R}^n$ generated by a front tracking algorithm, this functional can be defined as follows (Fig. 4). At each point $x$, we connect the states $u(x)$, $v(x)$ by means of $n$ shock curves. In other words, we construct intermediate states $\omega_0 = u(x), \omega_1, \ldots, \omega_n = v(x)$ such that each pair $\omega_{i-1}, \omega_i$ is connected by an $i$-shock. These states can be uniquely determined by the implicit function theorem. Call $q_1, \ldots, q_n$, the strengths of these shocks. We regard $q_i(x)$ as the $i$-th scalar component of the jump $\big(u(x),\ v(x)\big)$. For some constant $C'$, one clearly has

$$\frac{1}{C'} \cdot \big|v(x) - u(x)\big| \leq \sum_{i=1}^{n} \big|q_i(x)\big| \leq C' \cdot \big|v(x) - u(x)\big|. \tag{3.4}$$

The functional $\Phi$ is now defined as

$$\Phi(u, v) \doteq \sum_{i=1}^{n} \int_{-\infty}^{\infty} W_i(x) \big|q_i(x)\big|\ dx, \tag{3.5}$$

where the weights $W_i$ take the form

$$W_i(x) \doteq 1 + \kappa_1 \cdot [\text{total strength of waves in } u \text{ and in } v$$
$$\text{which approach the } i\text{-wave } q_i(x)]$$
$$+ \kappa_2 \cdot [\text{wave interaction potentials of } u \text{ and of } v] \tag{3.6}$$
$$\doteq 1 + \kappa_1 V_i(x) + \kappa_2 \big[Q(u) + Q(v)\big]$$

for suitable constants $\kappa_1, \kappa_2$. Notice that, by construction, $q_i(x)$ represents the strength of a fictitious shock wave located at $x$, travelling with a speed $\lambda_i(x)$ determined by the Rankine-Hugoniot equations. In (3.6), it is thus meaningful to consider the quantity

$$V_i(x) \doteq \sum_{\alpha \in \mathcal{A}_i(x)} |\sigma_\alpha|,$$

where the summation extends to all wave-fronts $\sigma_\alpha$ in $u$ and in $v$ which are *approaching* the $i$-shock $q_i(x)$. From (3.4) and the boundedness of the weights $W_i$, one easily derives (3.2). By careful estimates on the Riemann problem, one can prove that also (3.3) is approximately satisfied. In the end, by taking a limit of front tracking approximations, one obtains Theorem 2.

For general $n \times n$ systems, in (3.1) one finds a Lipschitz constant $L > 1$. Indeed, it is only in the scalar case that the semigroup is contractive and the theory of accretive operators and abstract evolution equations in Banach spaces can be applied, see [K], [C]. We refer to the flow generated by a system of conservation laws as a *Riemann semigroup*, because it is entirely determined by specifying how Riemann problems are solved. As proved in [B2], if two semigroups $S, S'$ yield the same solutions to all Riemann problems, then they coincide, up to the choice of their domains.

From (3.1) one can deduce the error bound

$$\left\| w(T) - S_T w(0) \right\|_{\mathbf{L}^1} \leq L \cdot \int_0^T \left\{ \liminf_{h \to 0+} \frac{\left\| w(t+h) - S_h w(t) \right\|_{\mathbf{L}^1}}{h} \right\} dt, \qquad (3.7)$$

valid for every Lipschitz continuous map $w : [0, T] \mapsto \mathcal{D}$ taking values inside the domain of the semigroup. We can think of $t \mapsto w(t)$ as an approximate solution of (1.1), while $t \mapsto S_t w(0)$ is the exact solution having the same initial data. According to (3.7), the distance at time $T$ is bounded by the integral of an *instantaneous error rate*, amplified by the Lipschitz constant $L$ of the semigroup.

Using (3.7), one can estimate the distance between a front tracking approximation and the corresponding exact solution. For approximate solutions constructed by the Glimm scheme, a direct application of this same formula is not possible because of the additional errors introduced by the restarting procedures at times $t_k \doteq k \, \Delta t$. However, relying on a careful analysis of Liu [L1], one can construct a front tracking approximate solution having the same initial and terminal values as the Glimm solution. By this technique, in [BM] the authors proved the estimate

$$\lim_{\Delta x \to 0} \frac{\left\| u^{\mathrm{Glimm}}(T, \cdot) - u^{\mathrm{exact}}(T, \cdot) \right\|_{\mathbf{L}^1}}{\sqrt{\Delta x} \cdot |\ln \Delta x|} = 0. \qquad (3.8)$$

In other words, letting the mesh sizes $\Delta x, \Delta t \to 0$ while keeping their ratio $\Delta x / \Delta t$ constant, the $\mathbf{L}^1$ norm of the error in the Glimm approximate solution tends to zero at a rate slightly slower than $\sqrt{\Delta x}$.

# 4. Uniqueness

The uniqueness and stability results stated in Theorem 2 refer to a special class of weak solutions: those obtained as limits of Glimm or front tracking approximations. For several applications, it is desirable to have a uniqueness theorem valid for general weak solutions, without reference to any particular constructive procedure. Results in this direction were proved in [BLF], [BG], [BLe]. They are all based on the error formula (3.7). In the proofs, one considers a weak solution $u = u(t,x)$ of the Cauchy problem (1.1)–(1.2). Assuming that $u$ satisfies suitable entropy and regularity conditions, one shows that

$$\liminf_{h \to 0+} \frac{\left\| u(t+h) - S_h u(t) \right\|_{\mathbf{L}^1}}{h} = 0 \qquad (4.1)$$

at almost every time $t$. By (3.7), $u$ thus coincides with the semigroup trajectory $t \mapsto S_t u(0) = S_t \bar{u}$. Of course, this implies uniqueness. As an example, we state below the result of [BLe]. Consider the following assumptions:

**(A1)** **(Conservation Equations)** The function $u = u(t,x)$ is a weak solution of the Cauchy problem (1.1)–(1.2), taking values within the domain $\mathcal{D}$ of the semigroup $S$. More precisely, $u : [0,T] \mapsto \mathcal{D}$ is continuous w.r.t. the $\mathbf{L}^1$ distance. The initial condition (1.2) holds, together with

$$\iint \left[ u\,\phi_t + f(u)\,\phi_x \right] dx dt = 0$$

for every $\mathcal{C}^1$ function $\phi$ with compact support contained inside the open strip $]0,T[\times \mathbb{R}$.

**(A2)** **(Lax Entropy Condition)** Let $u$ have an approximate jump discontinuity at some point $(\tau,\xi) \in ]0,T[\times \mathbb{R}$. In other words, assume that there exists states $u^-, u^+ \in \Omega$ and a speed $\lambda \in \mathbb{R}$ such that, calling

$$U(t,x) \doteq \begin{cases} u^- & \text{if} \quad x < \xi + \lambda(t - \tau), \\ u^+ & \text{if} \quad x > \xi + \lambda(t - \tau), \end{cases} \qquad (4.2)$$

there holds

$$\lim_{\rho \to 0+} \frac{1}{\rho^2} \int_{\tau-\rho}^{\tau+\rho} \int_{\xi-\rho}^{\xi+\rho} \left| u(t,x) - U(t,x) \right| dx dt = 0. \qquad (4.3)$$

Then, for some $i \in \{1,\dots,n\}$, one has the entropy inequality:

$$\lambda_i(u^-) \geq \lambda \geq \lambda_i(u^+). \qquad (4.4)$$

**(A3)** **(Bounded Variation Condition)** The function $x \mapsto u\big(\tau(x),x\big)$ has bounded variation along every Lipschitz continuous space-like curve $\{t = \tau(x)\}$, which satisfies $|d\tau/dx| < \delta$ a.e., for some constant $\delta > 0$ small enough.

**Theorem 3.** *Let $u = u(t, x)$ be a weak solution of the Cauchy problem (1.1)–(1.2) satisfying the assumptions (A1), (A2) and (A3). Then*

$$u(t, \cdot) = S_t \bar{u} \tag{4.5}$$

*for all $t$. In particular, the solution that satisfies the three above conditions is unique.*

An additional characterization of these unique solutions, based on local integral estimates, was given in [B2]. The underlying idea is as follows. In a forward neighborhood of a point $(\tau, \xi)$ where $u$ has a jump, the weak solution $u$ behaves much in the same way as the solution of the corresponding Riemann problem. On the other hand, on a region where its total variation is small, our solution $u$ can be accurately approximated by the solution of a linear hyperbolic system with constant coefficients.

To state the result more precisely, we introduce some notations. Given a function $u = u(t, x)$ and a point $(\tau, \xi)$, we denote by $U^{\sharp}_{(u;\tau,\xi)}$ the solution of the Riemann problem with initial data

$$u^{-} = \lim_{x \to \xi-} u(\tau, x), \qquad u^{+} = \lim_{x \to \xi+} u(\tau, x). \tag{4.6}$$

In addition, we define $U^{\flat}_{(u;\tau,\xi)}$ as the solution of the linear hyperbolic Cauchy problem with constant coefficients

$$w_t + \widehat{A} w_x = 0, \qquad w(0, x) = u(\tau, x). \tag{4.7}$$

Here $\widehat{A} \doteq A\big(u(\tau, \xi)\big)$. Observe that (4.7) is obtained from the quasilinear system

$$u_t + A(u)u_x = 0 \qquad (A = Df) \tag{4.8}$$

by "freezing" the coefficients of the matrix $A(u)$ at the point $(\tau, \xi)$ and choosing $u(\tau)$ as initial data. A new notion of "good solution" can now be introduced, by locally comparing a function $u$ with the self-similar solution of a Riemann problem and with the solution of a linear hyperbolic system with constant coefficients. More precisely, we say that a function $u = u(t, x)$ is a **viscosity solution** of the system (1.1) if $t \mapsto u(t, \cdot)$ is continuous as a map with values into $\mathbf{L}^1_{\mathrm{loc}}$, and moreover the following integral estimates hold.

(i) At every point $(\tau, \xi)$, for every $\beta' > 0$ one has

$$\lim_{h \to 0+} \frac{1}{h} \int_{\xi - \beta' h}^{\xi + \beta' h} \left| u(\tau + h, \ x) - U^{\sharp}_{(u;\tau,\xi)}(h, \ x - \xi) \right| \, dx \ = \ 0. \tag{4.9}$$

(ii) There exist constants $C, \beta > 0$ such that, for every $\tau \geq 0$ and $a < \xi < b$, one has

$$\limsup_{h \to 0+} \frac{1}{h} \int_{a + \beta h}^{b - \beta h} \left| u(\tau + h, \ x) - U^{\flat}_{(u;\tau,\xi)}(h, x) \right| \, dx \leq C \cdot \left( \mathrm{Tot.Var.}\{u(\tau); \ ]a, \, b[ \ \} \right)^2. \tag{4.10}$$

As proved in [B2], this concept of viscosity solution completely characterizes semigroup trajectories.

**Theorem 4.** *Let* $S : \mathcal{D} \times [0, \infty[ \times \mathcal{D}$ *be a semigroup generated by the system of conservation laws (1.1). A function* $u : [0, T] \mapsto \mathcal{D}$ *is a viscosity solution of (1.1) if and only if* $u(t) = S_t u(0)$ *for all* $t \in [0, T]$.

# 5. Vanishing viscosity approximations

A natural conjecture is that the entropic solutions of the hyperbolic system (1.1) actually coincide with the limits of solutions to the parabolic system

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon\, u_{xx}^\varepsilon \,, \tag{5.1}$$

letting the viscosity coefficient $\varepsilon \to 0$. In view of the previous uniqueness results, one expects that the vanishing viscosity limit should single out the unique "good" solution of the Cauchy problem, satisfying the appropriate entropy conditions. In earlier literature, results in this direction were based on three main techniques:

**1 - Comparison principles for parabolic equations.** For a *scalar* conservation law, the existence, uniqueness and global stability of vanishing viscosity solutions was first established by Oleinik [O] in one space dimension. The famous paper by Kruzhkov [K] covers the more general class of $\mathbf{L}^\infty$ solutions and is also valid in several space dimensions.

**2 - Singular perturbations.** Let $u$ be a piecewise smooth solution of the $n \times n$ system (1.1), with finitely many non-interacting, entropy admissible shocks. In this special case, using a singular perturbation technique, Goodman and Xin [GX] constructed a family of solutions $u^\varepsilon$ to (5.1), with $u^\varepsilon \to u$ as $\varepsilon \to 0$.

**3 - Compensated compactness.** If, instead of a *BV* bound, only a uniform bound on the $\mathbf{L}^\infty$ norm of solutions of (5.1) is available, one can still construct a weakly convergent subsequence $u^\varepsilon \rightharpoonup u$. In general, we cannot expect that this weak limit satisfies the nonlinear equations (1.1). However, for a class of $2 \times 2$ systems, in [DP2] DiPerna showed that this limit $u$ is indeed a weak solution of (1.1). The proof relies on a compensated compactness argument, based on the representation of the weak limit in terms of Young measures, which must reduce to a Dirac mass due to the presence of a large family of entropies.

Since the main existence and uniqueness results for hyperbolic systems of conservation laws are valid within the space of BV functions, it is natural to seek uniform BV bounds also for the viscous approximations $u^\varepsilon$ in (5.1). This is indeed the main goal accomplished in [BB]. As soon as these BV bounds are established, the existence of a vanishing viscosity limit follows by a standard compactness argument. The uniqueness of the limit can then be deduced from the uniqueness theorem in [BG]. By further analysis, one can also prove the continuous dependence on the

initial data for the viscous approximations $u^\varepsilon$, in the $\mathbf{L}^1$ norm. Remarkably, these results are valid for general $n \times n$ strictly hyperbolic systems, not necessarily in conservation form.

**Theorem 5.** *Consider the Cauchy problem for a strictly hyperbolic system with viscosity*

$$u_t^\varepsilon + A(u^\varepsilon)u_x^\varepsilon = \varepsilon\, u_{xx}^\varepsilon, \qquad u^\varepsilon(0,x) = \bar{u}(x)\,. \tag{5.2}$$

*Then there exist constants $C, L, L'$ and $\delta > 0$ such that the following holds. If*

$$Tot.\,Var.\{\bar{u}\} < \delta\,, \qquad \big\|\bar{u}(x)\big\|_{\mathbf{L}^\infty} < \delta\,, \tag{5.3}$$

*then for each $\varepsilon > 0$ the Cauchy problem (5.2) has a unique solution $u^\varepsilon$, defined for all $t \geq 0$. Adopting a semigroup notation, this will be written as $t \mapsto u^\varepsilon(t, \cdot) \doteq S_t^\varepsilon \bar{u}$. In addition, one has:*

**BV bounds** : $\qquad\qquad Tot.\,Var.\big\{S_t^\varepsilon \bar{u}\big\} \leq C\ Tot.\,Var.\{\bar{u}\}\,. \tag{5.4}$

**$\mathbf{L}^1$ stability** : $\qquad\qquad \big\|S_t^\varepsilon \bar{u} - S_t^\varepsilon \bar{v}\big\|_{\mathbf{L}^1} \leq L\,\big\|\bar{u} - \bar{v}\big\|_{\mathbf{L}^1}\,, \tag{5.5}$

$$\big\|S_t^\varepsilon \bar{u} - S_s^\varepsilon \bar{u}\big\|_{\mathbf{L}^1} \leq L'\left(|t - s| + \big|\sqrt{\varepsilon t} - \sqrt{\varepsilon s}\,\big|\right). \tag{5.6}$$

**Convergence.** *As $\varepsilon \to 0+$, the solutions $u^\varepsilon$ converge to the trajectories of a semigroup $S$ such that*

$$\big\|S_t \bar{u} - S_s \bar{v}\big\|_{\mathbf{L}^1} \leq L\,\|\bar{u} - \bar{v}\|_{\mathbf{L}^1} + L'\,|t - s|\,. \tag{5.7}$$

*These vanishing viscosity limits can be regarded as the unique* **vanishing viscosity solutions** *of the hyperbolic Cauchy problems*

$$u_t + A(u)u_x = 0, \qquad u(0,x) = \bar{u}(x)\,. \tag{5.8}$$

In the conservative case where $A(u) = Df(u)$ for some flux function $f$, the vanishing viscosity solution is a weak solution of

$$u_t + f(u)_x = 0, \qquad u(0,x) = \bar{u}(x)\,, \tag{5.9}$$

*satisfying the Liu admissibility conditions [L3]. Moreover, the vanishing viscosity solutions are precisely the same as the viscosity solutions defined at (4.9)–(4.10) in terms of local integral estimates.*

The key step in the proof is to establish a priori bounds on the total variation of solutions of

$$u_t + A(u)u_x = u_{xx} \tag{5.10}$$

uniformly valid for all times $t \in [0, \infty[$. We outline here the main ideas.

(i) At each point $(t, x)$ we decompose the gradient along a suitable basis of unit vectors $\tilde{r}_i$, say

$$u_x = \sum v_i \tilde{r}_i \, . \tag{5.11}$$

(ii) We then derive an equation describing the evolution of these gradient components

$$v_{i,t} + (\tilde{\lambda}_i v_i)_x - v_{i,xx} = \phi_i \, . \tag{5.12}$$

(iii) Finally, we show that all source terms $\phi_i = \phi_i(t, x)$ are integrable. Hence, for all $\tau > 0$,

$$\left\| v_i(\tau, \cdot) \right\|_{\mathbf{L}^1} \leq \left\| v_i(0, \cdot) \right\|_{\mathbf{L}^1} + \int_0^\infty \int_{I\!\!R} \left| \phi_i(t, x) \right| dx dt < \infty \, . \tag{5.13}$$

In this connection, it seems natural to decompose the gradient $u_x$ along the eigenvectors of the hyperbolic matrix $A(u)$. This approach however does NOT work. In the case where the solution $u$ is a travelling viscous shock profile, we would obtain source terms which are not identically zero. Hence they are certainly not integrable over the domain $\{ t > 0 \, , x \in I\!\!R \}$.

An alternative approach, proposed by S. Bianchini, is to decompose $u_x$ as a *sum of gradients of viscous travelling waves*. By a viscous travelling $i$-wave we mean a solution of (5.10) having the form

$$w(t, x) = U(x - \sigma t) \, , \tag{5.14}$$

where the speed $\sigma$ is close to the $i$-th eigenvalue $\lambda_i$ of the hyperbolic matrix $A$. Clearly, the function $U$ must provide a solution to the second order O.D.E.

$$U'' = \big( A(U) - \sigma \big) U' \, . \tag{5.15}$$

The underlying idea for the decomposition is as follows. At each point $(t, x)$, given $(u, u_x, u_{xx})$, we seek travelling wave profiles $U_1, \ldots, U_n$ such that

$$U_i(x) = u(x), \qquad\qquad i = 1, \ldots, n \, , \tag{5.16}$$

$$\sum_i U_i'(x) = u_x(x) \, , \qquad\qquad \sum_i U_i''(x) = u_{xx}(x) \, . \tag{5.17}$$

In general, the system of algebraic equations (5.16)–(5.17) admits infinitely many solutions. A unique solution is singled out by considering only those travelling profiles $U_i$ that lie on a suitable *center manifold* $\mathcal{M}_i$. We now call $\tilde{r}_i$ the unit vector parallel to $U_i'$, so that $U_i' = v_i \tilde{r}_i$ for some scalar $v_i$. The decomposition (5.11) is then obtained from the first equation in (5.17).

Toward the BV estimate, the second part of the proof consists in deriving the equation (5.12) and estimating the integrals of the source terms $\phi_i$. Here the main

idea is that these source terms can be regarded as generated by wave interactions. In analogy with the hyperbolic case considered by Glimm [G], the total amount of these interactions can be controlled by suitable Lyapunov functionals. We describe here the main ones.

**1.** Consider first two independent, scalar diffusion equations with strictly different drifts:

$$\begin{cases} z_t + \big[\lambda(t,x)z\big]_x - z_{xx} = 0\,, \\ z_t^* + \big[\lambda^*(t,x)z^*\big]_x - z_{xx}^* = 0\,, \end{cases}$$

assuming that

$$\inf_{t,x} \lambda^*(t,x) - \sup_{t,x} \lambda(t,x) \geq c > 0\,.$$

We regard $z$ as the density of waves with a slow speed $\lambda$ and $z^*$ as the density of waves with a fast speed $\lambda^*$. A *transversal interaction potential* is defined as

$$Q(z,z^*) \doteq \frac{1}{c} \iint_{I\!\!R^2} K(x_2 - x_1)\big|z(x_1)\big|\,\big|z^*(x_2)\big|\,dx_1 dx_2\,, \tag{5.18}$$

$$K(y) \doteq \begin{cases} e^{-cy/2} & \text{if} \quad y > 0\,, \\ 1 & \text{if} \quad y \leq 0\,. \end{cases} \tag{5.19}$$

One can show that this functional $Q$ is monotonically decreasing along every couple of solutions $z, z^*$. The total amount of interaction between fast and slow waves can now be estimated as

$$\int_0^\infty \int_{I\!\!R} \big|z(t,x)\big|\,\big|z^*(t,x)\big|\,dxdt \;\leq\; -\int_0^\infty \left[\frac{d}{dt}Q\big(z(t),\,z^*(t)\big)\right]\,dt$$

$$\leq Q\big(z(0),z^*(0)\big) \;\leq\; \frac{1}{c}\int_{I\!\!R} \big|z(0,x)\big|\,dx \cdot \int_{I\!\!R} \big|z^*(0,x)\big|\,dx\,.$$

By means of Lyapunov functionals of this type one can control all source terms in (5.12) due to the interaction of waves of different families.

**2.** To control the interactions between waves of the same family, we seek functionals which are decreasing along every solution of a scalar viscous conservation law

$$u_t + g(u)_x = u_{xx}\,. \tag{5.20}$$

For this purpose, to a scalar function $x \mapsto u(x)$ we associate the curve in the plane

$$\gamma \doteq \begin{pmatrix} u \\ g(u) - u_x \end{pmatrix} = \begin{pmatrix} \text{conserved quantity} \\ \text{flux} \end{pmatrix}\,. \tag{5.21}$$

In connection with a solution $u = u(t,x)$ of (5.20), the curve $\gamma$ evolves according to

$$\gamma_t + g'(u)\gamma_x = \gamma_{xx}\,. \tag{5.22}$$

Notice that the vector $g'(u)\gamma_x$ is parallel to $\gamma$, hence the presence of this term in (5.22) only amounts to a reparametrization of the curve, and does not affect its shape. The curve thus evolves in the direction of curvature. An obvious Lyapunov functional is the *length* of the curve. In terms of the variables

$$\gamma_x = \begin{pmatrix} v \\ w \end{pmatrix} \doteq \begin{pmatrix} u_x \\ -u_t \end{pmatrix}, \tag{5.23}$$

this length is given by

$$L(\gamma) \doteq \int |\gamma_x| \, dx = \int \sqrt{v^2 + w^2} \, dx. \tag{5.24}$$

We can estimate the rate of decrease in the length as

$$-\frac{d}{dt} L\big(\gamma(t)\big) = \int_{I\!R} \frac{|v| \left[ (w/v)_x \right]^2}{\left( 1 + (w/v)^2 \right)^{3/2}} \, dx \geq \frac{1}{(1 + \delta^2)^{3/2}} \int_{|w/v| \leq \delta} |v| \left[ (w/v)_x \right]^2 \, dx, \tag{5.25}$$

for any given constant $\delta > 0$. This yields a useful a priori estimate on the integral on the right hand side of (5.25).

**3.** In connection with the same curve $\gamma$ in (5.21), we now introduce another functional, defined in terms of a wedge product.

$$Q(\gamma) \doteq \frac{1}{2} \iint_{x < x'} \left| \gamma_x(x) \wedge \gamma_x(x') \right| \, dx \, dx'. \tag{5.26}$$

For any curve that moves in the plane in the direction of curvature, one can show that this functional is monotone decreasing and its decrease bounds the area swept by the curve: $|dA| \leq -dQ$.

Using (5.22)–(5.23) we now compute

$$-\frac{dQ}{dt} \geq \left| \frac{dA}{dt} \right| = \int |\gamma_t \wedge \gamma_x| \, dx = \int |\gamma_{xx} \wedge \gamma_x| \, dx = \int |v_x w - v w_x| \, dx.$$

Integrating w.r.t. time, we thus obtain another useful a priori bound:

$$\int_0^\infty \int |v_x w - v w_x| \, dx \, dt \leq \int_0^\infty \left| \frac{dQ\big(\gamma(t)\big)}{dt} \right| \, dt \leq Q\big(\gamma(0)\big).$$

Together, the functionals in (5.24) and (5.26) allow us to estimate all source terms in (5.12) due to the interaction of waves of the same family.

This yields the $\mathbf{L}^1$ estimates on the source terms $\phi_i$, in (5.12), proving the uniform bounds on the total variation of a solution $u$ of (5.10). See [BB] for details.

Next, to prove the uniform stability of all solutions of the parabolic system (5.10) having small total variation, we consider the linearized system describing the

evolution of a first order variation. Inserting the formal expansion $u = u_0 + \epsilon z + O(\epsilon^2)$ in (5.10), we obtain

$$z_t + \left[DA(u) \cdot z\right]u_x + A(u)z_x = z_{xx} \,. \tag{5.27}$$

Our basic goal is to prove the bound

$$\left\|z(t)\right\|_{\mathbf{L}^1} \leq L \left\|z(0)\right\|_{\mathbf{L}^1} \,, \tag{5.28}$$

for some constant $L$ and all $t \geq 0$ and every solution $z$ of (5.27). By a standard homotopy argument, from (5.28) one easily deduces the Lipschitz continuity of the solution of (5.8) on the initial data. Namely, for every couple of solutions $u, \tilde{u}$ with small total variation one has

$$\left\|u(t) - \tilde{u}(t)\right\|_{\mathbf{L}^1} \leq L \left\|u(0) - \tilde{u}(0)\right\|_{\mathbf{L}^1} \,. \tag{5.29}$$

To prove (5.28) we decompose the vector $z$ as a sum of scalar components: $z = \sum_i h_i \tilde{r}_i$, write an evolution equation for these components:

$$h_{i,t} + (\tilde{\lambda}_i h_i)_x - h_{i,xx} = \hat{\phi}_i \,,$$

and show that the source terms $\hat{\phi}_i$ are integrable on the domain $\{t > 0 \,, x \in \mathbb{R}\}$.

For every initial data $u(0, \cdot) = \bar{u}$ with small total variation, the previous arguments yield the existence of a unique global solution to the parabolic system (5.8), depending Lipschitz continuously on the initial data, in the $\mathbf{L}^1$ norm. Performing the rescaling $t \mapsto t/\varepsilon$, $x \mapsto x/\varepsilon$, we immediately obtain the same results for the Cauchy problem (5.2). Adopting a semigroup notation, this solution can be written as $u^\varepsilon(t, \cdot) = S_t^\varepsilon \bar{u}$. Thanks to the uniform bounds on the total variation, a compactness argument yields the existence of a strong limit in $\mathbf{L}_{\mathrm{loc}}^1$

$$u = \lim_{\varepsilon_m \to 0} u^{\varepsilon_m} \tag{5.30}$$

at least for some subsequence $\varepsilon_m \to 0$. Since the $u^\varepsilon$ depend continuously on the initial data, with a uniform Lipschitz constant, the same is true of the limit solution $u(t, \cdot) = S_t \bar{u}$. In the conservative case where $A(u) = Df(u)$, it is not difficult to show that this limit $u$ actually provides a weak solution to the Cauchy problem (1.1)–(1.2).

The only remaining issue is to show that the limit in (5.30) is unique, i.e. it does not depend on the subsequence $\{\varepsilon_m\}$. In the standard conservative case, this fact can already be deduced from the uniqueness result in [BG]. In the general case, uniqueness is proved in two steps. First we show that, in the special case of a Riemann problem, the solution obtained as vanishing viscosity limit is unique and can be completely characterized. To conclude the proof, we then rely on the same

general argument as in [B2]: if two Lipschitz semigroups $S, S'$ provide the same solutions to all Riemann problems, then they must coincide. See [BB] for details.

# 6. Concluding remarks

**1.** A classical tool in the analysis of first order hyperbolic systems is the *method of characteristics*. To study the system

$$u_t + A(u)u_x = 0\,,$$

one decomposes the solution along the eigenspaces of the matrix $A(u)$. The evolution of these components is then described by a family of O.D.E's along the *characteristic curves*. In the $t$-$x$ plane, these are the curves which satisfy $dx/dt = \lambda_i\big(u(t,x)\big)$. The local decomposition (5.16)–(5.17) in terms of viscous travelling waves makes it possible to implement this "hyperbolic" approach also in connection with the parabolic system (5.10). In this case, the projections are taken along the vectors $\tilde{r}_i$, while the characteristic curves are defined as $dx/dt = \sigma_i$, where $\sigma_i$ is the speed of the $i$-th travelling wave. Notice that in the hyperbolic case the projections and the wave speeds depend only on the state $u$, through the eigenvectors $r_i(u)$ and the eigenvalues $\lambda_i(u)$ of the matrix $A(u)$. On the other hand, in the parabolic case the construction involves the derivatives $u_x$, $u_{xx}$ as well.

**2.** In nearly all previous works on BV solutions for systems of conservation laws, following [G] the basic estimates on the total variation were obtained by a careful study of the Riemann problem and of elementary wave interactions. The Riemann problem also takes the center stage in all earlier proofs of the stability of solutions [BC1], [BCP], [BLY]. In this connection, the hypothesis (H) introduced by Lax [Lx1] is widely adopted in the literature. It guarantees that solutions of the Riemann problem have a simple structure, consisting of at most $n$ elementary waves (shocks, centered rarefactions or contact discontinuities). If the assumption (H) is dropped, some results on global existence [L3], and continuous dependence [AM] are still available, but their proofs become far more technical. On the other hand, the approach introduced in [BB] marks the first time where uniform $BV$ estimates are obtained without any reference to Riemann problems. Global existence and stability of weak solutions are obtained for the whole class of strictly hyperbolic systems, regardless of the hypothesis (H).

**3.** For the viscous system of conservation laws

$$u_t + f(u)_x = u_{xx}\,,$$

previous results in [L4], [SX], [SZ], [Yu] have established the stability of special types of solutions, for example travelling viscous shocks or viscous rarefactions. Taking $\varepsilon = 1$ in (5.2), from Theorem 5 we obtain the uniform Lipschitz stability (w..r.t. the $\mathbf{L}^1$ distance) of ALL viscous solutions with sufficiently small total variation. An

interesting alternative technique for proving stability of viscous solutions, based on spectral methods, was recently developed in [HZ].

**4.** In the present survey we only considered initial data with small total variation. This is a convenient setting, adopted in much of the current literature, which guarantees the global existence of *BV* solutions of (1.1) and captures the main features of the problem. A recent example constructed by Jenssen [J] shows that, for initial data with large total variation, the $\mathbf{L}^\infty$ norm of the solution can blow up in finite time. In this more general setting, one expects that the existence and uniqueness of weak solutions, together with the convergence of vanishing viscosity approximations, should hold locally in time as long as the total variation remains bounded. For the hyperbolic system (1.1), results on the local existence and stability of solutions with large *BV* data can be found in [Sc] and [BC2], respectively. Because of the counterexample in [BS], on the other hand, similar well posedness results are not expected in the general $\mathbf{L}^\infty$ case.

# References

[AM] F. Ancona and A. Marson, Well posedness for general 2 × 2 systems of conservation laws, *Amer. Math. Soc. Memoir*, to appear.

[BaJ] P. Baiti and H. K. Jenssen, On the front tracking algorithm, *J. Math. Anal. Appl.* **217** (1998), 395–404.

[BB] S. Bianchini and A. Bressan, Vanishing viscosity solutions of nonlinear hyperbolic systems, preprint S.I.S.S.A., Trieste 2001.

[B1] A. Bressan, Global solutions to systems of conservation laws by wave-front tracking, *J. Math. Anal. Appl.* **170** (1992), 414–432.

[B2] A. Bressan, The unique limit of the Glimm scheme, *Arch. Rational Mech. Anal.* **130** (1995), 205–230.

[B3] A. Bressan, *Hyperbolic Systems of Conservation Laws. The One Dimensional Cauchy Problem.* Oxford University Press, 2000.

[BC1] A. Bressan and R. M. Colombo, The semigroup generated by 2 × 2 conservation laws, *Arch. Rational Mech. Anal.* **133** (1995), 1–75.

[BC2] A. Bressan and R. M. Colombo, Unique solutions of 2 × 2 conservation laws with large data, *Indiana Univ. Math. J.* **44** (1995), 677–725.

[BCP] A. Bressan, G. Crasta and B. Piccoli, Well posedness of the Cauchy problem for $n \times n$ conservation laws, *Amer. Math. Soc. Memoir* **694** (2000).

[BG] A. Bressan and P. Goatin, Oleinik type estimates and uniqueness for $n \times n$ conservation laws, *J. Diff. Equat.* **156** (1999), 26–49.

[BLF] A. Bressan and P. LeFloch, Uniqueness of weak solutions to systems of conservation laws, *Arch. Rat. Mech. Anal.* **140** (1997), 301–317.

[BLe] A. Bressan and M. Lewicka, A uniqueness condition for hyperbolic systems of conservation laws, *Discr. Cont. Dynam. Syst.* **6** (2000), 673–682.

[BLY] A. Bressan, T. P. Liu and T. Yang, $L^1$ stability estimates for $n \times n$ conservation laws, *Arch. Rational Mech. Anal.* **149** (1999), 1–22.

[BM] A. Bressan and A. Marson, Error bounds for a deterministic version of the Glimm scheme, *Arch. Rat. Mech. Anal.* **142** (1998), 155–176.

[BS] A. Bressan and W. Shen, Uniqueness for discontinuous O.D.E. and conservation laws, *Nonlinear Analysis, T. M. A.* **34** (1998), 637–652.

[C] M. Crandall, The semigroup approach to first-order quasilinear equations in several space variables, *Israel J. Math.* **12** (1972), 108–132.

[D1] C. Dafermos, Polygonal approximations of solutions of the initial value problem for a conservation law, *J. Math. Anal. Appl.* **38** (1972), 33–41.

[D2] C. Dafermos, Hyperbolic systems of conservation laws, *Proceedings of the International Congress of Mathematicians, Zürich 1994*, Birchäuser (1995), 1096–1107.

[D3] C. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*, Springer-Verlag, Berlin 2000.

[DP1] R. DiPerna, Global existence of solutions to nonlinear hyperbolic systems of conservation laws, *J. Diff. Equat.* **20** (1976), 187–212.

[DP2] R. DiPerna, Convergence of approximate solutions to conservation laws, *Arch. Rational Mech. Anal.* **82** (1983), 27–70.

[G] J. Glimm, Solutions in the large for nonlinear hyperbolic systems of equations, *Comm. Pure Appl. Math.* **18** (1965), 697–715.

[GL] J. Glimm and P. Lax, Decay of solutions of systems of nonlinear hyperbolic conservation laws, *Amer. Math. Soc. Memoir* **101** (1970).

[GX] J. Goodman and Z. Xin, Viscous limits for piecewise smooth solutions to systems of conservation laws, *Arch. Rational Mech. Anal.* **121** (1992), 235–265.

[HR] H. Holden and N. H. Risebro *Front Tracking for Hyperbolic Conservation Laws*, Springer Verlag, New York 2002.

[HZ] P. Howard and K. Zumbrun, Pointwise semigroup methods for stability of viscous shock waves, *Indiana Univ. Math. J.* **47** (1998), 727–841.

[K] S. Kruzhkov, First order quasilinear equations with several space variables, *Math. USSR Sbornik* **10** (1970), 217–243.

[J] H. K. Jenssen, Blowup for systems of conservation laws, *SIAM J. Math. Anal.* **31** (2000), 894–908.

[Lx1] P. Lax, Hyperbolic systems of conservation laws II, *Comm. Pure Appl. Math.* **10** (1957), 537–566.

[Lx2] P. Lax, Problems solved and unsolved concerning nonlinear P.D.E., *Proccedings of the International Congress of Mathematicians, Warszawa 1983*. Elsevier Science Pub. (1984), 119–138.

[L1] T. P. Liu, The deterministic version of the Glimm scheme, *Comm. Math. Phys.* **57** (1977), 135–148.

[L2] T. P. Liu, Linear and nonlinear large time behavior of solutions of general systems of hyperbolic conservation laws, *Comm. Pure Appl. Math.* **30** (1977), 767–796.

[L3] T. P. Liu, Admissible solutions of hyperbolic conservation laws, *Amer. Math. Soc. Memoir* **240** (1981).

[L4] T. P. Liu, Nonlinear stability of shock waves, *Amer. Math. Soc. Memoir* **328** (1986).

[LY] T. P. Liu and T. Yang, $L^1$ stability for $2 \times 2$ systems of hyperbolic conservation laws, *J. Amer. Math. Soc.* **12** (1999), 729–774.

[M] A. Majda, *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables*, Springer-Verlag, New York, 1984.

[O] O. Oleinik, Discontinuous solutions of nonlinear differential equations (1957), *Amer. Math. Soc. Translations* **26**, 95–172.

[R] N. H. Risebro, A front-tracking alternative to the random choice method, *Proc. Amer. Math. Soc.* **117** (1993), 1125–1139.

[Sc] S. Schochet, Sufficient conditions for local existence via Glimm's scheme for large BV data, *J. Differential Equations* **89** (1991), 317–354.

[S1] D. Serre, *Systems of Conservation Laws I, II*, Cambridge University Press, 2000.

[S2] D. Serre, Systems of conservation laws : A challenge for the XXIst century, *Mathematics Unlimited - 2001 and beyond*, B. Engquist and W. Schmid eds., Springer-Verlag, 2001.

[SX] A. Szepessy and Z. Xin, Nonlinear stability abd viscous shocks, *Arch. Rational Mech. Anal.* **122** (1993), 53–103.

[SZ] A. Szepessy and K. Zumbrun, Stability of rarefaction waves in viscous media, *Arch. Rational Mech. Anal.* **133** (1996), 249–298.

[Yu] S. H. Yu, Zero-dissipation limit of solutions with shocks for systems of hyperbolic conservation laws, *Arch. Rational Mech. Anal.* **146** (1999), 275–370.