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Cohomology of Moduli Spaces

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Abstract

Some recent progress towards understanding the cohomology of moduli spaces of curves is described. Madsen and Weiss have announced a proof of a generalisation of Mumford's conjecture on the stable cohomology of these moduli spaces $\mathcal{M}_g$, and other contributors have made advances related to Faber's conjectures concerning the tautological ring of $\mathcal{M}_g$.

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Moduli spaces arise in classification problems in algebraic geometry (and other areas of geometry) when, as is typically the case, there are not enough discrete invariants to classify objects up to isomorphism. In the case of nonsingular complex projective curves (or compact Riemann surfaces) the genus g is a discrete invariant which classifies the curve regarded as a topological surface, but does not determine its complex structure when $g > 0$. For each $g \geq 0$ there is a moduli space $\mathcal{M}_g$ whose points correspond bijectively to isomorphism classes of nonsingular complex projective curves of genus g, and whose geometric structure reflects the way such curves can vary in families depending on parameters. The topology of these moduli spaces $\mathcal{M}_g$ and their compactifications has been studied for several decades, and important progress has been made recently on some long-standing questions concerning their cohomology.

In his fundamental paper [93] Mumford considered some tautological cohomological classes $\kappa_j \in H^{2j}(\mathcal{M}_g)$ for $j = 1, 2, \ldots$ which extend naturally to the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$. Much work on the cohomology of $\mathcal{M}_g$ has concentrated on its tautological ring, which is the subalgebra of its rational cohomology ring (or of its Chow ring) generated by these tautological classes.

One reason for the importance of the tautological ring of $\mathcal{M}_g$ is its relationship with the stable cohomology ring $H^*(\mathcal{M}_\infty; \mathbb{Q})$. It was proved by Harer [47] that the cohomology $H^k(\mathcal{M}_g; \mathbb{Q})$ of $\mathcal{M}_g$ in degree k is independent of the genus g when

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$g \gg k$, making it possible to define $H^k(\mathcal{M}_\infty;\mathbb{Q})$ as $H^k(\mathcal{M}_g;\mathbb{Q})$ for suitably large g. Mumford conjectured that the stable cohomology ring $H^*(\mathcal{M}_\infty;\mathbb{Q})$ is freely generated by the tautological classes $\kappa_1, \kappa_2, \ldots$ and Miller [83] and Morita [85] proved part of this conjecture by showing that the natural map

$$\mathbb{Q}[\kappa_1, \kappa_2, \ldots] \to H^*(\mathcal{M}_\infty;\mathbb{Q})$$

is injective. The remainder of Mumford's conjecture, that this map is surjective, remained unproved for nearly two decades. However Madsen and Tillmann [80] found an interpretation of Mumford's map on the level of homotopy, which they conjectured should be a homotopy equivalence. Very recently a proof of their conjecture, using h-principle arguments combined with Harer stabilisation, has been announced by Madsen and Weiss [81, 103], and from this Mumford's conjecture follows.

The tautological ring of $\mathcal{M}_g$ for finite g has many beautiful properties. Faber [26] conjectured that when $g \geq 2$ the tautological ring of $\mathcal{M}_g$ looks like the algebraic cohomology ring of a nonsingular complex projective variety of dimension $g-2$, and that it is generated by the tautological classes $\kappa_1, \kappa_2, \ldots, \kappa_{[g/3]}$ with no relations in degrees at most $[g/3]$. He also provided an explicit conjecture for a complete set of relations among these generators. Progress has been made by many contributors towards Faber's conjectures, and also related problems on moduli spaces linked to $\mathcal{M}_g$. In particular Morita [90, 91] has recently proved that the rational cohomological version of the tautological ring of $\mathcal{M}_g$ is indeed generated by $\kappa_1, \kappa_2, \ldots, \kappa_{[g/3]}$. The definition of the tautological ring has also been extended to the compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space $\mathcal{M}_{g,n}$ of nonsingular curves of genus g with n marked points (motivated by Witten's conjectures [107], proved by Kontsevich [71], on intersection pairings on $\overline{\mathcal{M}}_{g,n}$).

The moduli spaces $\mathcal{M}_g$ and $\mathcal{M}_{g,n}$ have other younger and more sophisticated relatives, such as the moduli spaces $\mathcal{M}_{g,n}(X,\beta)$ which parametrise holomomorphic maps $f : \Sigma \to X$ from a nonsingular complex projective curve Σ of genus g with n marked points satisfying $f_*[\Sigma] = \beta \in H_2(X)$, and their compactifications $\overline{\mathcal{M}}_{g,n}(X,\beta)$ which parametrise 'stable' maps. Intersection theory on $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is fundamental to Gromov-Witten theory and quantum cohomology for X, with numerous applications in the last decade to enumerative geometry. The Virasoro conjecture of Eguchi, Hori and Xiong provides relations among the descendent Gromov-Witten invariants of X, and its recent proof by Givental [39] for $X = \mathbb{P}^n$ implies part of Faber's conjecture by [37].

Other relatives of $\mathcal{M}_g$ include the moduli spaces of pairs (Σ, E) where Σ is a nonsingular curve and E is a stable vector bundle over Σ, and their compactifications; intersection theory on these relates intersection theory on $\overline{\mathcal{M}}_g$ and intersection theory on moduli spaces of bundles over a fixed curve, which is by now quite well understood.

1. Moduli spaces of curves

The study of algebraic curves, and how they vary in families, has been fundamental to algebraic geometry since the beginning of the subject, and has made huge advances in the last few decades [3, 52]. The concept of moduli as parameters describing as efficiently as possible the variation of geometric objects was initiated in Riemann's famous paper [99] of 1857, in which he observed that an isomorphism class of compact Riemann surfaces of genus $g \geq 2$ 'hängt ... von $3g-3$ stetig veränderlichen Grössen ab, welche die Moduln dieser Klasse genannt werden sollen'. In modern terminology, Riemann's observation is the statement that the dimension of $\mathcal{M}_g$ is $3g-3$ if $g \geq 2$. It was not until the 1960s that precise definitions and methods of constructing moduli spaces were given by Mumford in [92] following ideas of Grothendieck. Roughly speaking, the moduli space $\mathcal{M}_g$ is the set of isomorphism classes of nonsingular complex projective curves[1] of genus g, endowed with the structure of a complex variety in such a way that any family of nonsingular complex projective curves parametrised by a base space S induces a morphism from S to $\mathcal{M}_g$ which associates to each $s \in S$ the isomorphism class of the curve parametrised by s. The moduli spaces $\mathcal{M}_g$ can be constructed in several different ways, including

- as orbit spaces for group actions,
- via period maps and Torelli's theorem, and
- using Teichmüller theory.

The first of these is a standard method for constructing many different moduli spaces, using Mumford's geometric invariant theory [92, 95, 105] or more recent ideas due to Kollár [70] and to Mori and Keel [64]. Geometric invariant theory provides a beautiful compactification of $\mathcal{M}_g$ known as the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$ [15]. This compactification is itself modular: it is the moduli space of (Deligne-Mumford) stable curves (i.e. complex projective curves with only nodal singularities and finitely many automorphisms). $\overline{\mathcal{M}}_g$ is singular but in a relatively mild way; it is the quotient of a nonsingular variety by a finite group action [77].

The moduli space $\mathcal{M}_{g,n}$ of nonsingular complex projective curves of genus g with n marked points has a similar compactification $\overline{\mathcal{M}}_{g,n}$ which is the moduli space of complex projective curves with n marked nonsingular points and with only nodal singularities and finitely many automorphisms. Finiteness of the automorphism group of such a curve Σ is equivalent to the requirement that any irreducible component of genus 0 (respectively 1) has at least 3 (respectively 1) special points, where 'special' means either marked or singular in Σ (and the condition on genus 1 components here is redundant when $g \geq 2$).

The second method of construction using the period matrices of curves leads to a different compactification $\tilde{\mathcal{M}}_g$ of $\mathcal{M}_g$ known as the Satake (or Satake-Baily-Borel) compactification. Like the Deligne-Mumford compactification, $\tilde{\mathcal{M}}_g$ is a complex projective variety, but the boundary $\tilde{\mathcal{M}}_g \setminus \mathcal{M}_g$ of $\mathcal{M}_g$ in $\tilde{\mathcal{M}}_g$ has (complex) codimension 2 for $g \geq 3$ whereas the boundary $\Delta = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ of $\mathcal{M}_g$ in $\overline{\mathcal{M}}_g$ has codimension 1. Each of the irreducible components $\Delta_0, \ldots, \Delta_{[g/2]}$ of Δ is the closure of a locus of curves with exactly one node (irreducible curves with one node

[1] All complex curves and real surfaces will be assumed to be connected.

in the case of Δ_0, and in the case of any other Δ_i the union of two nonsingular curves of genus i and $g-i$ meeting at a single point). The divisors Δ_i meet transversely in $\overline{\mathcal{M}}_g$, and their intersections define a natural decomposition of Δ into connected strata which parametrise stable curves of a fixed topological type. The boundary of $\mathcal{M}_{g,n}$ in $\overline{\mathcal{M}}_{g,n}$ has a similar description, but now as well as the genus of each irreducible component it is necessary to keep track of which marked points it contains.

The third method of constructing $\mathcal{M}_g$, via Teichmüller theory, leaves algebraic geometry altogether.

2. Teichmüller theory and mapping class groups

Important recent advances concerning the cohomology of the moduli spaces $\mathcal{M}_g$ (in particular [80, 81, 90, 91, 103]) have been proved by topologists via the link between these moduli spaces and mapping class groups of compact surfaces.

Let us fix a compact oriented smooth surface Σ_g of genus $g \geq 2$, and let $\mathrm{Diff}_+\Sigma_g$ be the group of orientation preserving diffeomorphisms of Σ_g. Then the mapping class group Γ_g of Σ_g is the group

$$\Gamma_g = \pi_0(\mathrm{Diff}_+\Sigma_g)$$

of connected components of $\mathrm{Diff}_+\Sigma_g$. It acts properly and discontinuously on the Teichmüller space $\mathcal{T}_g$ of Σ_g, which is the space of conformal structures on Σ_g up to isotopy. The Teichmüller space $\mathcal{T}_g$ is homeomorphic to $\mathbb{R}^{6g-6}$, and its quotient by the action of the mapping class group Γ_g can be identified naturally with the moduli space $\mathcal{M}_g$. This means that there is a natural isomorphism of rational cohomology

$$H^*(\mathcal{M}_g;\mathbb{Q}) \cong H^*(\Gamma_g;\mathbb{Q}). \tag{2.1}$$

The corresponding integral cohomology groups are not in general isomorphic because of the existence of nonsingular complex projective curves with nontrivial automorphisms. If, however, we work with the moduli spaces $\mathcal{M}_{g,n}$ of nonsingular complex projective curves of genus g with n marked points, then when n is large enough such marked curves have no nontrivial automorphisms (cf. [52] p 37) and

$$H^*(\mathcal{M}_{g,n};\mathbb{Z}) \cong H^*(\Gamma_{g,n};\mathbb{Z})$$

where $\Gamma_{g,n}$ is the group of connected components of the group $\mathrm{Diff}_+\Sigma_{g,n}$ of orientation preserving diffeomorphisms of Σ_g which fix n chosen points on Σ_g.

In fact [20] the components of $\mathrm{Diff}_+\Sigma_g$ are contractible when $g \geq 2$, so there is also a natural isomorphism

$$H^*(\Gamma_g;\mathbb{Z}) \cong H^*(B\mathrm{Diff}_+\Sigma_g;\mathbb{Z})$$

where $B\mathrm{Diff}_+\Sigma_g$ is the universal classifying space for $\mathrm{Diff}_+\Sigma_g$. This means that any cohomology class of the mapping class group Γ_g can be regarded as a characteristic class of oriented surface bundles, while any rational cohomology class of Γ_g can be regarded as a rational cohomology class of the moduli space $\mathcal{M}_g$.

The mapping class group Γ_g can be described in a group theoretical way. Γ_g acts faithfully by outer automorphisms (that is, the action is defined modulo inner automorphisms) on the fundamental group $\pi_1(\Sigma_g)$ of Σ_g, which is generated by $2g$ elements $a_1, \ldots, a_g, b_1, \ldots, b_g$ subject to one relation $\prod_j a_j b_j a_j^{-1} b_j^{-1} = 1$, and the image of Γ_g in $\mathrm{Out}(\pi_1(\Sigma_g))$ is the group of outer automorphisms of $\pi_1(\Sigma_g)$ which act trivially on $H_2(\pi_1(\Sigma_g); \mathbb{Z})$ [110]. Γ_g has a finite presentation [106] with generators represented by Dehn twists (diffeomorphisms of Σ_g obtained by cutting Σ_g along a regularly embedded circle, twisting one of the resulting boundary circles through 2π and reglueing). There are similar descriptions of $\Gamma_{g,n}$ [110, 33].

3. Stable cohomology

Harer [47] proved in the 1980s that $H^k(\Gamma_g; \mathbb{Z})$ and $H^k(\Gamma_{g+1}; \mathbb{Z})$ are isomorphic when $g \geq 3k-1$, and the same is true for $\Gamma_{g,n}$ and $\Gamma_{g+1,n}$. This bound was improved by Ivanov [55] and Harer [50] made a further improvement. Since $H^*(\Gamma_g; \mathbb{Q})$ is isomorphic to $H^*(\mathcal{M}_g; \mathbb{Q})$, this means that the rational cohomology group $H^k(\mathcal{M}_g; \mathbb{Q})$ is independent of g for $g \gg k$, and we can define the stable cohomology ring

$$H^*(\mathcal{M}_\infty; \mathbb{Q})$$

so that $H^k(\mathcal{M}_\infty; \mathbb{Q}) \cong H^k(\mathcal{M}_g; \mathbb{Q})$ for $g \gg k$.

Harer's stabilisation map can be defined as follows. We choose a smooth identification of Σ_{g+1} with a connected sum of a smooth surface Σ_g of genus g and a surface Σ_1 of genus 1 (and if we have marked points we make sure they all correspond to points in Σ_g). Let Γ_{g+1,Σ_1} be the subgroup of Γ_{g+1} consisting of mapping classes represented by diffeomorphisms from Σ_{g+1} to itself which fix all the points coming from Σ_1. The result of collapsing all such points in Σ_{g+1} together is diffeomorphic to Σ_g, so there is a homomorphism from Γ_{g+1,Σ_1} to Γ_g as well as an inclusion of Γ_{g+1,Σ_1} in Γ_{g+1}. Harer showed that both of these induce isomorphisms

$$H^k(\Gamma_g; \mathbb{Z}) \cong H^k(\Gamma_{g+1,\Sigma_1}; \mathbb{Z}) \text{ and } H^k(\Gamma_{g+1}; \mathbb{Z}) \cong H^k(\Gamma_{g+1,\Sigma_1}; \mathbb{Z})$$

when $g \gg k$, and likewise we have

$$H^k(\Gamma_{g,n}; \mathbb{Z}) \cong H^k(\Gamma_{g+1,\Sigma_1,n}; \mathbb{Z}) \text{ and } H^k(\Gamma_{g+1,n}; \mathbb{Z}) \cong H^k(\Gamma_{g+1,\Sigma_1,n}; \mathbb{Z}) \qquad (3.1)$$

when $g \gg k$.

A similar construction can be made to describe the stabilisation isomorphism

$$H^k(\mathcal{M}_{g,n}; \mathbb{Q}) \cong H^k(\mathcal{M}_{g+1,n}; \mathbb{Q})$$

for the moduli spaces $\mathcal{M}_{g,n}$ (cf. [28, p.31]). Identifying the last marked point of a smooth nonsingular complex projective curve of genus g with a marked point on a curve of genus 1 gives a stable curve of genus $g+1$ with n marked points. This defines for us a morphism

$$\phi : \mathcal{M}_{g,n+1} \times \mathcal{M}_{1,1} \to \overline{\mathcal{M}}_{g+1,n}$$

whose image is an open subset of an irreducible component of the boundary of $\mathcal{M}_{g+1,n}$ in $\overline{\mathcal{M}}_{g+1,n}$, and there is a normal bundle $\mathcal{N}_\phi$ which is a complex line bundle (in the sense of orbifolds) over $\mathcal{M}_{g,n+1} \times \mathcal{M}_{1,1}$. Using $\mathcal{N}_\phi^*$ to denote the complement of the zero section of $\mathcal{N}_\phi$ we can compose projection maps with the forgetful map from $\mathcal{M}_{g,n+1}$ to $\mathcal{M}_{g,n}$ to get

$$\mathcal{N}_\phi^* \to \mathcal{M}_{g,n+1} \times \mathcal{M}_{1,1} \to \mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n}$$

which induces

$$H^k(\mathcal{M}_{g,n}; \mathbb{Q}) \to H^k(\mathcal{N}_\phi^*; \mathbb{Q}). \tag{3.2}$$

On the other hand, using a tubular neighbourhood of the image of ϕ in $\overline{\mathcal{M}}_{g+1,n}$ we obtain a natural homotopy class of maps from $\mathcal{N}_\phi^*$ to $\mathcal{M}_{g+1,n}$ which induces

$$H^k(\mathcal{M}_{g+1,n}; \mathbb{Q}) \to H^k(\mathcal{N}_\phi^*; \mathbb{Q}). \tag{3.3}$$

Here (3.2) and (3.3) represent Harer's maps (3.1) and hence they are isomorphisms if $g \gg k$.

4. Tautological classes

When $g \geq 2$ Mumford [93] and Morita [84] independently defined tautological classes

$$\kappa_i \in H^{2i}(\overline{\mathcal{M}}_g; \mathbb{Q}) \text{ and } e_i \in H^{2i}(\Gamma_g; \mathbb{Z})$$

which correspond up to a sign $(-1)^{i+1}$ in $H^*(\mathcal{M}_g; \mathbb{Q})$ under the isomorphism (2.1). The subalgebra $R^*(\mathcal{M}_g)$ of $H^*(\mathcal{M}_g; \mathbb{Q})$ generated by the κ_i, or equivalently by the e_i, is called its tautological ring.

The classes κ_i are defined using the natural forgetful map $\pi : \mathcal{M}_{g,1} \to \mathcal{M}_g$ which takes an element $[\Sigma, p]$ of $\mathcal{M}_{g,1}$ represented by a nonsingular complex projective curve Σ with one marked point p to the element $[\Sigma]$ of $\mathcal{M}_g$ represented by Σ. This is often called the universal curve over $\mathcal{M}_g$, since for generic choices of Σ the fibre $\pi^{-1}([\Sigma])$ is a copy of Σ. However if Σ has nontrivial automorphisms then $\pi^{-1}([\Sigma])$ is not a copy of Σ but is instead the quotient of Σ by its automorphism group $\mathrm{Aut}(\Sigma)$ (which has size at most $84(g-1)$ when $g \geq 2$).

From the topologists' viewpoint the rôle of $\pi : \mathcal{M}_{g,1} \to \mathcal{M}_g$ is played by the universal oriented Σ_g-bundle

$$\Pi : E\mathrm{Diff}_+\Sigma_g \to B\mathrm{Diff}_+\Sigma_g.$$

Its relative tangent bundle is an oriented real vector bundle of rank 2 on $E\mathrm{Diff}_+\Sigma_g$ (whose fibre at $x \in E\mathrm{Diff}_+\Sigma_g$ is the tangent space at x to the oriented surface $\Pi^{-1}(x)$), so it has an Euler class $e \in H^2(E\mathrm{Diff}_+\Sigma_g; \mathbb{Z})$. Morita defined his tautological classes

$$e_i \in H^{2i}(\Gamma_g; \mathbb{Z}) \cong H^{2i}(B\mathrm{Diff}_+\Sigma_g; \mathbb{Z})$$

by setting e_i to be the pushforward (or integral over the fibres) $\Pi_!(e^{i+1})$ of e^{i+1}.

To define his tautological classes κ_i Mumford used essentially the same procedure with the forgetful map $\pi : \mathcal{M}_{g,1} \to \mathcal{M}_g$, except that he used cotangent spaces instead of tangent spaces (which is the reason that κ_i and e_i only correspond up to a sign $(-1)^{i+1}$) and the relative cotangent bundle (or relative dualising sheaf) for $\pi : \mathcal{M}_{g,1} \to \mathcal{M}_g$ exists as a complex line bundle over $\mathcal{M}_{g,1}$ only in the sense of orbifold line bundles (or line bundles over stacks) because of the existence of nontrivial automorphism groups $\mathrm{Aut}(\Sigma)$.

The forgetful map $\pi : \mathcal{M}_{g,1} \to \mathcal{M}_g$ can be generalised to $\pi : \mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n}$ for any $n \geq 0$ by forgetting the last marked point of an $n+1$-pointed curve, and this can be extended to $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$. Care is needed here when the last marked point lies on an irreducible component with genus 0 and only two other special points; such an irreducible component needs to be collapsed in order to produce a stable n-pointed curve of genus g. This collapsing procedure gives us a forgetful map $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ whose fibre at $[\Sigma, p_1, \dots, p_n] \in \overline{\mathcal{M}}_{g,n}$ can be identified with the quotient of Σ by the automorphism group of $(\Sigma, p_1, ..., p_n)$. Mumford's tautological classes can be extended to classes $\kappa_i \in H^{2i}(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$ (in fact to classes in the rational Chow ring of $\overline{\mathcal{M}}_{g,n}$) defined by

$$\kappa_i = \pi_!(c_1(\omega_{g,n})^{i+1})$$

where $\omega_{g,n}$ is the relative dualising sheaf of $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ and $c_1(\omega_{g,n}) \in H^2(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$ is its first Chern class.

When $n > 0$ there are other interesting tautological classes on $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ exploited by Witten. The forgetful map $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ has tautological sections $s_j : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n+1}$ for $1 \leq j \leq n$ such that $s_j([\Sigma, p_1, \dots, p_n])$ is the element of $\pi^{-1}([\Sigma, p_1, \dots, p_n]) = \Sigma/\mathrm{Aut}(\Sigma)$ represented by p_j. The Witten classes $\psi_j \in H^2(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$ for $j = 1, \dots, n$ can then be defined by

$$\psi_j = c_1(s_j^*(\omega_{g,n})).$$

Roughly speaking, ψ_j is the first Chern class of the (orbifold) line bundle on $\overline{\mathcal{M}}_{g,n}$ whose fibre at $[\Sigma, p_1, \dots, p_n]$ is the cotangent space $T^*_{p_j}\Sigma$ to Σ at p_j.

The boundary $\Delta = \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ of $\mathcal{M}_{g,n}$ in $\overline{\mathcal{M}}_{g,n}$ is the union of finitely many divisors which meet transversely in $\overline{\mathcal{M}}_{g,n}$. The intersection of any nonempty set of these divisors is the closure of a subset of $\mathcal{M}_{g,n}$ parametrising stable n-pointed curves of some fixed topological type, and is the image of a finite-to-one map to $\overline{\mathcal{M}}_{g,n}$ from a product of moduli spaces $\prod_k \overline{\mathcal{M}}_{g_k,n_k}$ which glues together stable curves of genus g_k with n_k marked points at certain of the marked points. These glueing maps induce pushforward maps on cohomology

$$H^*(\prod_k \overline{\mathcal{M}}_{g_k,n_k}; \mathbb{Q}) \to H^*(\overline{\mathcal{M}}_{g,n}; \mathbb{Q}) \tag{4.1}$$

and the tautological ring $R^*(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$ is defined inductively to be the subalgebra of $H^*(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$ generated by the Mumford classes, the Witten classes and the images of the tautological classes in $H^*(\prod_k \overline{\mathcal{M}}_{g_k,n_k}; \mathbb{Q})$ under the pushforward maps (4.1) from the boundary of $\mathcal{M}_{g,n}$. Its restriction to $H^*(\mathcal{M}_{g,n}; \mathbb{Q})$ is the tautological ring of $\mathcal{M}_{g,n}$ and is generated by the Mumford and Witten classes.

5. Mumford's conjecture

Mumford's tautological classes $\kappa_i \in H^{2i}(\mathcal{M}_g;\mathbb{Q})$ are preserved by Harer stabilisation when g is sufficiently large, and so they define elements of the stable cohomology $H^*(\mathcal{M}_\infty;\mathbb{Q})$. Mumford conjectured in [93] that $H^*(\mathcal{M}_\infty;\mathbb{Q})$ is freely generated by $\kappa_1,\kappa_2,\ldots$, or in other words that the obvious map

$$\mathbb{Q}[\kappa_1,\kappa_2,\ldots] \to H^*(\mathcal{M}_\infty;\mathbb{Q}) \tag{5.1}$$

is an isomorphism. Miller [83] and Morita [85] soon proved that this map is injective, so it remained to prove surjectivity. Not long ago Madsen and Tillmann [80] found a homotopy version of Mumford's map (5.1) which they conjectured to be a homotopy equivalence, and very recently Madsen and Weiss [81] have announced a proof of their conjecture, from which Mumford's conjecture follows.

The Madsen-Tillmann map involves the stable mapping class group Γ_∞ rather than the moduli spaces $\mathcal{M}_g$. From the description of $\mathcal{M}_g$ as the quotient of the Γ_g action on Teichmüller space $\mathcal{T}_g$ it follows that when $g \geq 2$ there is a continuous map

$$B\Gamma_g \to \mathcal{M}_g \tag{5.2}$$

uniquely determined up to homotopy. It is known that Γ_g is a perfect group when $g \geq 3$ [46], so we can apply Quillen's plus construction to $B\Gamma_g$ to obtain a simply connected space $B\Gamma_g^+$ with the same homology as $B\Gamma_g$. The moduli space $\mathcal{M}_g$ is also simply connected, so (5.2) factors through a map $B\Gamma_g^+ \to \mathcal{M}_g$ which induces the isomorphism $H^*(\Gamma_g;\mathbb{Q}) \to H^*(\mathcal{M}_g;\mathbb{Q})$ discussed above at (2.1). Moreover Harer stabilisation gives us maps $B\Gamma_g^+ \to B\Gamma_{g+1}^+$ between simply connected spaces which are homology equivalences (and hence also homotopy equivalences) in a range up to some degree which tends to infinity with g. If $B\Gamma_\infty^+$ denotes the homotopy direct limit of these maps as $g \to \infty$, then Mumford's conjecture becomes the statement that

$$H^*(B\Gamma_\infty^+;\mathbb{Q}) \cong \mathbb{Q}[\kappa_1,\kappa_2,\ldots].$$

The conjecture of Madsen and Tillmann [80] describes the homotopy type of $B\Gamma_\infty^+$ (or rather $\mathbb{Z}\times B\Gamma_\infty^+$), giving Mumford's conjecture as a corollary.

Tillmann [103] had already shown that $\mathbb{Z}\times B\Gamma_\infty^+$ is an infinite loop space, in the sense that there exists a sequence of spaces E_n with $E_n = \Omega E_{n+1}$ and $\mathbb{Z}\times B\Gamma_\infty^+ = E_0$. This was an encouraging result because infinite loop spaces have many good properties. Subsequently Madsen and Tillmann [80] found an Ω^∞ map α_∞ from $\mathbb{Z}\times B\Gamma_\infty^+$ to an infinite loop space which they denoted by $\Omega^\infty\mathbb{CP}^\infty_{-1}$ and whose connected component has rational cohomology isomorphic to $\mathbb{Q}[\kappa_1,\kappa_2,\ldots]$.

The infinite loop space $\Omega^\infty\mathbb{CP}^\infty_{-1}$ is related to the limit $\mathbb{CP}^\infty$ of the complex projective spaces $\mathbb{CP}^k$ as $k\to\infty$. Over $\mathbb{CP}^k$ there is a tautological complex line bundle L_k, whose fibre at $x \in \mathbb{CP}^k$ is the one-dimensional subspace of $\mathbb{C}^{k+1}$ represented by x, and a complex vector bundle $L_k^\perp$ of rank k which is its complement in the trivial bundle of rank $k+1$ over $\mathbb{CP}^k$. The restriction of $L_{k+1}^\perp$ to $\mathbb{CP}^k$ is the direct sum of $L_k^\perp$ and a trivial complex line bundle, giving us maps $\mathrm{Th}(L_k^\perp) \to \Omega^2\mathrm{Th}(L_{k+1}^\perp)$ and $\Omega^{2k+2}\mathrm{Th}(L_k^\perp) \to \Omega^{2k+4}\mathrm{Th}(L_{k+1}^\perp)$ where $\mathrm{Th}(L_k^\perp)$ is the Thom space (or one-point compactification) of the bundle $L_k^\perp$. Madsen and Tillmann define $\Omega^\infty\mathbb{CP}^\infty_{-1}$ to be the direct limit of the spaces $\Omega^{2k+2}\mathrm{Th}(L_k^\perp)$ as $k\to\infty$.

Homotopy classes of maps from an n-dimensional manifold X to $\Omega^\infty\mathbb{CP}^\infty_{-1}$ are represented by proper maps $\phi: M\to X$ from an $(n+2)$-dimensional manifold M together with an 'artificial differential' $\Phi: TM\to\phi^*TX$ and an orientation of $\ker\Phi$. Here Φ is a stable vector bundle surjection; that is, it may be that Φ is defined and becomes a surjective bundle map only once a trivial bundle of sufficiently large rank has been added to TM and ϕ^*TX. Any smooth oriented surface bundle $\phi: E\to X$ induces a homotopy class of maps from X to $\Omega^\infty\mathbb{CP}^\infty_{-1}$ represented by ϕ together with its differential $\Phi = d\phi: TE\to\phi^*TX$, and this effectively defines the Madsen-Tillmann map $\alpha_\infty:\mathbb{Z}\times B\Gamma^+_\infty\to\Omega^\infty\mathbb{CP}^\infty_{-1}$.

Submersion theory suggests a way to tackle the problem of showing that α_∞ is a homotopy equivalence, but compactness of X creates a difficulty for this. Therefore Madsen and Weiss replace X with $X\times\mathbb{R}$. They study a commutative diagram

$$\begin{array}{ccccc}
\mathcal{V} & \to & \mathcal{W} & \to & \mathcal{W}_{\mathrm{loc}}\\
\downarrow & & \downarrow & & \downarrow\\
h\mathcal{V} & \to & h\mathcal{W} & \to & h\mathcal{W}_{\mathrm{loc}}
\end{array}$$

of contravariant functors from smooth manifolds to sets with the sheaf property for open coverings, and the induced diagram

$$\begin{array}{ccccc}
|\mathcal{V}| & \to & |\mathcal{W}| & \to & |\mathcal{W}_{\mathrm{loc}}|\\
\downarrow & & \downarrow & & \downarrow\\
|h\mathcal{V}| & \to & |h\mathcal{W}| & \to & |h\mathcal{W}_{\mathrm{loc}}|
\end{array}$$

of the associated spaces, where homotopy classes of maps from X to $|\mathcal{F}|$ correspond naturally to concordance classes in $\mathcal{F}(X)$, and $s_0, s_1\in\mathcal{F}(X)$ are concordant if $s_0 = t|_{X\times\{0\}}$ and $s_1 = t|_{X\times\{1\}}$ for some $t\in\mathcal{F}(X\times\mathbb{R})$.

If X is any smooth manifold then elements of $\mathcal{V}(X)$ are given by smooth oriented surface bundles E (that is, proper submersions whose fibres are connected oriented surfaces) over $X\times\mathbb{R}$, together with identifications $\partial E\cong\partial(S^1\times[0,1]\times X\times\mathbb{R})$ compatible with the maps to $X\times\mathbb{R}$. These identifications on the boundary are crucial, because they give $\mathcal{V}$ and the other functors involved the structure of monoids, and thus the associated spaces become topological monoids.

In one version of the bottom row $h\mathcal{V}\to h\mathcal{W}\to h\mathcal{W}_{\mathrm{loc}}$ of the commutative diagram, elements of $h\mathcal{V}(X)$ are given by $(n+3)$-dimensional manifolds E, where $n = \dim X$, and smooth maps $\pi: E\to X$ and $f, g: E\to\mathbb{R}$ such that $(\pi, f): E\to X\times\mathbb{R}$ is a submersion and $(\pi, g): E\to X\times\mathbb{R}$ is proper, together with an identification $\partial E\cong\partial(S^1\times[0,1]\times X\times\mathbb{R})$ compatible with the maps to X and $\mathbb{R}$. If $(\pi, f): E\to X\times\mathbb{R}$ represents an element of $\mathcal{V}(X)$ then we get an element of $h\mathcal{V}(X)$ by setting $g = f$. The functors $\mathcal{W}$ and $h\mathcal{W}$ are defined similarly, except that the requirement that $(\pi, f): E\to X\times\mathbb{R}$ should be a submersion is weakened to the requirements that $\pi: E\to X$ should be a submersion and that the restriction of $f: E\to\mathbb{R}$ to any fibre of π should be a Morse function. For $\mathcal{W}_{\mathrm{loc}}$ and $h\mathcal{W}_{\mathrm{loc}}$ the requirements are weakened again, so that 'proper' is replaced by 'proper when restricted to the set of singularities of f on fibres of π'.

The strategy of Madsen and Weiss is to deduce that α_∞ is a homotopy equivalence from the following properties of the commutative diagram above:

(i) the first vertical map represents the Madsen-Tillmann map α_∞;
(ii) the second vertical map is a homotopy equivalence (by a corollary to Vassiliev's h-principle [104]);
(iii) the third vertical map is also a homotopy equivalence (by a much easier argument);
(iv) the bottom row is a homotopy fibre sequence;
(v) the top row becomes a homotopy fibre sequence after group completion (using stratifications of $|\mathcal{W}|$ and $|\mathcal{W}_{\mathrm{loc}}|$ and a subtle application of Harer stabilisation).

6. Faber's conjectures

Although Mumford's conjecture tells us that the tautological classes κ_i generate the stable cohomology ring $H^*(\mathcal{M}_\infty;\mathbb{Q})$, they do not generate $H^*(\mathcal{M}_g;\mathbb{Q})$ for finite g, and in fact $H^*(\mathcal{M}_g;\mathbb{Q})$ has lots of unstable cohomology (at least when g is large enough). This follows from the calculation of Euler characteristics by Harer and Zagier [51] (see also [71]). They show that the orbifold Euler characteristic of $\mathcal{M}_{g,n}$ is

$$(-1)^{n-1}\frac{(2g+n-3)!}{(2g-2)!}\zeta(1-2g)$$

where ζ denotes the Riemann ζ-function, and their work implies that when $g \geq 15$ the Euler characteristic of $\mathcal{M}_g$ is too large in absolute value for $H^*(\mathcal{M}_g;\mathbb{Q})$ to be generated by $\kappa_1, \kappa_2, \dots$ (cf. also [41, 76]). Nonetheless the tautological ring $R^*(\mathcal{M}_g)$ generated by $\kappa_1, \kappa_2, \dots$ has many beautiful properties.

Faber [26] has conjectured that $R^*(\mathcal{M}_g)$ has the structure of the algebraic cohomology ring of a nonsingular complex projective variety of dimension $g-2$. More precisely, he conjectured that

(i) $R^k(\mathcal{M}_g)$ is zero when $k > g-2$ and is one-dimensional when $k = g-2$, and the natural pairing $R^k(\mathcal{M}_g) \times R^{g-2-k}(\mathcal{M}_g) \to R^{g-2}(\mathcal{M}_g)$ is perfect. In addition $R^k(\mathcal{M}_g)$ satisfies the Hard Lefschetz property and the Hodge index theorem with respect to the class κ_1.
(ii) The classes $\kappa_1, \dots, \kappa_{[g/3]}$ generate $R^*(\mathcal{M}_g)$ with no relations in degrees up to and including $[g/3]$.
(iii) Faber also gave an explicit conjecture for a complete set of relations between these generators (in terms of the proportionalities between monomials in $R^{g-2}(\mathcal{M}_g)$).

When $g \leq 15$ Faber [26] has proved all these conjectures concerning $R^*(\mathcal{M}_g)$, and for general g Looijenga [78] and Faber [27] have shown that $R^k(\mathcal{M}_g)$ is zero when $k > g-2$ and is one-dimensional when $k = g-2$. Their proofs apply to both the cohomological version and the Chow ring version of $R^*(\mathcal{M}_g)$. Using topological methods, Morita [90, 91] has recently proved that the classes $\kappa_1, \dots, \kappa_{[g/3]}$ generate the cohomological version of $R^*(\mathcal{M}_g)$ (and the rest of (ii) then follows essentially from [50]).

The mapping class group Γ_g acts naturally on $H_1(\Sigma_g;\mathbb{Z})$ in a way which preserves the intersection pairing. This representation gives us an exact sequence of groups

$$1 \to \mathcal{I}_g \to \Gamma_g \to Sp(2g;\mathbb{Z}) \to 1$$

where $\mathcal{I}_g$ denotes the subgroup of Γ_g which acts trivially on $H_1(\Sigma_g;\mathbb{Z})$ and which is called the Torelli group. In [58, 59, 60, 61, 62] Johnson showed that $\mathcal{I}_g$ is finitely generated for $g \geq 3$ (in contrast with the case $g = 2$ [82]), introduced a surjective homomorphism

$$\tau : \mathcal{I}_g \to \wedge^3 H_1(\Sigma_g;\mathbb{Z})/H_1(\Sigma_g;\mathbb{Z})$$

whose kernel is the subgroup of Γ_g generated by all Dehn twists along separating embedded circles, and used τ to determine the abelianisation of $\mathcal{I}_g$. Morita [88] extended the Johnson homomorphism τ to a representation

$$\rho_1 : \Gamma_g \to (\tfrac{1}{2}\wedge^3 H_1(\Sigma_g;\mathbb{Z})/H_1(\Sigma_g;\mathbb{Z})) \rtimes Sp(2g;\mathbb{Z})$$

of the mapping class group Γ_g. Via the cohomology of semi-direct products this induces

$$\rho_1^* : \mathrm{Hom}(\wedge^* U, \mathbb{Q})^{Sp(2g;\mathbb{Z})} \to H^*(\Gamma_g;\mathbb{Q}) \cong H^*(\mathcal{M}_g;\mathbb{Q})$$

where $U = \wedge^3 H_1(\Sigma_g;\mathbb{Q})/H_1(\Sigma_g;\mathbb{Q})$, and the image of ρ_1^* is the tautological ring $R^*(\mathcal{M}_g)$ [63, 79]. By finding suitable relations in $\mathrm{Hom}(\wedge^* U,\mathbb{Q})^{Sp(2g;\mathbb{Z})}$ and exploiting the map $H_1(\Sigma_g;\mathbb{Q}) \to H_1(\Sigma_{g-1};\mathbb{Q})$ induced by collapsing a handle of Σ_g, Morita [90, 91] is able to prove that the classes $\kappa_1,\ldots,\kappa_{[g/3]}$ generate the cohomological version of $R^*(\mathcal{M}_g)$.

Faber, Getzler, Hain, Looijenga, Pandharipande, Vakil and others (cf. [23, 24, 25, 31, 42, 44, 78]) have also made conjectures about the structure of the tautological rings of the compact moduli spaces $\overline{\mathcal{M}}_{g,n}$, which are generated not just by the Mumford classes κ_i but also by the Witten classes ψ_j and the pushforwards of tautological classes from the boundary of $\overline{\mathcal{M}}_{g,n}$. For example, it is expected that $R^*(\overline{\mathcal{M}}_{g,n})$ looks like the algebraic cohomology ring of a nonsingular complex projective variety of dimension $3g-3+n$, while Getzler has conjectured that if $g > 0$ then the monomials of degree g or higher in the Witten classes ψ_j should all come from the boundary of $\overline{\mathcal{M}}_{g,n}$ (a cohomological version of this has been proved by Ionel [54]), and Vakil has made a closely related conjecture that any tautological class in $R^k(\overline{\mathcal{M}}_{g,n})$ with $k \geq g$ should come from classes supported on boundary strata corresponding to stable curves with at least $k-g+1$ components of genus 0.

7. The Virasoro conjecture

The geometry of a nonsingular complex projective variety X can be studied by examining curves in X. Intersection theory on moduli spaces of curves in X, or more precisely moduli spaces of maps from curves to X, leads to Gromov-Witten theory and the quantum cohomology of X, with numerous applications in the last decade to enumerative geometry (cf. [14, 32, 71, 72, 73]).

Let us assume for simplicity that $2g-2+n>0$. For any $\beta \in H_2(X;\mathbb{Z})$ there is a moduli space $\mathcal{M}_{g,n}(X,\beta)$ of n-pointed nonsingular complex projective curves Σ of genus g equipped with maps $f:\Sigma \to X$ satisfying $f_*[\Sigma]=\beta$. This moduli space has a compactification $\overline{\mathcal{M}}_{g,n}(X,\beta)$ which classifies 'stable maps' of type β from n-pointed curves of genus g into X [32]. Here a map $f:\Sigma \to X$ from an n-pointed complex projective curve Σ satisfying $f_*[\Sigma]=\beta$ is called stable if Σ has only nodal singularities and $f:\Sigma \to X$ has only finitely many automorphisms, or equivalently every irreducible component of Σ of genus 0 (respectively genus 1) which is mapped to a single point in X by f contains at least 3 (respectively 1) special points. The forgetful map from $\mathcal{M}_{g,n}(X,\beta)$ to $\mathcal{M}_{g,n}$ which sends $[\Sigma,p_1,\ldots,p_n,\, f:\Sigma\to X]$ to $[\Sigma,p_1,\ldots,p_n]$ extends to a forgetful map $\pi:\overline{\mathcal{M}}_{g,n}(X,\beta)\to\overline{\mathcal{M}}_{g,n}$ which collapses components of Σ with genus 0 and at most two special points.

Of course, when X is itself a single point, $\mathcal{M}_{g,n}(X,\beta)$ and $\overline{\mathcal{M}}_{g,n}(X,\beta)$ are simply the moduli spaces $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$. In general $\overline{\mathcal{M}}_{g,n}(X,\beta)$ has more serious singularities than $\overline{\mathcal{M}}_{g,n}$ and may indeed have many different irreducible components with different dimensions (cf. [66]). Nonetheless, it is a remarkable fact [7, 8, 75] that $\overline{\mathcal{M}}_{g,n}(X,\beta)$ has a 'virtual fundamental class' $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}$ lying in the expected dimension

$$3g-3+n+(1-g)\dim X+\int_\beta c_1(TX)$$

of $\overline{\mathcal{M}}_{g,n}(X,\beta)$. Gromov-Witten invariants (originally developed mainly in the case $g=0$ when $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is more tractable, but now also studied when $g>0$) are obtained by evaluating cohomology classes on $\overline{\mathcal{M}}_{g,n}(X,\beta)$ against this virtual fundamental class.

The cohomology classes used are of two types. Recall that if $1\le j\le n$ the Witten class $\psi_j\in H^2(\overline{\mathcal{M}}_{g,n};\mathbb{Q})$ is the first Chern class of $s_j^*(\omega_{g,n})$, where s_j is the jth tautological section of the forgetful map from $\overline{\mathcal{M}}_{g,n+1}$ to $\overline{\mathcal{M}}_{g,n}$ and $\omega_{g,n}$ is the relative dualising sheaf of this forgetful map. In a similar way, using the forgetful map from $\overline{\mathcal{M}}_{g,n+1}(X,\beta)$ to $\overline{\mathcal{M}}_{g,n}(X,\beta)$, we can define $\Psi_j\in H^2(\overline{\mathcal{M}}_{g,n}(X,\beta);\mathbb{Q})$ (and Ψ_j is not quite the pullback of ψ_j via the forgetful map $\pi:\overline{\mathcal{M}}_{g,n}(X,\beta)\to\overline{\mathcal{M}}_{g,n}$ because of the collapsing process in the definition of π). We can also pull back cohomology classes on X via the evaluation maps $ev_j:\overline{\mathcal{M}}_{g,n}(X,\beta)\to X$ which send a stable map $f:\Sigma\to X$ to the image $f(p_j)$ of the jth marked point p_j of Σ for $1\le j\le n$.

Gromov-Witten invariants for X are given by integrals

$$\int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} ev_1^*(\alpha_1)\ldots ev_n^*(\alpha_n)$$

of classes of the second type $ev_j^*(\alpha_j)$, where $\alpha_1,\ldots,\alpha_n\in H^*(X;\mathbb{Q})$, against the virtual fundamental class of $\overline{\mathcal{M}}_{g,n}(X,\beta)$, while descendent Gromov-Witten invariants are of the form

$$\int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \Psi_1^{k_1}\ldots\Psi_n^{k_n}\, ev_1^*(\alpha_1)\ldots ev_n^*(\alpha_n)$$

for nonnegative integers $k_1, \ldots, k_n$, not all zero. More generally, instead of integrating against $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}$ to get rational numbers one can consider the image in $H^*(\overline{\mathcal{M}}_{g,n};\mathbb{Q})$ of the product $\Psi_1^{k_1}\ldots\Psi_n^{k_n} ev_1^*(\alpha_1)\ldots ev_n^*(\alpha_n)$ under the virtual push-forward map associated to $\pi : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n}$.

When X is a single point, the descendent Gromov-Witten invariants reduce to the integrals

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1}\ldots\psi_n^{k_n}.$$

Witten [107] conjectured relations between these integrals (later proved by Kontsevich [71] via a combinatorial description of $\overline{\mathcal{M}}_{g,n}$) which enable them to be calculated recursively. Witten's conjecture can be formulated in terms of the formal power series

$$F_g = \sum_{n\geq 0}\frac{1}{n!}\sum_{k_1,\ldots,k_n\geq 0}\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{k_1}\ldots\psi_n^{k_n}t_{k_1}\ldots t_{k_n}$$

in $\mathbb{Q}[[t_0,t_1,\ldots]]$: it says that $\exp(\sum_{g\geq 0}F_g)$ satisfies a system of differential equations called the Virasoro relations.

Witten's conjecture has been generalised by Eguchi, Hori and Xiong (with an extension by Katz) [14, 22, 34, 37] to provide relations between Gromov-Witten invariants and their descendents for general nonsingular projective varieties X. Their generalisation is called the Virasoro conjecture for X, since it says that a certain formal expression (the 'total Gromov-Witten potential') Z^X in the Gromov-Witten invariants and their descendents satisfies a system of differential equations

$$\mathcal{L}_k Z^X = 0 \text{ for } k \geq -1$$

where the differential operators $\mathcal{L}_k$ satisfy the commutation relations $[\mathcal{L}_k,\mathcal{L}_\ell] = (k-\ell)\mathcal{L}_{k+\ell}$ and hence span a Lie subalgebra of the Virasoro algebra isomorphic to the Lie algebra of polynomial vector fields in one variable (with $\mathcal{L}_k$ corresponding to $-x^{k+1}\mathrm{d}/\mathrm{d}x$). Dubrovin and Zhang [18] have proved that the Virasoro conjecture determines the Gromov-Witten invariants of X when X is homogeneous.

Getzler and Pandharipande [37] showed that part of Faber's conjectures on the structure of the tautological ring of $\mathcal{M}_g$ (the proportionality formulas) would follow from the Virasoro conjecture for $X = \mathbb{CP}^2$, and Givental [39] has recently found a proof of the Virasoro conjecture for a class of varieties which includes all complex projective spaces, thus completing the proof of the proportionality formulas.

Other methods for finding relations between Gromov-Witten invariants include the Toda conjecture [35, 36, 96, 97] and exploitation of intersection theory on $\overline{\mathcal{M}}_{g,n}$ and localisation methods [9, 10, 14, 21, 29, 30, 40, 43, 72], which have been very powerful in enumerative geometry.

8. Moduli spaces of bundles over curves

Another very well studied family of moduli spaces is given by the moduli spaces $\mathcal{B}_\Sigma(r,d)$ of stable holomorphic vector bundles E of rank r and degree d over a fixed nonsingular complex projective curve Σ of genus $g \geq 2$. When r and d are coprime $\mathcal{B}_\Sigma(r,d)$ is a nonsingular complex projective variety; when r and d have a common factor then $\mathcal{B}_\Sigma(r,d)$ is nonsingular but not projective, and it has a natural compactification $\overline{\mathcal{B}}_\Sigma(r,d)$ which is projective but singular (except when $g = r = 2$) [38, 95]. If the curve Σ is allowed to vary as well as the bundle E over Σ then we obtain a 'universal' moduli space of bundles $\mathcal{B}_g(r,d)$, which maps to the moduli space $\mathcal{M}_g$ of nonsingular curves of genus g with fibre $\mathcal{B}_\Sigma(r,d)$ over $[\Sigma]$. Pandharipande [98] has shown that $\mathcal{B}_g(r,d)$ has a compactification $\overline{\mathcal{B}}_g(r,d)$ which maps to $\overline{\mathcal{M}}_g$ with the fibre over $[\Sigma] \in \mathcal{M}_g$ given by $\overline{\mathcal{B}}_\Sigma(r,d)$.

In the case when r and d are coprime we have a good understanding of the structure of the cohomology ring $H^*(\mathcal{B}_\Sigma(r,d);\mathbb{Z})$, and this understanding is particularly thorough when $r = 2$ [6, 67, 100, 109]. For arbitrary r it is known that the cohomology has no torsion [4] and inductive formulas [4, 16, 45] as well as explicit formulas [5, 74] for computing the Betti numbers are available. There is a simple set of generators for the cohomology ring [4] and there are explicit formulas for the intersection pairings between polynomial expressions in these generators, which in principle determine all the relations by Poincaré duality [17, 56, 101]. There is also an elegant description of a complete set of relations among the generators when $r = 2$ [6, 67, 100, 109], partially motivated by a conjecture of Mumford [69], and there is a generalisation when $r > 2$ which is somewhat less elegant [19].

When r and d are not coprime the structure of the cohomology ring $H^*(\mathcal{B}_\Sigma(r,d);\mathbb{Z})$ is a little more difficult to describe; for example, the induced Torelli group action on $H^*(\mathcal{B}_\Sigma(r,d);\mathbb{Q})$ is nontrivial [13], whereas when r and d are coprime the Torelli action is trivial and the mapping class group acts via representations of $Sp(2g;Z)$ which are easy to determine. However even in this case information is available on the intersection cohomology of the compactification $\overline{\mathcal{B}}_\Sigma(r,d)$ of $\mathcal{B}_\Sigma(r,d)$ and the cohomology of another compactification $\tilde{\mathcal{B}}_\Sigma(r,d)$ of $\mathcal{B}_\Sigma(r,d)$ with only orbifold singularities: for example, there are formulas for the Betti numbers in both cases [68] and their intersection pairings [57, 65], and the mapping class group again acts via representations of $Sp(2g;\mathbb{Z})$ [94].

One of the main reasons for our good understanding of the moduli spaces $\mathcal{B}_\Sigma(r,d)$ (and their compactifications $\overline{\mathcal{B}}_\Sigma(r,d)$ and $\tilde{\mathcal{B}}_\Sigma(r,d)$ when r and d have a common factor) is that they can be constructed as quotients, in the sense of geometric invariant theory [92], of well behaved spaces whose properties are relatively easy to understand. Similar techniques could in principle be used to study the moduli spaces of stable curves $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_{g,n}$, as well as Pandharipande's compactification $\overline{\mathcal{B}}_g(r,d)$ of the universal moduli space of bundles $\mathcal{B}_g(r,d)$, since they too can be constructed using geometric invariant theory. In practice this has not succeeded except in very special cases because, in contrast to the case of $\mathcal{B}_\Sigma(r,d)$, we do not have quotients of well behaved spaces which are easy to analyse. However as our understanding of the moduli spaces $\overline{\mathcal{M}}_{g,n}(X,\beta)$ of stable maps becomes increasingly well developed, and in particular localisation techniques are used with greater and greater effect, perhaps the techniques available for studying the cohomology of geometric invariant theoretic quotients will provide an additional approach to the cohomology of the moduli spaces $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_{g,n}$ which can be added to the plethora of methods already available.

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