Interactions Between Ergodic Theory, Lie Groups, and Number Theory

MARINA RATNER

Department of Mathematics
Stanford University
Stanford, CA 94305, USA

and

Department of Mathematics
University of California
Berkeley, CA 94720, USA

Introduction

In this paper we discuss the use of dynamical and ergodic-theoretic ideas and methods to solve some long-standing problems originating from Lie groups and number theory. These problems arise from looking at actions of Lie groups on their homogeneous spaces. Such actions, viewed as dynamical systems, have long been interesting and rich objects of ergodic theory and geometry. Since the 1930s ergodic-theoretic methods have been applied to the study of geodesic and horocycle flows on unit tangent bundles of compact surfaces of negative curvature. From the algebraic point of view the latter flows are examples of semisimple and unipotent actions on finite-volume homogeneous spaces of real Lie groups. It was established in the 1960s through the fundamental work of D. Ornstein that typical semisimple actions are all statistically the same due to their extremal randomness caused by exponential instability of orbits. Their algebraic nature has little to do with the isomorphism problem for such actions: they are measure-theoretically isomorphic as long as their entropies coincide.

In striking contrast, unipotent actions (all having zero entropy), though random and chaotic from a dynamical point of view, were found to be rigidly linked to the algebraic structure of the underlying homogeneous space. In 1981 it was shown by the author that measure theoretic isomorphisms of horocycle flows must be algebraic and imply the isometry of the underlying surfaces. Subsequently, further "rigidities" of an algebraic nature have been found.

While the study of this "rigidity" phenomenon was underway, a powerful impetus came from number theory. Around 1980 Raghunathan made a remarkable observation that the long-standing Oppenheim conjecture on the density of values of irrational quadratic forms at integral points would follow if it were true that closures of orbits of certain unipotent subgroups $\mathbf{U} \subset SL(3, \mathbb{R})$ acting on $SL(3, \mathbb{Z})\backslash SL(3, \mathbb{R})$ were merely orbits of larger groups containing $\mathbf{U}$. The latter result was proved by Margulis in 1986.

Raghunathan's observation led him to propose a general conjecture on orbit closures of unipotent actions. In 1990 it was shown by the author that ergodic-theoretic methods (some of which we developed previously for horocycle flows) can

Proceedings of the International Congress
of Mathematicians, Zürich, Switzerland 1994

be applied to solve this and other related conjectures. This made it possible to answer further number theoretic questions and stimulated subsequent developments in ergodic theory and dynamics of subgroup actions on homogeneous spaces.

This paper consists of six sections. In Sections 1 and 2 we introduce the necessary defintions and state conjectures and results prior to 1990. In Section 3 we state and discuss new results for real Lie groups and in Section 4 we give p-adic and $\mathbb{S}$-arithmetic generalizations of these results. In Section 5 we discuss applications to number theory and in Section 6 applications to ergodic theory and the "rigidity" phenomenon of unipotent actions.

It should be noted that this paper discusses only some of the many important topics that fall under its title.

I thank A. Borel, G. Prasad and D. Witte for their valuable comments on the preliminary version of this paper.

1 Definitions

Let $\mathbf{G}$ be a locally compact second countable topological group, $\mathbf{\Gamma}$ a discrete subgroup of $\mathbf{G}$, and $\mathbf{\Gamma}\backslash\mathbf{G} = \{\mathbf{\Gamma h} : \mathbf{h} \in \mathbf{G}\}$. We shall denote by $\pi : \mathbf{G} \to \mathbf{\Gamma}\backslash\mathbf{G}$ the covering projection $\pi(\mathbf{h}) = \mathbf{\Gamma h}$, $\mathbf{h} \in \mathbf{G}$. The group $\mathbf{G}$ acts on $\mathbf{\Gamma}\backslash\mathbf{G}$ by right translations: $x \to x\mathbf{g}$, $x \in \mathbf{\Gamma}\backslash\mathbf{G}$, $\mathbf{g} \in \mathbf{G}$. We study the dynamics of this action.

Let $\{\mathbf{x}_n\}$ be a sequence in $\mathbf{G}$ and let $\mathbf{e}$ denote the identity element of $\mathbf{G}$. We say that $x_n = \pi(\mathbf{x}_n)$ *cuspidally diverges* in $\mathbf{\Gamma}\backslash\mathbf{G}$ if there are $\mathbf{e} \neq \gamma_n \in \mathbf{\Gamma}$, $n = 1, 2, \ldots$ such that $\mathbf{x}_n^{-1}\gamma_n\mathbf{x}_n \to \mathbf{e}$ as $n \to \infty$. (This means that a left invariant distance between $\mathbf{x}_n$ and $\gamma_n\mathbf{x}_n$ tends to zero as $n \to \infty$ or that the sequence $\{x_n\}$ escapes to the cusps of $\mathbf{\Gamma}\backslash\mathbf{G}$.) For $\mathbf{g} \in \mathbf{G}$ the set

$$\mathcal{D}(\mathbf{g}) = \{x \in \mathbf{\Gamma}\backslash\mathbf{G} : x\mathbf{g}^n \text{ cuspidally diverges as } n \to \infty\}$$

is called the *divergent* set of $\mathbf{g}$. It is clear that if $\mathcal{D}(\mathbf{g}) \neq \emptyset$ for some $\mathbf{g} \in \mathbf{G}$ then $\mathbf{\Gamma}\backslash\mathbf{G}$ is not compact.

The group $\mathbf{\Gamma}$ is called a *lattice* in $\mathbf{G}$ if there is a *finite* $\mathbf{G}$-invariant measure $\nu_{\mathbf{G}}$ on $\mathbf{\Gamma}\backslash\mathbf{G}$. (In this case we shall assume that $\nu_{\mathbf{G}}$ is a *probability* measure, i.e. $\nu_{\mathbf{G}}(\mathbf{\Gamma}\backslash\mathbf{G}) = 1$.) Then a sequence $\{x_n\}$ in $\mathbf{\Gamma}\backslash\mathbf{G}$ cuspidally diverges if and only if it eventually leaves every compact subset of $\mathbf{\Gamma}\backslash\mathbf{G}$ (see [R]).

Now let $\mathbf{U}$ be a subgroup of $\mathbf{G}$ and $x \in \mathbf{\Gamma}\backslash\mathbf{G}$. The set $x\mathbf{U} = \{x\mathbf{u} : \mathbf{u} \in \mathbf{U}\}$ is called the $\mathbf{U}$-orbit of x. A typical orbit $x\mathbf{U}$ in $\mathbf{\Gamma}\backslash\mathbf{G}$ is random and chaotic.

We pose the following questions:

(1) What are the *closures* of orbits $x\mathbf{U}$ in $\mathbf{\Gamma}\backslash\mathbf{G}$?

(2) What are the *ergodic* $\mathbf{U}$-invariant Borel probability measures on $\mathbf{\Gamma}\backslash\mathbf{G}$? (A $\mathbf{U}$-invariant probability measure μ on $\mathbf{\Gamma}\backslash\mathbf{G}$ is ergodic if every $\mathbf{U}$-invariant measurable subset of $\mathbf{\Gamma}\backslash\mathbf{G}$ has μ-measure zero or one.)

Let us give a few natural examples. Suppose $\mathbf{G}$ is a real Lie group, $\mathbf{U} = \{\mathbf{u}(t) : t \in \mathbb{R}\}$ a one-parameter subgroup of $\mathbf{G}$, and $x\mathbf{U}$ a periodic orbit. Then $x\mathbf{U} = \overline{x\mathbf{U}}$ and the normalized length measure on $x\mathbf{U}$ is $\mathbf{U}$-invariant and ergodic.

For a more general example suppose that the closure $\overline{x\mathbf{U}}$ coincides with the orbit of a larger group $\mathbf{H}$ containing $\mathbf{U}$, i.e. $\overline{x\mathbf{U}} = x\mathbf{H}$. In addition, it might happen that $x\mathbf{H}$ is the support of an $\mathbf{H}$-invariant Borel probability measure $\nu_{\mathbf{H}}$ (this happens if and only if $\mathbf{xHx}^{-1} \cap \mathbf{\Gamma}$ is a lattice in $\mathbf{xHx}^{-1}$, $\mathbf{x} \in \pi^{-1}\{x\}$) that is ergodic for the action of $\mathbf{U}$.

These examples motivate the following definitions.

DEFINITION 1. A subset $A \subset \mathbf{\Gamma}\backslash\mathbf{G}$ is called *homogeneous* if there exists a closed subgroup $\mathbf{H} \subset \mathbf{G}$ and a point $x \in \mathbf{\Gamma}\backslash\mathbf{G}$ such that $A = x\mathbf{H}$ and $x\mathbf{H}$ is the support of an $\mathbf{H}$-invariant Borel probability measure $\nu_{\mathbf{H}}$.

We emphasize that this definition of $x\mathbf{H}$ being homogeneous is different from the commonly used one where the existence of a *finite* $\mathbf{H}$-invariant measure on $x\mathbf{H}$ is not required.

DEFINITION 2. A Borel probability measure μ on $\mathbf{\Gamma}\backslash\mathbf{G}$ is *algebraic* if there exist $x \in \mathbf{\Gamma}\backslash\mathbf{G}$ and a closed subgroup $\mathbf{H} \subset \mathbf{G}$ such that $x\mathbf{H}$ is homogeneous and $\mu = \nu_{\mathbf{H}}$.

Equivalently, μ is algebraic if there is $x \in \mathbf{\Gamma}\backslash\mathbf{G}$ such that $\mu(x\mathbf{\Lambda}(\mu)) = 1$, where

$$\mathbf{\Lambda}(\mu) = \{\mathbf{g} \in \mathbf{G} : \text{ the action of } \mathbf{g} \text{ on } \mathbf{\Gamma}\backslash\mathbf{G} \text{ preserves } \mu\}.$$

It is rather exceptional for a subgroup $\mathbf{U}$ to have homogeneous orbit closures or algebraic ergodic measures. However, there are some $\mathbf{U}$ for which this happens. To characterize these $\mathbf{U}$ we need the following definitions.

Let $\mathbf{G}$ be a Lie group over a field κ (where κ is either the real field or a p-adic field) with the Lie algebra $\mathfrak{G}$. For $\mathbf{g} \in \mathbf{G}$ let $\mathrm{Ad}_{\mathbf{g}} : \mathfrak{G} \to \mathfrak{G}$ denote the differential at the identity of the map $\mathbf{h} \to \mathbf{g}^{-1}\mathbf{hg}$, $\mathbf{h} \in \mathbf{G}$. Then $\mathrm{Ad}_{\mathbf{g}}$ (called the adjoint map of $\mathbf{g}$) is a linear automorphism of $\mathfrak{G}$.

It is a fact that there is a neighborhood $\mathfrak{O}$ of zero in $\mathfrak{G}$ such that the exponential map $\exp : \mathfrak{O} \to \mathbf{G}$ is well defined on $\mathfrak{O}$ and maps $\mathfrak{O}$ diffeomorphically onto a neighborhood of $\mathbf{e}$ in $\mathbf{G}$. (When $\kappa = \mathbb{R}$ the map exp is defined on all of $\mathfrak{G}$.) If $\mathbf{x}, \mathbf{y} \in \mathbf{G}$ and $\mathbf{y} = \mathbf{x} \exp v$ for some $v \in \mathfrak{O}$ with $\mathrm{Ad}_{\mathbf{g}^r}(v) \in \mathfrak{O}$ for all $r = 1, \dots, n$ and some $0 \leq n \in \mathbb{Z}$ then $\mathbf{yg}^r = \mathbf{xg}^r \exp(\mathrm{Ad}_{\mathbf{g}^r}(v))$ for all $r = 1, \dots, n$. Thus $\mathrm{Ad}_{\mathbf{g}^r}$ characterizes the divergence of $\mathbf{yg}^r$ from $\mathbf{xg}^r$ when r runs from 1 to n.

An element $\mathbf{u} \in \mathbf{G}$ is called Ad-*unipotent* if $\mathrm{Ad}_{\mathbf{u}}$ is a unipotent element of $GL(\mathrm{n}, \kappa)$, $\mathrm{n} = \dim \mathfrak{G}$, i.e. every eigenvalue of $\mathrm{Ad}_{\mathbf{u}}$ equals one. Then $\mathrm{Ad}_{\mathbf{u}^r} = \sum_{k=0}^{m}(r^k T_{\mathbf{u}}^k)/k!$ for all $r \in \mathbb{Z}$ and some integer $m \geq 0$, where $T_{\mathbf{u}}$ is a nilpotent endomorphism of $\mathfrak{G}$. This polynomial (in r) form of $\mathrm{Ad}_{\mathbf{u}^r}$ plays a crucial role in all of the results stated below. It shows that Ad-unipotent orbits diverge *polynomially*.

A subgroup $\mathbf{U} \subset \mathbf{G}$ is Ad-*unipotent* if each $\mathbf{u} \in \mathbf{U}$ is Ad-unipotent. A subgroup $\mathbf{U} \subset GL(\mathrm{n}, \kappa)$ is *unipotent* if each $\mathbf{u} \in \mathbf{U}$ is unipotent. A unipotent $\mathbf{U} \subset GL(\mathrm{n}, \kappa)$ is Ad-unipotent.

Now let u be an ad-nilpotent element of $\mathfrak{G}$ (this means that the map $\mathrm{ad}_u : \mathfrak{G} \to \mathfrak{G}$, $\mathrm{ad}_u(v) = [v, u]$ is a nilpotent linear transformation of $\mathfrak{G}$). An element $a \in \mathfrak{G}$ is called "*diagonal*" for u if there exists an ad-nilpotent element $u^* \in \mathfrak{G}$ (called an "*opposite*" for u) such that

$$\mathrm{ad}_{u^*}(u) = a, \ \mathrm{ad}_a(u) = -2u, \ \mathrm{ad}_a(u^*) = 2u^*.$$

This terminology is motivated by the fact that u, u^* generate a Lie subalgebra $sl_2(u,a)$ of $\mathfrak{G}$ isomorphic to $sl(2,\kappa)$.

Now let $t \to \mathbf{u}(t)$ be a continuous (hence analytic) homomorphism from κ (as an additive group) to $\mathbf{G}$ with $u = d\mathbf{u}(t)/dt|_{t=0} \neq 0$. The latter condition implies that if κ is a p-adic field then the map $t \to \mathbf{u}(t)$ is one-to-one. We call $\mathbf{U} = \{\mathbf{u}(t) : t \in \kappa\}$ a *one-parameter* subgroup of $\mathbf{G}$ with tangent $u \in \mathfrak{G}$. Then $\mathbf{U}$ is Ad-unipotent if and only if u is ad-nilpotent.

2 Conjectures and Results Prior to 1990

CONJECTURE 1. (Raghunathan's Topological Conjecture) Let $\mathbf{G}$ be a real connected Lie group and $\mathbf{U}$ an Ad-unipotent subgroup of $\mathbf{G}$. Then given any lattice $\mathbf{\Gamma}$ of $\mathbf{G}$ and any $x \in \mathbf{\Gamma}\backslash\mathbf{G}$ the closure of the orbit $x\mathbf{U}$ in $\mathbf{\Gamma}\backslash\mathbf{G}$ is homogeneous.

CONJECTURE 2. (Raghunathan's Measure Conjecture) Let $\mathbf{G}$ and $\mathbf{U}$ be as in Conjecture 1. Then given any lattice $\mathbf{\Gamma}$ in $\mathbf{G}$ every ergodic $\mathbf{U}$-invariant Borel probability measure on $\mathbf{\Gamma}\backslash\mathbf{G}$ is algebraic.

Actually Raghunathan proposed a weaker version of Conjecture 1 and showed its connection with the long-standing Oppenheim conjecture on the density of values of irrational quadratic forms at integral points (see Section 5 below). The latter version as well as Conjecture 2 were stated by Dani [D1] in 1981 for reductive $\mathbf{G}$ and one-parameter $\mathbf{U}$ and by Margulis [M1, Conjectures 2 and 3] in 1986 for general $\mathbf{G}$ and $\mathbf{U}$. (Raghunathan did not propose Conjecture 2. We gave the latter his name because it represents a natural measure-theoretic analogue of his topological conjecture.)

CONJECTURE 3. (Margulis [M1, Conjecture 1], [M2, Conjecture 2]) Let $\mathbf{G}$ be a real connected Lie group and $\mathbf{U}$ a subgroup of $\mathbf{G}$ generated by Ad-unipotent elements of $\mathbf{G}$. Then given any lattice $\mathbf{\Gamma}$ in $\mathbf{G}$ and any $x \in \mathbf{\Gamma}\backslash\mathbf{G}$ the closure of $x\mathbf{U}$ in $\mathbf{\Gamma}\backslash\mathbf{G}$ is homogeneous.

In fact, Margulis proposed a weaker version of this conjecture. Conjecture 3 generalizes Conjecture 1 to a class of subgroups $\mathbf{U}$ much larger than Ad-unipotent subgroups. For example, every connected semisimple $\mathbf{U}$ without compact factors is generated by Ad-unipotent elements of $\mathbf{G}$.

It was shown earlier by Furstenberg [Fu1] and Parry [P1] (see also [AGH]) that Conjectures 1 and 2 hold for one-parameter and one-generator subgroups of *nilpotent* $\mathbf{G}$. Also Starkov [St2] proved Conjecture 1 for one-parameter Ad-unipotent subgroups of *solvable* $\mathbf{G}$ with $\mathbf{\Gamma}$ being an arbitrary closed subgroup of $\mathbf{G}$ such that $\mathbf{\Gamma}\backslash\mathbf{G}$ has finite $\mathbf{G}$-invariant measure. Conjecture 2 for the latter case follows from [St2] and [P1] (in fact, without the assumption of $\mathbf{\Gamma}\backslash\mathbf{G}$ being of finite $\mathbf{G}$-invariant measure).

As for *semisimple* $\mathbf{G}$, Hedlund [H] showed that if $\mathbf{G} = SL(2,\mathbb{R})$ and $\mathbf{\Gamma}\backslash\mathbf{G}$ is compact (in this case $\mathbf{\Gamma}$ is called a uniform lattice in $\mathbf{G}$) then the action of a unipotent one-parameter subgroup $\mathbf{U}$ of $\mathbf{G}$ on $\mathbf{\Gamma}\backslash\mathbf{G}$ is minimal (i.e. every orbit of $\mathbf{U}$ is dense). Subsequently, Furstenberg [Fu2] proved that in this case the action of $\mathbf{U}$ is uniquely ergodic.

It is a fact that one-parameter unipotent subgroups of $\mathbf{G} = SL(2,\mathbb{R})$ are horospherical. A subgroup $\mathbf{U}$ of a Lie group $\mathbf{G}$ is called horospherical if there exists $\mathbf{g} \in \mathbf{G}$ such that

$$\mathbf{U} = \{\mathbf{u} \in \mathbf{G} : \mathbf{g}^{-n}\mathbf{u}\mathbf{g}^{n} \to \mathbf{e} \text{ as } n \to \infty\}$$

where $\mathbf{e}$ denotes the identity element of $\mathbf{G}$.

Generalizing Furstenberg's Theorem, Bowen [Bw], Veech [V] and Ellis and Perrizo [EPe] showed that if $\mathbf{\Gamma}$ is a uniform lattice in a connected semisimple Lie group $\mathbf{G}$ without compact factors then ergodic actions of horospherical subgroups on $(\mathbf{\Gamma}\backslash\mathbf{G}, \nu_{\mathbf{G}})$ are uniquely ergodic. Adapting the method of Furstenberg and Veech, Dani [D1] proved Conjecture 2 when $\mathbf{G}$ is reductive and $\mathbf{U}$ is a maximal horospherical subgroup of $\mathbf{G}$.

As for Conjecture 1, Dani [D3] proved it for horospherical subgroups of reductive $\mathbf{G}$. Also Dani and Margulis [DM2, 3] showed that Conjecture 1 holds for one-parameter unipotent subgroups of $SL(3,\mathbb{R})$.

Dani and Margulis [DM1] proved Conjecture 3 for $\mathbf{G} = SL(3,\mathbb{R})$, $\mathbf{\Gamma} = SL(3,\mathbb{Z})$, and $\mathbf{U} = SO(2,1)^0$.

It should be noted that in 1986 Dani showed [D2, Theorem 3.5] that if $\mathbf{G}$ is a connected semisimple Lie group and $\mathbf{\Gamma}$ a lattice in $\mathbf{G}$ then given $\varepsilon > 0$ there is a compact $K(\varepsilon) \subset \mathbf{\Gamma}\backslash\mathbf{G}$ such that for any $x \in \mathbf{\Gamma}\backslash\mathbf{G}$ and any one-parameter Ad-unipotent subgroup $\mathbf{U} = \{\mathbf{u}(t) : t \in \mathbb{R}\}$ of $\mathbf{G}$ either $\lambda\{t \in [0,T] : x\mathbf{u}(t) \in K(\varepsilon)\} > (1-\varepsilon)T$ for all large T or $x\mathbf{L}$ is homogeneous for some *proper* closed connected subgroup $\mathbf{L}$ of $\mathbf{G}$ containing $\mathbf{U}$. (Here λ denotes the Lebesgue measure on $\mathbb{R}$.) This important result is used in the proofs of Theorems 6 and 8–10 below.

3 New Results (1990 and After)

All Lie groups in this section are assumed to be real, and, unless otherwise stated, the results below are due to the author.

THEOREM 1 (Classification of ergodic invariant measures for Ad-unipotent actions). *Let $\mathbf{G}$ be a connected Lie group and $\mathbf{U}$ an Ad-unipotent subgroup of $\mathbf{G}$. Then given any discrete subgroup $\mathbf{\Gamma}$ (not necessarily a lattice) of $\mathbf{G}$ every ergodic $\mathbf{U}$-invariant Borel probability measure on $\mathbf{\Gamma}\backslash\mathbf{G}$ is algebraic.*

THEOREM 2. *Let $\mathbf{G}$ be a connected Lie group and $\mathbf{U}$ a Lie subgroup of $\mathbf{G}$ of the form $\mathbf{U} = \cup_{i=1}^{\infty}\mathbf{u}_i\mathbf{U}^0$, where $\mathbf{u}_i$ are Ad-unipotent in $\mathbf{G}$, $i = 1,2,\ldots$, $\mathbf{U}/\mathbf{U}^0$ is finitely generated, and the identity component $\mathbf{U}^0$ is generated by Ad-unipotent elements of $\mathbf{G}$ contained in $\mathbf{U}^0$. Then Theorem 1 holds for $\mathbf{U}$.*

Theorem 1 is stronger than Conjecture 2 and Theorem 2 extends it to groups generated by Ad-unipotent elements.

THEOREM 3 (Orbit closures for Ad-unipotent actions). *Let $\mathbf{G}$ and $\mathbf{U}$ be as in Theorem 1. Then given any lattice $\mathbf{\Gamma}$ in $\mathbf{G}$ and any $x \in \mathbf{\Gamma}\backslash\mathbf{G}$ the closure of the orbit $x\mathbf{U}$ in $\mathbf{\Gamma}\backslash\mathbf{G}$ is homogeneous.*

THEOREM 4. *Let* $\mathbf{G}$ *and* $\mathbf{U}$ *be as in Theorem 2. Then Theorem 3 holds for* $\mathbf{U}$. *Moreover, if* $\mathbf{U}$ *is connected, then for any lattice* $\mathbf{\Gamma}$ *in* $\mathbf{G}$ *and any* $x \in \mathbf{\Gamma}\backslash\mathbf{G}$ *there exists a closed connected subgroup* $\mathbf{H}$ *of* $\mathbf{G}$ *containing* $\mathbf{U}$ *and a one-parameter subgroup* $\mathbf{V}$ *of* $\mathbf{U}$ *Ad-unipotent in* $\mathbf{G}$ *such that* $\overline{x\mathbf{V}} = \overline{x\mathbf{U}} = x\mathbf{H}$ *is homogeneous and* $\mathbf{V}$ *acts ergodically on* $(x\mathbf{H}, \nu_{\mathbf{H}})$, *where* $\nu_{\mathbf{H}}$ *denotes the* $\mathbf{H}$*-invariant Borel probability measure on* $\mathbf{\Gamma}\backslash\mathbf{G}$ *supported on* $x\mathbf{H}$.

Conjecture 1 is implied by Theorem 3 and Theorem 4 extends it to groups generated by Ad-unipotent elements.

THEOREM 5. *Let* $\mathbf{G}$ *be a connected Lie group,* $\mathbf{\Gamma}$ *a discrete subgroup of* $\mathbf{G}$, *and* $x \in \mathbf{\Gamma}\backslash\mathbf{G}$. *Let* $\mathcal{A}_x$ *denote the set of all closed connected* $\mathbf{H} \subset \mathbf{G}$ *such that* $x\mathbf{H}$ *is homogeneous and there is a one-parameter subgroup* $\mathbf{U} \subset \mathbf{H}$ *Ad-unipotent in* $\mathbf{G}$ *acting ergodically on* $(x\mathbf{H}, \nu_{\mathbf{H}})$. *Then* $\mathcal{A}_x$ *is countable.*

We show that in order to prove Theorem 3 for general Ad-unipotent $\mathbf{U}$ it suffices to prove it for one-parameter Ad-unipotent $\mathbf{U}$. But for such $\mathbf{U}$ we have the far stronger Theorem 6 below. To state it we need to introduce a definition.

DEFINITION 3. Let $\mathbf{U} = \{\mathbf{u}(t) : t \in \mathbb{R}\}$ be an arbitrary one-parameter subgroup of $\mathbf{G}$. A point $x \in \mathbf{\Gamma}\backslash\mathbf{G}$ is called *generic* for $\mathbf{U}$ if there exists a closed subgroup $\mathbf{H} \subset \mathbf{G}$ such that $\overline{x\mathbf{U}} = x\mathbf{H}$ is homogeneous and $\frac{1}{t}\int_0^t f(x\mathbf{u}(s))\,ds \xrightarrow[t\to\infty]{} \int_{\mathbf{\Gamma}\backslash\mathbf{G}} f\,d\nu_{\mathbf{H}}$ for every bounded continuous function f on $\mathbf{\Gamma}\backslash\mathbf{G}$.

A similar definition can be given for a one-generator $\mathbf{U} = \{\mathbf{u}^k : k \in \mathbb{Z}\}$ replacing the integral by the sum $\sum_{k=0}^{n-1} f(x\mathbf{u}^k)/n$.

THEOREM 6 (Uniform distribution of Ad-unipotent flows). *Let* $\mathbf{G}$ *be a connected Lie group and* $\mathbf{U}$ *a one-parameter or one-generator* Ad*-unipotent subgroup of* $\mathbf{G}$. *Then given any lattice* $\mathbf{\Gamma}$ *of* $\mathbf{G}$ *every point* $x \in \mathbf{\Gamma}\backslash\mathbf{G}$ *is generic for* $\mathbf{U}$ *and* $\mathbf{U}$ *acts ergodically on* $(\overline{x\mathbf{U}} = x\mathbf{H}, \nu_{\mathbf{H}})$.

This theorem was proved one month before it was conjectured by Margulis at the ICM 1990 in Kyoto, Japan [M2, Conjectures 3 and 4].

Theorem 6 for nilpotent $\mathbf{G}$ follows from [P1] (see also [L]) and for $\mathbf{G} = SL(2,\mathbb{R})$ it was proved earlier by Dani and Smillie [DSm]. Also Shah [S1] proved it for semisimple $\mathbf{G}$ of real rank 1. Their methods are totally different from the author's.

To derive the results stated above we first prove Theorem 1 for one-parameter Ad-unipotent $\mathbf{U}$. Theorem 7 below plays a crucial role in this proof. To state it we introduce the following definition. Let $\mathbf{U} = \{\mathbf{u}(t) = \exp tu : t \in \mathbb{R}\}$ be a one-parameter Ad-unipotent subgroup of $\mathbf{G}$ and assume there is a "diagonal" element $a \in \mathfrak{G}$ for u (see Section 1). Then we call $\mathbf{A} = \{\mathbf{a}(t) = \exp ta : t \in \mathbb{R}\}$ "diagonal" for $\mathbf{U}$ and denote by $SL_2(\mathbf{U}, \mathbf{A})$ the connected subgroup of $\mathbf{G}$ with the Lie algebra $sl_2(u, a)$ (see Section 1). It is clear that $\mathbf{A}$ is "diagonal" for $\mathbf{U}$ if and only if $\mathbf{c}\mathbf{A}\mathbf{c}^{-1}$ is so for every $\mathbf{c} \in \mathbf{C}(\mathbf{U})$ — the centralizer of $\mathbf{U}$ in $\mathbf{G}$.

THEOREM 7. *Let* $\mathbf{G}$ *be a Lie group,* $\mathbf{\Gamma}$ *a discrete subgroup of* $\mathbf{G}$, *and* $\mathbf{U} = \{\mathbf{u}(t) : t \in \mathbb{R}\}$ *a one-parameter* Ad*-unipotent subgroup of* $\mathbf{G}$. *Suppose there is a "diagonal"* $\mathbf{A} = \{\mathbf{a}(t) : t \in \mathbb{R}\}$ *for* $\mathbf{U}$ *in* $\mathbf{G}$ *and let* μ *be an ergodic* $\mathbf{U}$*-invariant Borel probability* *measure on* $\Gamma\backslash\mathbf{G}$. *Then either* (1) $\mu(\mathcal{D}(\mathbf{a}(t))) = 1$ *for all* $t > 0$ *or* (2) μ *is algebraic and is preserved by* $\mathbf{c}SL_2(\mathbf{U}, \mathbf{A})\mathbf{c}^{-1}$ *for some* $\mathbf{c} \in \mathbf{C}(\mathbf{U})$.

Recall that $\mathcal{D}(\mathbf{g})$, $\mathbf{g} \in \mathbf{G}$, denotes the divergent set of $\mathbf{g}$ (see Section 1).

The central role in the proof of Theorem 7 in [Ra9] is played by a dynamical property of Ad-unipotent actions which we introduced in [Ra8, Theorem 3.1] and called the R-property. It is a consequence of the polynomial divergence of Ad-unipotent orbits. Also it is a generalization of the H-property for horocycle flows introduced in [Ra4] (see Section 6 below).

The R-property states, roughly speaking, that given $0 < \varepsilon < 1$ there exists $0 < \eta(\varepsilon) < 1$ such that if $\mathbf{F}$ is an appropriate sufficiently large open rectangular subset of a closed connected simply connected Ad-unipotent subgroup $\mathbf{U}$ of $\mathbf{G}$ with $\mathbf{e} \in \mathbf{F}$ and $\sup\{d_{\mathbf{G}}(\mathbf{u}, \mathbf{xU}) : \mathbf{u} \in \mathbf{F}\} = \theta$ for some small $\theta > 0$ and some $\mathbf{x} \notin \mathbf{U}$, $d_{\mathbf{G}}(\mathbf{e}, \mathbf{x}) \leq \theta$, then there exists $\mathbf{A} \subset \mathbf{F}$ such that $(1-\varepsilon)\theta \leq d_{\mathbf{G}}(\mathbf{u}, \mathbf{xU}) \leq \theta$ for all $\mathbf{u} \in \mathbf{A}$ and $\lambda(\mathbf{A}) \geq \eta(\varepsilon)\lambda(\mathbf{F})$, where λ denotes a Haar measure on $\mathbf{U}$ and $d_{\mathbf{G}}$ denotes a left invariant metric on $\mathbf{G}$. Moreover, if $\mathbf{u} \in \mathbf{F}$ and $d_{\mathbf{G}}(\mathbf{u}, \mathbf{xU}) = d_{\mathbf{G}}(\mathbf{u}, \mathbf{ur}(\mathbf{u}))$ for some $\mathbf{r}(\mathbf{u}) \in \mathbf{G}$ with $\mathbf{ur}(\mathbf{u}) \in \mathbf{xU}$ and $d_{\mathbf{G}}(\mathbf{e}, \mathbf{r}(\mathbf{u})) \leq \theta$ then $\mathbf{r}(\mathbf{u})$ is close to the normalizer of $\mathbf{U}$ in $\mathbf{G}$ and this closeness tends to zero as the sides of the rectangular set $\mathbf{F}$ tend to infinity. (When $\mathbf{U} = \{\mathbf{u}(t) : t \in \mathbb{R}\}$ is a one-parameter subgroup of $\mathbf{G}$ we can take $\mathbf{F} = \{\mathbf{u}(t) : 0 \leq t \leq T\}$ for large $T > 0$.)

The rectangular sets $\mathbf{F}$ in the description of the R-property are Fϕlner subsets of $\mathbf{U}$ (see [Ra8]) and the Birkhoff Ergodic Theorem for measure preserving actions of $\mathbf{U}$ holds for averages performed over $\mathbf{F}$.

It should be noted that the R-property and the Birkhoff Ergodic Theorem are *the only* basic facts used in the proof of Theorem 7.

Using Theorem 1 (proved in [Ra8–10]) and Theorem 5 (whose proof is simple [Ra10, Theorem 1.1]) we deduce Theorem 6 [Ra11]. These two proofs (of Theorem 1 and of Theorem 6 from Theorems 1 and 5) are central. Note that Theorem 6 implies Theorem 3 for one-parameter Ad-unipotent $\mathbf{U}$. In [Ra14, Section 8] we outlined (using Theorem 5) how the validity of Theorems 1 and 3 for one-parameter Ad-unipotent $\mathbf{U}$ implies their validity for higher-dimensional connected $\mathbf{U}$ generated by Ad-unipotent elements of $\mathbf{G}$ (see [Ra10] and [Ra11]).

Let us outline the main idea used in [Ra11, Proof of Theorem 2.1] to deduce Theorem 6 from Theorems 1 and 5.

Let $\mathbf{U} = \{\mathbf{u}(t) : t \in \mathbb{R}\}$ be a one-parameter Ad-unipotent subgroup of $\mathbf{G}$. For $x \in \Gamma\backslash\mathbf{G} = X$ and a sequence $t_n \to \infty$ we denote by μ_n the normalized length measure on the orbit interval $L_n = \{x\mathbf{u}(t) : 0 \leq t \leq t_n\}$, assuming that μ_n is a measure on X supported on L_n. Thus $t_n^{-1} \int_0^{t_n} f(x\mathbf{u}(s))\, ds = \int_X f\, d\mu_n$ for every bounded continuous function f on X. Also the sequence μ_n contains a weak* convergent subsequence (as the closed unit ball in the space of Borel measures on X is weak* compact).

Suppose μ_n converges weak* to μ for some $t_n \to \infty$. Then μ is $\mathbf{U}$-invariant, $\mathrm{Supp}(\mu) \subset \overline{x\mathbf{U}}$ and $\mu(X) = 1$ (by [D2, Theorem 3.5]). Let $\{(C(y), \mu_{C(y)}) : y \in X\}$ be the ergodic decomposition of the action of $\mathbf{U}$ on (X, μ). Here each $\mu_{C(y)}$ is an ergodic $\mathbf{U}$-invariant measure supported on $C(y)$ and μ is the direct integral of the measures $\mu_{C(y)}$, $y \in X$. By Theorem 1 each $\mu_{C(y)}$ is algebraic and $C(y)$

is homogeneous. It follows from Theorem 5 that there exist $y_0 \in X$ and a small $\delta > 0$ such that the set $\Omega = \cup\{C(y) : C(y) = C(y_0)\mathbf{z}, d_{\mathbf{G}}(\mathbf{z}, \mathbf{e}) \leq \delta\}$ has positive μ-measure. It follows from the definition of μ that the proportion of time spent by L_n in every small neighborhood of Ω is close to $\mu(\Omega)$ for all sufficiently large n. Using the polynomial form of $\mathrm{Ad}_{\mathbf{u}(t)}$ (via a version of the R-property for $\mathbf{U}$) we show that this may happen only if $x \in C(y) \subset \Omega$ for some y and $\mu(C(y)) = 1$. Then $\overline{x\mathbf{U}} = C(y)$ is homogeneous and $\mu = \mu_{C(y)}$ is algebraic. Thus there exists $\mathbf{H} \subset \mathbf{G}$ such that $\overline{x\mathbf{U}} = x\mathbf{H}$ is homogeneous and $\mu = \nu_{\mathbf{H}}$. Because this is true for all sequences $t_n \to \infty$ with μ_n weak* convergent, x is generic for $\mathbf{U}$.

Recently, Dani and Margulis [DM4] offered a linearized version of this argument using the action of the adjoint representation of $\mathbf{G}$ on the mth exterior power of $\mathfrak{G}$ with $m = \dim C(y_0)$. Using this version they offered an alternative proof of Theorem 6 (and Theorem 8 below) and showed that the convergence in Theorem 6 (and Theorem 8) is uniform on compact subsets of $\mathbf{\Gamma}\backslash\mathbf{G}$. Also this linearization method is basic for the proofs of Theorems 9, 10, and C2 below.

Now let $\mathbf{U}_n = \{\mathbf{u}_n(t) : t \in \mathbb{R}\}$, $n = 1, 2, \ldots$ and $\mathbf{U} = \{\mathbf{u}(t) : t \in \mathbb{R}\}$ be one-parameter Ad-unipotent subgroups of $\mathbf{G}$. We say that $\mathbf{U}_n \to \mathbf{U}$ if $\mathbf{u}_n(t) \to \mathbf{u}(t)$ for all $t \in \mathbb{R}$.

The argument given above can be applied to derive the following more general version of Theorem 6. (This was pointed out to the author by Marc Burger in December 1990.)

THEOREM 8. *Let $\mathbf{U}_n \to \mathbf{U}$ and $x_n \to x \in \mathbf{\Gamma}\backslash\mathbf{G}$ with $\mathbf{\Gamma}$ being a lattice in $\mathbf{G}$. Suppose that there exists no proper closed connected subgroup $\mathbf{L}$ of $\mathbf{G}$ such that $\mathbf{U} \subset \mathbf{L}$ and $x\mathbf{L}$ is homogeneous. Then*

$$\lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} f(x_n \mathbf{u}_n(s))\, ds = \int_{\mathbf{\Gamma}\backslash\mathbf{G}} f\, d\nu_{\mathbf{G}}$$

for every bounded continuous function f on $\mathbf{\Gamma}\backslash\mathbf{G}$ and every sequence $t_n \to \infty$ when $n \to \infty$, where $\nu_{\mathbf{G}}$ denotes the $\mathbf{G}$-invariant Borel probability measure on $\mathbf{\Gamma}\backslash\mathbf{G}$.

Theorem 6 follows from Theorem 8 if we set $\mathbf{U}_n = \mathbf{U}$, $x_n = x$ for all n and use induction on the dimension of $\mathbf{G}$. The main part of the proof of Theorem 8 is given in [Ra14, Section 7].

Now let $\mathbf{\Gamma}$ be a discrete subgroup of $\mathbf{G}$, $X = \mathbf{\Gamma}\backslash\mathbf{G}$, and let $\mathcal{P}(X)$ denote the set of all Borel probability measures on X. Recall that a sequence $\{\mu_n\}$ in $\mathcal{P}(X)$ weak* converges to a measure μ on X if $\int f\, d\mu_n \to \int f\, d\mu$ for every bounded continuous function f on X. Define

$$\mathcal{Q}(X) = \{\mu \in \mathcal{P}(X) : \text{there exists a one-parameter Ad-unipotent subgroup of } \mathbf{G} \text{ that preserves } \mu \text{ and acts ergodically on } (X, \mu)\}.$$

By Theorem 1 every member of $\mathcal{Q}(X)$ is algebraic.

Recently Mozes and Shah proved the following theorem.

THEOREM 9 (Mozes, Shah [MoS]). *Let $\{\mu_n\}$ be a sequence of measures in $\mathcal{Q}(X)$ weak* converging to $\mu \in \mathcal{P}(X)$. Then $\mu \in \mathcal{Q}(X)$. Moreover, there exist $x \in$* Supp(μ) *and $\mathbf{g}_n \in \mathbf{G}$, $\mathbf{g}_n \to \mathbf{e}$ such that $x\mathbf{g}_n \in$* Supp$(\mu_n) \subset$ Supp$(\mu)\mathbf{g}_n$ *for all large n.*

The proof of this theorem uses Theorems 1 and 5 and the method of [DM4].

Theorem 9 implies the following extension of Theorem 6 to the case when $\mathbf{\Gamma}$ is not a lattice: if $\mathbf{U}$ is a one-parameter Ad-unipotent subgroup of $\mathbf{G}$ and $\overline{x\mathbf{U}}$ is compact in $\mathbf{\Gamma}\backslash\mathbf{G}$, then x is generic for $\mathbf{U}$. This was conjectured in [Ra12, Conjecture D] and proved there for $\mathbf{G} = SL(2, \mathbb{R})$.

It is clear that if $\{\mathbf{u}(t) : t \in \mathbb{R}\}$ is a one-parameter unipotent subgroup of $GL(n, \mathbb{R})$ then each entry of the matrix $\mathbf{u}(t)$ is a polynomial in t.

Recently, Shah extended Theorem 6 to more general polynomial actions. A map $\boldsymbol{\theta} : \mathbb{R}^k \to SL(n, \mathbb{R})$, $n \in \mathbb{Z}^+$, is called *polynomial* if every entry of the matrix $\boldsymbol{\theta}(t_1, \ldots, t_k) \in SL(n, \mathbb{R})$ is a polynomial in $(t_1, \ldots, t_k) \in \mathbb{R}^k$ and $\boldsymbol{\theta}$ maps the origin to the identity element of $SL(n, \mathbb{R})$.

THEOREM 10 (Shah [S2]). *Let $\boldsymbol{\theta} : \mathbb{R}^k \to SL(n, \mathbb{R})$ be polynomial and let $\mathbf{G}$ be a closed subgroup of $SL(n, \mathbb{R})$ containing $\boldsymbol{\theta}(\mathbb{R}^k)$. Then given any lattice $\mathbf{\Gamma}$ in $\mathbf{G}$ and any $x \in \mathbf{\Gamma}\backslash\mathbf{G}$ there is a closed subgroup $\mathbf{H} \subset \mathbf{G}$ such that $\overline{x\boldsymbol{\theta}(\mathbb{R}^k)} = x\mathbf{H}$ is homogeneous and*

$$\lim_{R\to\infty} \frac{1}{\lambda(B_R)} \int_{B_R} f(x\boldsymbol{\theta}(t))d\lambda(t) = \int_{\mathbf{\Gamma}\backslash\mathbf{G}} f d\nu_{\mathbf{H}}$$

for every bounded continuous function f on $\mathbf{\Gamma}\backslash\mathbf{G}$, where B_R denotes the ball of radius R in $\mathbb{R}^k$ centered at the origin and λ denotes the Lebesgue measure on $\mathbb{R}^k$.

The proof of Theorem 10 uses Theorems 1 and 5 and the method of [DM4]. Shah also showed that if $\boldsymbol{\theta}(t_1, \ldots, t_k) = \boldsymbol{\theta}_1(t_1) \ldots \boldsymbol{\theta}_k(t_k)$ for some polynomial maps $\boldsymbol{\theta}_i : \mathbb{R} \to SL(n, \mathbb{R})$, $i = 1, \ldots, k$, then the conclusion of Theorem 10 holds also for B_n being of the form $[0, T_n^{(1)}] \times \cdots \times [0, T_n^{(k)}]$ with $T_n^{(i)} \to \infty$, $i = 1, \ldots, k$. This implies, in particular, that Theorem 6 holds for higher-dimensional connected simply connected Ad-unipotent $\mathbf{U}$ with averages performed over large rectangular subsets of $\mathbf{U}$ and, in particular, over Fϕlner subsets of $\mathbf{U}$ (see [Ra11]). This gives an affirmative answer to a question raised in [Ra14, Problem 2].

Next we address the following question: Are there subgroups of $\mathbf{G}$ not generated by Ad-unipotent elements of $\mathbf{G}$ for which Theorems 1 and 3 hold? Theorems 11–13 below give an affirmative answer to this question.

Indeed, let $\mathbf{\Gamma}$ be a discrete subgroup of $\mathbf{G}$ and μ a Borel probability measure on $\mathbf{\Gamma}\backslash\mathbf{G}$. Also let $\mathbf{U}$ be a one-parameter Ad-unipotent subgroup of $\mathbf{G}$ and $\mathbf{A}$ be "diagonal" for $\mathbf{U}$. Using Theorem 7 we showed in [Ra10, Proposition 2.1] that if μ is preserved by both $\mathbf{U}$ and $\mathbf{A}$ then μ is preserved by $SL_2(\mathbf{U}, \mathbf{A})$. Note that $SL_2(\mathbf{U}, \mathbf{A})$ is generated by Ad-unipotent elements of $\mathbf{G}$. This and Theorems 2 and 4 imply the following

THEOREM 11. *Let $\mathbf{G}$ be a connected Lie group and $\mathbf{U}$ a connected subgroup of $\mathbf{G}$ generated by* Ad*-unipotent elements of $\mathbf{G}$. Let $\mathbf{A}_1, \ldots, \mathbf{A}_n$ be "diagonal" for some* *one-parameter* Ad-*unipotent subgroups* $\mathbf{U}_1, \dots, \mathbf{U}_n$ *of* $\mathbf{U}$. *Then Theorems* 1 *and* 3 *hold for the subgroup* $\mathbf{H}$ *of* $\mathbf{G}$ *generated by* $\mathbf{U}$ *and* $\mathbf{A}_1, \dots, \mathbf{A}_n$.

Indeed, if μ is an ergodic $\mathbf{H}$-invariant Borel probability measure on $\Gamma\backslash\mathbf{G}$ then μ is invariant under the action of the group $\mathbf{H}'$ generated by $\mathbf{U}$ and $SL_2(\mathbf{U}_i, \mathbf{A}_i)$, $i = 1, \dots, n$. Because $\mathbf{H} \subset \mathbf{H}'$, μ is ergodic for $\mathbf{H}'$. Because $\mathbf{H}'$ is generated by Ad-unipotent elements of $\mathbf{G}$, μ is algebraic by Theorem 2. This gives Theorem 1 for $\mathbf{H}$. (This argument was brought to the author's attention by Mozes.) To derive Theorem 3 for $\mathbf{H}$ we show that when Γ is a lattice in $\mathbf{G}$, then $\overline{x\mathbf{H}} = \overline{x\mathbf{H}'}$ for all $x \in \Gamma\backslash\mathbf{G}$. Hence $\overline{x\mathbf{H}}$ is homogeneous by Theorem 4.

This implies the following

THEOREM 12. *Let* $\mathbf{G}$ *be a connected Lie group and* $\mathbf{G}_1$ *a connected semisimple subgroup of* $\mathbf{G}$ *without compact factors. Let* $\mathbf{H}$ *be a parabolic subgroup of* $\mathbf{G}_1$. *Then Theorems* 1 *and* 3 *hold for* $\mathbf{H}$.

Theorem 12 implies, in particular, that Theorems 1 and 3 hold for the subgroup $\mathbf{H}$ of $\mathbf{G} = SL(n, \mathbb{R})$ consisting of all upper triangular matrices in $\mathbf{G}$.

We say that a subgroup $\mathbf{L}$ of $\mathbf{G}$ is *epimorphic* with respect to $\mathbf{G}$ if for every finite-dimensional representation of $\mathbf{G}$ every vector v fixed by $\mathbf{L}$ is also fixed by $\mathbf{G}$. It is a fact that the group $\mathbf{H}$ described in Theorem 11 is epimorphic with respect to $\mathbf{H}'$ generated by $\mathbf{U}$ and $SL_2(\mathbf{U}_i, \mathbf{A}_i)$, $i = 1, \dots, n$. Recently, Mozes has generalized Theorems 11 and 12 in the following form.

THEOREM 13 (Mozes [Mo2]). *Let* $\mathbf{G}$ *be a connected Lie group and* $\mathbf{L}$ *a subgroup of* $\mathbf{G}$ *epimorphic with respect to a connected semisimple subgroup* $\mathbf{G}_1$ *of* $\mathbf{G}$ *without compact factors. Then Theorem* 1 *holds for* $\mathbf{L}$.

Mozes' proof uses Theorem 2 and a recent result of Bien and Borel [BBo].

PROBLEM. Let $\mathbf{G}$ and $\mathbf{L}$ be as in Theorem 13. Does Theorem 3 hold for $\mathbf{L}$?

In [Ra14] we incorrectly stated that Raghunathan had a counterexample to this problem.

It is a fact that, in general, Theorem 3 does not hold for non-Ad-unipotent one-parameter $\mathbf{U}$. However, using Theorem 3, Starkov proved the following

THEOREM 14 (Starkov [St3]). *Let* Γ *be a lattice in* $\mathbf{G}$ *and* $\mathbf{U}$ *a one-parameter subgroup of* $\mathbf{G}$. *Then the following statements are equivalent:* (1) *for every* $x \in \Gamma\backslash\mathbf{G}$ *the closure* $\overline{x\mathbf{U}}$ *is a smooth submanifold of* $\Gamma\backslash\mathbf{G}$; (2) $|\lambda| = 1$ *for every eigenvalue* λ *of* $\mathrm{Ad}_{\mathbf{u}}$ *and every* $\mathbf{u} \in \mathbf{U}$.

Finally we mention that the validity of Theorems 1 and 2 for *discrete* subgroups Γ implies their validity for *arbitrary closed* $\Gamma \subset \mathbf{G}$. This was shown by Witte in [W3] (see also [St1] for a related result). Witte also showed (in a recent correspondence with the author) that the validity of Theorems 2 and 4 with the assumption of $\mathbf{U}/\mathbf{U}^0$ being finitely generated implies their validity without this assumption (because the assumption holds for the closure of $\mathbf{U}$).

In [Ra10] Theorem 2 is also proved for *disconnected* $\mathbf{G}$ with the additional assumption (which was omitted in [Ra10], though used in the proofs) that $\mathbf{U}/\mathbf{U}^0$

is nilpotent. This assumption automatically holds when $\mathbf{G}$ is connected. (See [W4] for more on the disconnected $\mathbf{G}$ case.)

In closing, we note that the following question remains unanswered.

QUESTION. Do Theorems 2 and 4 hold for *arbitrary* disconnected non-Ad-unipotent subgroups $\mathbf{U}$ of $\mathbf{G}$, generated by Ad-unipotent elements of $\mathbf{G}$?

4 Generalizations to the p-adic and $\mathbb{S}$-Arithmetic Cases

The problem of extending Raghunathan's conjectures to cartesian products of *algebraic* groups over local fields of characteristic zero (this is referred to as the $\mathbb{S}$-arithmetic setting) was raised by Borel and Prasad in [BoPr] (see also [Pr]). They pointed out that the validity of Conjecture 3 for the $\mathbb{S}$-arithmetic case (see Theorem S2 below) would solve the density problem in the Oppenheim conjecture for this case (see Section 5 below).

It turns out that the ideas and methods developed in [Ra8–11] for *real* Lie groups can be applied to prove Conjectures 1–3 for a more general (than the $\mathbb{S}$-arithmetic setting) case, namely, cartesian products of real and p-adic Lie groups. (If κ is a local field of characteristic zero then κ is (isomorphic to) either $\mathbb{R}$, or $\mathbb{C}$ or a finite extension of a p-adic field. Then a Lie group over κ can be viewed as either a real Lie group or a p-adic Lie group.) Also our results allow us to understand the structure of p-adic Lie groups $\mathbf{G}$ that carry discrete subgroups $\mathbf{\Gamma}$ (in particular, lattices) admitting finite Borel measures on $\mathbf{\Gamma}\backslash\mathbf{G}$ preserved by one-parameter subgroups of $\mathbf{G}$ (see Theorem S6 below).

More specifically, let $\mathbb{S}$ be a finite set and for each $\mathrm{s} \in \mathbb{S}$ let $\mathbb{Q}_{\mathrm{S}}$ be either the real field $\mathbb{R}$ or the field of p_{S}-adic numbers for some prime p_{S}. In the latter case we call s ultrametric, otherwise s is called real. The set $\mathbb{S}$ is ultrametric if each $\mathrm{s} \in \mathbb{S}$ is ultrametric.

For $\mathrm{s} \in \mathbb{S}$ let $\mathbf{G}_{\mathrm{S}}$ be a Lie group over $\mathbb{Q}_{\mathrm{S}}$ with the Lie algebra $\mathfrak{G}_{\mathrm{S}}$ and let $\mathbf{G}_{\mathbb{S}} = \prod\{\mathbf{G}_{\mathrm{S}} : \mathrm{s} \in \mathbb{S}\}$ denote the cartesian product of $\mathbf{G}_{\mathrm{S}}$, $\mathrm{s} \in \mathbb{S}$.

Let $\eta : \mathbf{G}_{\mathrm{S}} \to \mathbf{G}_{\mathbb{S}}$ denote the natural embedding of $\mathbf{G}_{\mathrm{S}}$ in $\mathbf{G}_{\mathbb{S}}$ and let $\mathbf{U}_{\mathrm{S}} = \{\mathbf{u}_{\mathrm{S}}(t) : t \in \mathbb{Q}_{\mathrm{S}}\}$ be a one-parameter Ad-unipotent subgroup of $\mathbf{G}_{\mathrm{S}}$. Then $\mathbf{U} = \eta(\mathbf{U}_{\mathrm{S}}) = \{\mathbf{u}(t) = \eta(\mathbf{u}_{\mathrm{S}}(t)) : t \in \mathbb{Q}_{\mathrm{S}}\}$ is called a one-parameter Ad-unipotent subgroup of $\mathbf{G}_{\mathbb{S}}$.

It is a fact (see [Ra15, Theorem 1.1]) that *every* one-parameter subgroup of a p-adic Lie group $\mathbf{G}$ is Ad-unipotent (this was recently proved independently by Lubotzky and Prasad). Also $\mathbf{G}$ is totally disconnected and small neighborhoods of the identity of $\mathbf{G}$ do not generate $\mathbf{G}$. Because of this, $\mathbf{G}$ might contain two *distinct* one-parameter subgroups $\mathbf{U}_1$ and $\mathbf{U}_2$ that have the same tangent (and hence coincide in a neighborhood of $\mathbf{e}$ in $\mathbf{G}$). This motivates the following definitions.

For an ultrametric $\mathrm{s} \in \mathbb{S}$ we call $\mathbf{G}_{\mathrm{S}}$ *Ad-regular* if $\ker \mathrm{Ad}_{\mathbf{G}_{\mathrm{S}}} = \mathbf{Z}(\mathbf{G}_{\mathrm{S}})$, where $\mathrm{Ad}_{\mathbf{G}_{\mathrm{S}}}$ denotes the adjoint representation of $\mathbf{G}_{\mathrm{S}}$ and $\mathbf{Z}(\mathbf{G}_{\mathrm{S}})$ the center of $\mathbf{G}_{\mathrm{S}}$. An Ad-regular $\mathbf{G}_{\mathrm{S}}$ is called *regular* if the orders of all finite subgroups of $\mathbf{G}_{\mathrm{S}}$ do not exceed a constant depending only on $\mathbf{G}_{\mathrm{S}}$.

We show that if two one-parameter subgroups $\mathbf{U}_1 = \{\mathbf{u}_1(t) : t \in \mathbb{Q}_{\mathrm{S}}\}$ and $\mathbf{U}_2 = \{\mathbf{u}_2(t) : t \in \mathbb{Q}_{\mathrm{S}}\}$ of a regular $\mathbf{G}_{\mathrm{S}}$ have the same tangent then $\mathbf{U}_1 = \mathbf{U}_2$ (i.e. $\mathbf{u}_1(t) = \mathbf{u}_2(t)$ for all $t \in \mathbb{Q}_{\mathrm{S}}$).

It is a fact that if κ is a finite extension of $\mathbb{Q}_s$ with an ultrametric s then $GL(n, \kappa)$, $n \in \mathbb{Z}^+$, and its Zariski closed and connected subgroups (viewed as Lie groups over $\mathbb{Q}_s$) are regular.

Also we showed in [Ra15] that if $\mathbf{G}_s$ is a Lie subgroup of a *regular* p_s-adic Lie group then there exists an open subgroup $\mathbf{G}_s^0$ of $\mathbf{G}_s$ such that $\mathbf{G}_s^0$ is *regular* and contains every one-parameter subgroup of $\mathbf{G}_s$. (This implies that if Theorems S1–S6 below hold for $\mathbf{G}_s^0$ in place of $\mathbf{G}_s$ then they hold for $\mathbf{G}_s$. Thus one can reduce these theorems to the case when $\mathbf{G}_s$ is regular for every ultrametric $s \in \mathbb{S}$.)

Henceforth we assume that $\mathbf{G}_s$ is a Lie subgroup of a regular p_s-adic Lie group for every ultrametric $s \in \mathbb{S}$.

THEOREM S1 (Ergodic measures). *Let* $\mathbf{H}$ *be a closed subgroup of* $\mathbf{G}_\mathbb{S}$ *and* $\mathbf{U}$ *a subgroup of* $\mathbf{H}$ *generated by one-parameter Ad-unipotent subgroups of* $\mathbf{G}_\mathbb{S}$. *Then given any discrete subgroup* Γ *of* $\mathbf{H}$ *every ergodic* $\mathbf{U}$*-invariant Borel probability measure on* $\Gamma\backslash\mathbf{H}$ *is algebraic.*

THEOREM S2 (Orbit closures). *Let* $\mathbf{H}$ *and* $\mathbf{U}$ *be as in Theorem* 1. *Then given any lattice* Γ *of* $\mathbf{H}$ *and any* $x \in \Gamma\backslash\mathbf{H}$ *the closure* $\overline{x\mathbf{U}}$ *of the orbit* $x\mathbf{U}$ *in* $\Gamma\backslash\mathbf{H}$ *is homogeneous.*

Theorems S1 and S2 proved in [Ra15, Theorems 1 and 2] extend Theorems 2 and 4 to $\mathbf{G}_\mathbb{S}$. To extend Theorems 6 and 8 we need to introduce the following notation.

Let $\mathbf{H}$ be a closed subgroup of $\mathbf{G}_\mathbb{S}$, Γ a discrete subgroup of $\mathbf{H}$, and $\mathbf{U} = \eta(\mathbf{U}_s) = \{\mathbf{u}(t) : t \in \mathbb{Q}_s\}$, $s \in \mathbb{S}$, a one-parameter Ad-unipotent subgroup of $\mathbf{G}_\mathbb{S}$ contained in $\mathbf{H}$. For $\tau > 0$ let

$$F_s(\tau) = \{t \in \mathbb{Q}_s : |t|_s \leq \tau\},$$

where $|\cdot|_s$ denotes the normalized absolute value on $\mathbb{Q}_s$. When s is ultrametric, $F_s(\tau)$ is a compact open subgroup of $\mathbb{Q}_s$. We denote by λ_s a Haar measure on $\mathbb{Q}_s$.

THEOREM S3 (Uniform Distribution [Ra15, Theorem 3]). *Given any lattice* Γ *of* $\mathbf{H}$ *and any* $x \in \Gamma\backslash\mathbf{H}$ *there exists a closed subgroup* $\mathbf{L}$ *of* $\mathbf{H}$ *such that* $\overline{x\mathbf{U}} = x\mathbf{L}$ *is homogeneous,* $\mathbf{U}$ *acts ergodically on* $(\overline{x\mathbf{U}} = x\mathbf{L}, \nu_\mathbf{L})$, *and*

$$S_f(x, \tau) = \frac{1}{\lambda_s(F_s(\tau))} \int_{F_s(\tau)} f(x\mathbf{u}(t))\, d\lambda_s(t) \to \int_{\Gamma\backslash\mathbf{H}} f\, d\nu_\mathbf{L} = \nu_\mathbf{L}(f) \text{ as } \tau \to \infty,$$

for every bounded continuous function f *on* $\Gamma\backslash\mathbf{H}$.

THEOREM S4 ([Ra15, Theorem 4]). *Let* $x_n \to x \in \Gamma\backslash\mathbf{H}$ *with* Γ *being a lattice in* $\mathbf{H}$. *Suppose there exists no closed nonopen subgroup* $\mathbf{L}$ *of* $\mathbf{H}$ *such that* $\mathbf{U} \subset \mathbf{L}$ *and* $x\mathbf{L}$ *is homogeneous. Then there exists an algebraic measure* ν *on* $\Gamma\backslash\mathbf{H}$ *such that* $\Lambda(\nu)$ *is an open subgroup of* $\mathbf{H}$, $\nu(x\Lambda(\nu)) = 1$, $\mathbf{U}$ *acts ergodically on* $(x\Lambda(\nu), \nu)$, *and*

$$\lim_{n\to\infty} S_f(x_n, \tau_n) = \nu(f)$$

for every bounded continuous function f on $\mathbf{\Gamma}\backslash\mathbf{H}$ and every sequence $\tau_n \to \infty$ when $n \to \infty$. If $\overline{x\mathbf{U}} = \mathbf{\Gamma}\backslash\mathbf{H}$ then ν is **H***-invariant.*

Recall (see Section 1) that $\mathbf{\Lambda}(\nu) = \{\mathbf{h} \in \mathbf{H} : \text{the action of } \mathbf{h} \text{ on } \mathbf{\Gamma}\backslash\mathbf{H} \text{ preserves } \nu\}$.

Theorem S3 follows from Theorem S4 if we set $x_n = x$ for all n and use induction on the dimension of **H**.

Note that Theorem 5 has also been extended to $\mathbf{G}_{\mathbb{S}}$ (see [Ra15, Theorem 1.3]).

Recently Margulis and Tomanov [MT1,2] published a particular case of Theorem S1 when for each $s \in \mathbb{S}$ the group $\mathbf{G}_s$ is the set of κ_s-rational points of an algebraic group defined over a local field κ_s of characteristic zero. (They also formulated Theorem S3 with $\mathbf{H} = \mathbf{G}_{\mathbb{S}}$ and a weaker version of Theorem S2 for this algebraic case.) As does the author's their proof uses in the most essential way the basic ideas and methods from [Ra8,9] (though they give no specific references to [Ra8,9] in [MT2] and no references to [Ra8,9] at all in [MT1]). In fact, for the most part their proof can be viewed as a translation (with modifications and substantial simplifications possible because $\mathbf{G}_s$ are algebraic) of these ideas and methods to the algebraic group setting. The basic Lemma 7.5 in [MT2] uses the fundamental idea from the proofs of [Ra8, Lemma 4.2], [Ra9, Lemma 3.1] of using the polynomial divergence of Ad-unipotent orbits through the normalizer and the ergodic theorem. Also the results in [MT2, Proposition 6.1] and [MT2, Propositions 6.7 and 8.3] are analogous to [Ra8, Theorem 3.1, Lemma 3.3] and [Ra9, Lemma 3.1].

Next we generalize the notion of a "diagonal" subgroup for a one-parameter Ad-unipotent subgroup $\mathbf{U}_S = \{\mathbf{u}(t) : t \in \mathbb{Q}_S\}$, $u = d\mathbf{u}(t)/dt|_{t=0}$. Suppose there is an "opposite" u^* and a "diagonal" $a = \mathrm{ad}_{u^*}(u)$ for u in $\mathfrak{G}_S$ (see Section 1) and let $\mathbf{A}_S$ be a one-dimensional Lie subgroup of $\mathbf{G}_S$ normalizing $\mathbf{U}_S$ whose Lie algebra is spanned by a.

DEFINITION. The group $\mathbf{A}_S$ is called "*diagonal*" for $\mathbf{U}_S$ if there exists a one-parameter Ad-unipotent $\mathbf{U}_S^* = \{\mathbf{u}^*(t) : t \in \mathbb{Q}_S\}$, $d\mathbf{u}^*(t)/dt|_{t=0} = u^*$ normalized by $\mathbf{A}_S$ such that if we denote by $\mathbf{S} = \langle \mathbf{U}_S, \mathbf{U}_S^* \rangle$ the subgroup of $\mathbf{G}_S$ generated by $\mathbf{U}_S, \mathbf{U}_S^*$ then $\mathbf{A}_S \subset \mathbf{S}$ and $\mathrm{Ad}_{\mathbf{G}_S}$ maps $\mathbf{A}_S$ homomorphically onto the multiplicative one-parameter subgroup $\{\mathfrak{a}(\tau) : \tau \in \mathbb{Q}_S^*\}$ of $\mathrm{Ad}_{\mathbf{G}_S}(\mathbf{S})$ with $d\mathfrak{a}(\tau)/d\tau|_{\tau=1} = \mathrm{ad}_a$. Here $\mathbb{Q}_S^* = \mathbb{Q}_S - \{0\}$.

We write $\mathbf{S} = SL_2(\mathbf{U}_S, \mathbf{A}_S)$ and $\mathbf{A}_S = \cup\{\mathbf{A}_S(\tau) : \tau \in \mathbb{Q}_S^*\}$, where

$$\mathbf{A}_S(\tau) = \{\mathbf{a} \in \mathbf{A}_S : \mathrm{Ad}_{\mathbf{a}} = \mathfrak{a}(\tau)\}.$$

Now let $\mathbf{U} = \eta(\mathbf{U}_S) \subset \mathbf{G}_{\mathbb{S}}$. Then we call $\mathbf{A} = \eta(\mathbf{A}_S)$ "diagonal" for $\mathbf{U}$ in $\mathbf{G}_{\mathbb{S}}$ and write $SL_2(\mathbf{U}, \mathbf{A}) = \eta(SL_2(\mathbf{U}_S, \mathbf{A}_S))$.

As in the real case, the central role in the proof of Theorem S1 is played by the following Theorem S5 [Ra15, Theorem 6], which generalizes Theorem 7.

THEOREM S5. *Let* **U** *be a one-parameter* Ad*-unipotent subgroup of $\mathbf{G}_{\mathbb{S}}$ and assume that $\mathbf{G}_{\mathbb{S}}$ contains a "diagonal" subgroup* **A** *for* **U**. *Let* $\mathbf{\Gamma}$ *be a discrete subgroup of $\mathbf{G}_{\mathbb{S}}$ and μ an ergodic* **U***-invariant Borel probability measure on $\mathbf{\Gamma}\backslash\mathbf{G}_{\mathbb{S}}$.*

Then either (1) $\mu(\mathcal{D}(\mathbf{a}(\tau))) = 1$ *for every* $\mathbf{a}(\tau) \in \mathbf{A}(\tau)$ *with* $|\tau| > 1$ *or* (2) $\mathbf{c}SL_2(\mathbf{U}, \mathbf{A})\mathbf{c}^{-1} \subset \Lambda(\mu)$ *for some* $\mathbf{c} \in \mathbf{C}(\mathbf{U})$ *and* μ *is algebraic.*

Recall that $\mathcal{D}(\mathbf{a}(\tau))$ denotes the divergent set of $\mathbf{a}(\tau)$ (see Section 1). It is a fact that if $\mathbb{S}$ is ultrametric then there are no cuspidally divergent sequences in $\Gamma\backslash\mathbf{G}_\mathbb{S}$ (see [Ra13, Proposition 2]). Thus when $\mathbb{S}$ is ultrametric and there is a "diagonal" $\mathbf{A}$ for $\mathbf{U}$, then conclusion (2) holds in Theorem S5. The following theorem shows that the presence of a "diagonal" subgroup for $\mathbf{U}$ is necessary for $\mathbf{U}$ to preserve a finite measure on $\Gamma\backslash\mathbf{G}_\mathbb{S}$.

THEOREM S6. *Assume* $\mathbb{S}$ *is ultrametric. Let* Γ *be a discrete subgroup of* $\mathbf{G}_\mathbb{S}$ *and* $\mathbf{U}$ *a one-parameter subgroup of* $\mathbf{G}_\mathbb{S}$ *preserving a Borel probability measure on* $\Gamma\backslash\mathbf{G}_\mathbb{S}$. *Then there is a "diagonal" subgroup* $\mathbf{A}$ *for* $\mathbf{U}$ *in* $\mathbf{G}_\mathbb{S}$.

COROLLARY S1. *Assume* $\mathbb{S}$ *is ultrametric and* $\mathbf{G}_\mathbb{S}$ *admits a lattice. Then for every* $s \in \mathbb{S}$ *and every one-parameter subgroup* $\mathbf{U}_s$ *of* $\mathbf{G}_s$ *there is a "diagonal"* $\mathbf{A}_s$ *in* $\mathbf{G}_s$.

This corollary can be viewed as a generalization of [T, Theorem 3] stating that if an *algebraic* p-adic group $\mathbf{G}$ admits a lattice then $\mathbf{G}$ is reductive.

Finally, we mention that Theorem S5 allows us to extend Theorem 11 to $\mathbf{G}_\mathbb{S}$ (see [Ra15, Corollary 3]).

5 Applications to Number Theory

The Oppenheim Conjecture

THEOREM O1 (Margulis). *Let* B *be a real nondegenerate indefinite quadratic form in* n *variables,* $n \geq 3$. *Suppose that the ratio of some two coefficients of* B *is irrational. Then the set of values of* B *at integral points is dense in* $\mathbb{R}$.

This is an equivalent version of the Oppenheim Conjecture proved by Margulis [M1] in 1986. (The original Oppenheim Conjecture asserts that zero is a limit point of $B(\mathbb{Z}^n)$.) In fact, it was Raghunathan who noticed that in order to derive this theorem for $n = 3$ one only needs to prove a weaker version of Theorem 4 for $\mathbf{G} = SL(3,\mathbb{R})$, $\Gamma = SL(3,\mathbb{Z})$, and $\mathbf{U} = SO(2,1)^0$. This is precisely what Margulis did. (He also observed that Theorem O1 for $n > 3$ can be reduced to the case $n = 3$.) Subsequently he and Dani [DM1,3] showed that the values of B at the primitive elements of $\mathbb{Z}^n$ are dense in $\mathbb{R}$. In 1990–91 Borel and Prasad [BoPr] obtained a remarkable strengthening of this fact, implied by Theorem 4.

THEOREM O2 (Borel, Prasad [BoPr]). *Let* B *be as in Theorem* O1. *Then given* $c_1, \ldots, c_{n-1} \in \mathbb{R}$ *and* $\varepsilon > 0$ *there are* $x_1, \ldots, x_{n-1} \in \mathbb{Z}^n$ *that are part of a basis in* $\mathbb{Z}^n$ *(and hence are primitive elements of* $\mathbb{Z}^n$*) such that* $|B(x_i) - c_i| < \varepsilon$ *for all* $i = 1, \ldots, n-1$.

In fact, Borel and Prasad [BoPr] have generalized the Oppenheim Conjecture and Theorem O2 to the following more general setting.

Let κ be a number field and $\mathfrak{o}$ the ring of integers of κ. For every normalized absolute value $|\cdot|_v$ on κ, let κ_v be the completion of κ at v. Let $\mathbb{S}$ be a finite set of places of κ containing the set $\mathbb{S}_\infty$ of archimedean ones, $\kappa_\mathbb{S}$ the direct sum of the fields κ_s, $s \in \mathbb{S}$, and $\mathfrak{o}_{\mathbb{S}}$ the ring of $\mathbb{S}$-integers of κ (i.e. of elements $x \in \kappa$ with $|x|_v \leq 1$ for all $v \notin \mathbb{S}$).

A quadratic form F on $\kappa_{\mathbb{S}}^n$ is a collection (F_s), $s \in \mathbb{S}$, where F_s is a quadratic form on κ_s^n. The form is nondegenerate if and only if each F_s is nondegenerate. The form is isotropic if each F_s is so, i.e. if there exists for each $s \in \mathbb{S}$ an element $x_s \in \kappa_s^n - \{0\}$ such that $F_s(x_s) = 0$. If s is a real place, this condition is equivalent to F_s being indefinite. The form F is said to be *rational* if there exists a unit $\lambda = (\lambda_s) \in \kappa_{\mathbb{S}}^*$ and a form F_0 on κ^n such that $F_s = \lambda_s F_0$ for all $s \in \mathbb{S}$, and *irrational* otherwise.

THEOREM O3 (Borel, Prasad [BoPr, Theorem A]). *Let F be irrational, nondegenerate, and isotropic, and $n \geq 3$. Then given $\varepsilon > 0$ there exists $x \in \mathfrak{o}_{\mathbb{S}}^n$ such that $0 < |F_s(x)| < \varepsilon$ for all $s \in \mathbb{S}$.*

THEOREM O4 (Borel, Prasad [BoPr, Corollary 7.9]). *Assume $\mathbb{S} = \mathbb{S}_\infty$ and let F be as in Theorem O3. Let $\lambda_1, \ldots, \lambda_{n-1} \in \kappa_{\mathbb{S}}$. Then for each $j = 1, 2, \ldots$ there are $x_{j,1}, \ldots, x_{j,n-1} \in \mathfrak{o}^n = \mathfrak{o}_{\mathbb{S}}^n$ that are part of a basis of $\mathfrak{o}^n$ over $\mathfrak{o}$ (and hence are primitive elements of $\mathfrak{o}^n$) such that $\lim_{j\to\infty} F(x_{j,i}) = \lambda_i$ for all $i = 1, \ldots, n-1$. In particular, the set of values of F on the primitive elements of $\mathfrak{o}^n$ is dense in $\kappa_{\mathbb{S}}$.*

Theorems O3 and O4 in [BoPr] are deduced by means of Theorem 4, geometry of numbers, and strong approximation in algebraic groups. In [BoPr] Borel and Prasad pointed out that the density of $F(\mathfrak{o}_{\mathbb{S}}^n)$ (and Theorem O4) for non-archimedean $\mathbb{S}$ would follow from the $\mathbb{S}$-arithmetic version of Theorem 4 (see Theorem S2 above). Indeed, the deduction of Theorem O5 below from Theorem S2 is given in [Bo].

THEOREM O5 (Borel, Prasad). *Theorem O4 holds also for non-archimedean $\mathbb{S}$ with $\mathfrak{o}$ replaced by $\mathfrak{o}_{\mathbb{S}}$.*

To illustrate the connection between the Oppenheim conjecture and the orbit closures Theorem 4 let us present the deduction of Theorem O1 from Theorem 4. This deduction is a simplified version of the argument originally given by Raghunathan.

Let B be a quadratic form as specified in Theorem O1. Also let $\mathbf{G} = SL(n, \mathbb{R})$, $\mathbf{\Gamma} = SL(n, \mathbb{Z})$, and $L = \mathbb{Z}^n$ be the lattice of integral points in $\mathbb{R}^n$. Let $L\mathbf{g}$ denote the lattice in $\mathbb{R}^n$ obtained by applying the linear transformation $\mathbf{g} \in \mathbf{G}$ to L. Then $X = \{L\mathbf{g} : \mathbf{g} \in \mathbf{G}\}$ is a set of lattices endowed with the natural Hausdorff topology. Note that if $\boldsymbol{\gamma} \in \mathbf{\Gamma}$ then $L\boldsymbol{\gamma} = L$. This says that we can identify $L\mathbf{g} \in X$ with the coset $\mathbf{\Gamma g} \in \mathbf{\Gamma}\backslash\mathbf{G}$. The identification $L\mathbf{g} \to \mathbf{\Gamma g}$ is a homeomorphism from X onto $\mathbf{\Gamma}\backslash\mathbf{G}$.

Now let $\mathbf{H}$ denote the subgroup of $\mathbf{G}$ preserving the quadratic form B. Then $\mathbf{H}$ is conjugate to $SO(p,q)$, $p+q = n$, $pq \neq 0$, and hence consists of two connected components. Also $\mathbf{H}^0$ is generated by unipotent elements of $\mathbf{G}$ (because $n > 2$). For each $\mathbf{h} \in \mathbf{H}$ the set of values of B on $L\mathbf{h}$ is the same as on L. To prove Theorem O1 it suffices to show that the orbit $L\mathbf{H}^0$ is dense in X or, equivalently, the orbit $z\mathbf{H}^0$ is dense in $\mathbf{\Gamma}\backslash\mathbf{G}$, where $z = \mathbf{\Gamma e}$ and $\mathbf{e}$ denotes the identity element of $\mathbf{G}$.

By Theorem 4 the closure $\overline{z\mathbf{H}^0} = z\mathbf{F}$ is homogeneous for some closed connected subgroup $\mathbf{F} \subset \mathbf{G}$, containing $\mathbf{H}^0$. But the only closed connected subgroups of $\mathbf{G}$ containing $\mathbf{H}^0$ are $\mathbf{G}$ and $\mathbf{H}^0$. Hence either $\mathbf{F} = \mathbf{G}$ or $\mathbf{F} = \mathbf{H}^0$.

We have to show that $\mathbf{F} = \mathbf{G}$, i.e. $\overline{z\mathbf{H}^0} = \mathbf{\Gamma}\backslash\mathbf{G}$. Suppose to the contrary that $\mathbf{F} = \mathbf{H}^0$. Then $\mathbf{H}^0 \cap \mathbf{\Gamma}$ is a lattice in $\mathbf{H}^0$ (by Definition 1) and hence $\mathbf{H} \cap \mathbf{\Gamma}$ is a lattice in $\mathbf{H}$ because $\mathbf{H}/\mathbf{H}_0$ is finite. Because $\mathbf{\Gamma} = SL(n, \mathbb{Z})$, it follows from the Borel density theorem that $\mathbf{H}$ is a $\mathbb{Q}$-subgroup of $\mathbf{G}$. This means that $\mathbf{H}$ is the set of real zeros of some ring of polynomials with rational coefficients. This implies by an elementary argument that B is proportional to a quadratic form with rational coefficients. This contradicts the conditions of Theorem O1 and proves the Theorem.

This proof shows that $L\mathbf{H}$ is dense in X. This is used to prove Theorem O2. Indeed, given $c_1, \dots, c_{n-1} \in \mathbb{R}$ there is a unimodular basis $y_1, \dots, y_n \in \mathbb{R}^n$ such that $B(y_i) = c_i$ for all $i = 1, \dots, n-1$ (because the level surface $B(x) = c$, $x \in \mathbb{R}^n$ is not contained in any hyperplane). Then the $\mathbb{Z}$-span of this basis belongs to X. Hence there are $x_1, \dots, x_n \in L$ and $\mathbf{h} \in \mathbf{H}$ such that $x_1\mathbf{h}, \dots, x_n\mathbf{h}$ are close to $y_1, \dots, y_n$ and $|B(x_i\mathbf{h}) - c_i| < \varepsilon$ for all $i = 1, \dots, n-1$. This implies that $x_1, \dots, x_n$ form a basis in L and $|B(x_i) - c_i| < \varepsilon$ for all $i = 1, \dots, n-1$, because $B(x_i) = B(x_i\mathbf{h})$, $i = 1, \dots, n$. This yields Theorem O2. (This proof is given in [BoPr, Proof of Corollary 7.9].)

Finally we mention the following problem. Let B be a quadratic form as in Theorem O1. Given $0 < a < b$ and $r > 0$, let $E_r(a,b) = \{x \in \mathbb{Z}^n : a \leq |B(x)| < b, \|x\| < r\}$. Then card $E_r(a,b) \to \infty$ when $r \to \infty$. It seems plausible that Theorems 1, 5 can be used to find the asymptotic growth rate for this number. It is believed that card $E_r(a,b) \sim c(a,b)r^{n-2}$ for some $c(a,b) > 0$. A lower bound of this type has already been found by Dani and Margulis in [DM4]. Also recently Eskin and Mozes have informed the author that using the latter lower bound they can prove this asymptotic growth for $n = 4$ and B of the signature (3,1) (and disprove it for the signature (2,2) and the case $n = 3$) and with a modification suggested by Margulis the proof works for any B with $n > 4$.

Counting Integral Points on Homogeneous Varieties

The discussion in this section is related to the following problem recently studied in [DuRuSa] and [EsMc].

Let W be a real finite-dimensional vector space and let $W(\mathbb{Q})(W(\mathbb{Z}))$ denote the set of all vectors in W with rational (integer) coordinates relative to a fixed basis in W. Let $\mathfrak{G}$ be a connected algebraic reductive group defined over $\mathbb{Q}$ and let $\mathfrak{G}(\mathbb{R})$ denote the set of real points of $\mathfrak{G}$, i.e. the set of real zeros of the polynomials defining $\mathfrak{G}$. Similarly, one defines $\mathfrak{G}(\mathbb{Q})$ and $\mathfrak{G}(\mathbb{Z})$. We assume that $\mathfrak{G}$ is homomorphic via a surjective morphism ρ defined over $\mathbb{Q}$ to an algebraic subgroup of $GL(\mathfrak{W})$ defined over $\mathbb{Q}$. (Here $\mathfrak{W}$ denotes the complexification of W.) Then $\mathfrak{G}(\mathbb{R})$ acts linearly on W by $w \to w\mathbf{g} = \rho(\mathbf{g})(w)$, $w \in W$, $\mathbf{g} \in \mathfrak{G}(\mathbb{R})$, and $\mathfrak{G}(\mathbb{Q})$ preserves $W(\mathbb{Q})$.

Now let $V \subset W$ be the set of real points of an affine subvariety of $\mathfrak{W}$ defined over $\mathbb{Q}$. Assume that V has finitely many connected components and $\mathbf{G} = \mathfrak{G}(\mathbb{R})^0$ acts transitively on each of these components. Suppose there exists $v_0 \in V(\mathbb{Z}) =$

$W(\mathbb{Z}) \cap V$ and let $\mathbf{H} = \{\mathbf{g} \in \mathbf{G} : v_0\mathbf{g} = v_0\}$ be the stabilizer of v_0 in $\mathbf{G}$. It is a fact that $\mathbf{H} = \mathfrak{H}(\mathbb{R}) \cap \mathbf{G}$ for some reductive algebraic subgroup $\mathfrak{H}$ of $\mathfrak{G}$ defined over $\mathbb{Q}$. Suppose $\mathfrak{G}$ and $\mathfrak{H}$ do not admit nontrivial characters defined over $\mathbb{Q}$. Then $\mathfrak{G}(\mathbb{Z}) \cap \mathbf{G}$ is a lattice in $\mathbf{G}$ and $\mathfrak{G}(\mathbb{Z}) \cap \mathbf{H}$ a lattice in $\mathbf{H}$. Let $\mathbf{\Gamma}$ be a subgroup of finite index in $\mathfrak{G}(\mathbb{Z}) \cap \mathbf{G}$ whose action on W preserves $W(\mathbb{Z})$.

We denote by $\| \quad \|$ a norm on W and for $T > 0$ define $B_T = \{w \in W : \|w\| < T\}$. We are interested in the asymptotics of the number of points in $V(\mathbb{Z}) \cap B_T$ as $T \to \infty$. The group $\mathbf{\Gamma}$ acts on $V(\mathbb{Z})$ and it was shown in [BoHC] that $V(\mathbb{Z})$ consists of finitely many $\mathbf{\Gamma}$-orbits. Thus it suffices to know the asymptotics of the number $N(T, V, \mathcal{O})$ of points in $\mathcal{O} \cap B_T$, where $\mathcal{O} = v_0\mathbf{\Gamma}$.

Theorem C1 below recently proved by Eskin, Mozes and Shah [EsMoS] generalizes an earlier result of Duke, Rudnick, and Sarnak [DuRuSa] where an asymptotic of $N(T, V, \mathcal{O})$ was first found. To state the theorem we need the following definition.

Let $\{E_n\}$, $n = 1, 2, \dots$, be an increasing sequence of open subsets of $\mathbf{H}\backslash\mathbf{G} = \cup\{E_n,\ n = 1, 2, \dots\}$ and let $\hat{E}_n$ denote the natural lifting of E_n to $\mathbf{H}^0\backslash\mathbf{G}$. Also let λ denote the $\mathbf{G}$-invariant volume on $\mathbf{H}^0\backslash\mathbf{G}$, p the natural projection from $\mathbf{G}$ onto $\mathbf{H}^0\backslash\mathbf{G}$, and $\mathfrak{H}^0$ the Zariski closure of $\mathbf{H}^0$.

DEFINITION. [EsMoS] The sequence $\{E_n\}$ is said *to be focused* in $\mathbf{H}\backslash\mathbf{G}$ as $n \to \infty$ if there is a compact $\mathbf{C} \subset \mathbf{G}$ and a proper $\mathbb{Q}$-subgroup $\mathfrak{L}$ of $\mathfrak{G}$ containing $\mathfrak{H}^0$ such that

$$\limsup_{n\to\infty} \frac{\lambda(\mathrm{p}((\mathbf{Z}_{\mathbf{G}}(\mathbf{H}^0) \cap \mathbf{\Gamma})\mathbf{L}\mathbf{C}) \cap \hat{E}_n)}{\lambda(\hat{E}_n)} > 0,$$

where $\mathbf{L} = \mathfrak{L}(\mathbb{R})^0$ and $\mathbf{Z}_{\mathbf{G}}(\mathbf{H}^0)$ denotes the centralizer of $\mathbf{H}^0$ in $\mathbf{G}$.

THEOREM C1 [EsMoS]. *Suppose that every $\mathbb{Q}$-subgroup of $\mathfrak{G}$ containing $\mathfrak{H}^0$ is reductive and for every sequence $T_n \uparrow \infty$ the sequence $R_{T_n} = \{\mathbf{Hg} : v_0\mathbf{g} \in B_{T_n}\}$ is not focused in $\mathbf{H}\backslash\mathbf{G}$. Then*

$$N(T, V, \mathcal{O}) \sim \frac{\mathrm{vol}_{\mathbf{H}}((\mathbf{H} \cap \mathbf{\Gamma})\backslash\mathbf{H})}{\mathrm{vol}_{\mathbf{G}}(\mathbf{\Gamma}\backslash\mathbf{G})} \mathrm{vol}_{\mathbf{H}\backslash\mathbf{G}}(R_T) \tag{1}$$

as $T \to \infty$, where the volumes in (1) *are induced by a left invariant Riemannian metric on* $\mathbf{G}$.

COROLLARY C1 [EsMoS]. *Suppose $\mathfrak{H}^0$ is a maximal proper connected $\mathbb{Q}$-subgroup of $\mathfrak{G}$. Then relation* (1) *holds for $N(T, V, \mathcal{O})$ as $T \to \infty$.*

For the particular case when $\mathfrak{H}$ is an affine symmetric subgroup of $\mathfrak{G}$ (i.e. $\mathfrak{H}$ is a fixed point set of an involution of $\mathfrak{G}$ defined over $\mathbb{Q}$) Corollary C1 was proved earlier in [DuRuSa] by other methods (subsequently a simpler proof appeared in [EsMc]).

To give an application of Theorem C1 the authors of [EsMoS] denote by $M_n(\mathbb{Z})$ the set of all $n \times n$ integer matrices and consider the set

$$V_p(\mathbb{Z}) = \{A \in M_n(\mathbb{Z}) : \det(tI - A) = p(t)\},$$

where $p(t)$ denotes a monic polynomial of degree n with integer coefficients irreducible over $\mathbb{Q}$. Theorem C1 implies that

$$N(T, V_p) \sim c_p T^{n(n-1)/2}$$

for some $c_p > 0$, where $N(T, V_p)$ denotes the number of points in $V_p(\mathbb{Z})$ of Hilbert-Schmidt norm less than T.

Theorem C1 is deduced in [EsMoS] from the results on the limit behavior of algebraic measures under translations. More specifically, let $x\mathbf{H}, x = \mathbf{\Gamma e}$ be a homogeneous subset of $\mathbf{\Gamma}\backslash\mathbf{G}$ with $\mathbf{\Gamma}$ being a lattice in a Lie group $\mathbf{G}$ and let $\{\mathbf{g}_i\}_{i\in\mathbb{N}}$ be a sequence in $\mathbf{G}$. We denote by $\nu_{\mathbf{H}}\mathbf{g}_i$ the $\mathbf{g}_i^{-1}\mathbf{H}\mathbf{g}_i$-invariant probability measure supported on $x\mathbf{H}\mathbf{g}_i$. One asks what are the possible weak* limits of the sequence $\{\nu_{\mathbf{H}}\mathbf{g}_i\}_{i\in\mathbb{N}}$ and, in particular, when does the sequence converge to the $\mathbf{G}$-invariant probability measure $\nu_{\mathbf{G}}$?

Using Theorems 1 and 5 and the linearization method of [DM4], Eskin, Mozes, and Shah [EsMoS] showed that if $\mathbf{G}$, $\mathbf{H}$, and $\mathbf{\Gamma}$ satisfy the conditions of Theorem C1 and the sequence $\{\nu_{\mathbf{H}^0}\mathbf{g}_i\}$ weak* converges to a probability measure ν then $\nu = \nu_{\mathbf{L}}\mathbf{c}$ for some $\mathbf{c} \in \mathbf{G}$, where $\nu_{\mathbf{L}}$ is the $\mathbf{L}$-invariant probability measure supported on a homogeneous set $x\mathbf{L}$ with $\mathbf{L} = \mathfrak{L}(\mathbb{R})^0$ for some reductive $\mathbb{Q}$-subgroup $\mathfrak{L}$ of $\mathfrak{G}$ containing $\mathfrak{H}^0$. Also they proved the following

THEOREM C2 [EsMoS]. *Let* $\mathbf{G}$, $\mathbf{H}$, *and* $\mathbf{\Gamma}$ *be as in Theorem C1 and let* $\{E_n\}$ *be an increasing sequence of open subsets of* $\mathbf{H}\backslash\mathbf{G} = \cup\{E_n : n = 1, 2, \dots\}$. *Suppose that* $\{E_n\}$ *is not focused in* $\mathbf{H}\backslash\mathbf{G}$ *as* $n \to \infty$. *Then given any* $\varepsilon > 0$ *there exists an open set* $A \subset \mathbf{H}\backslash\mathbf{G}$ *such that*

$$\liminf_{n\to\infty} \frac{\mathrm{vol}_{\mathbf{H}\backslash\mathbf{G}}(A \cap E_n)}{\mathrm{vol}_{\mathbf{H}\backslash\mathbf{G}}(E_n)} > 1 - \varepsilon$$

and $\{\nu_{\mathbf{H}}\mathbf{g}_i\}$ *weak* converges to* $\nu_{\mathbf{G}}$ *for every sequence* $\{\mathbf{g}_i\}$ *with* $\{\mathbf{H}\mathbf{g}_i\}$ *being a sequence in* A *containing no subsequences convergent in* $\mathbf{H}\backslash\mathbf{G}$.

COROLLARY C2 [EsMoS]. *Suppose* $\mathfrak{H}^0$ *is a maximal connected* $\mathbb{Q}$*-subgroup of* $\mathfrak{G}$ *and let* $\{\mathbf{g}_i\}$ *be a sequence in* $\mathbf{G}$ *such that* $\{\mathbf{H}\mathbf{g}_i\}$ *contains no subsequences convergent in* $\mathbf{H}\backslash\mathbf{G}$. *Then* $\{\nu_{\mathbf{H}}\mathbf{g}_i\}$ *weak* converges to* $\nu_{\mathbf{G}}$.

To deduce Theorem C1 from Theorem C2 one denotes by χ_T the characteristic function of the ball B_T, and for $\mathbf{g} \in \mathbf{G}$ one defines

$$F_T(\mathbf{g}) = \sum \{\chi_T(v_0\boldsymbol{\gamma}\mathbf{g}) : \boldsymbol{\gamma} \in \mathbf{H} \cap \mathbf{\Gamma}\backslash\mathbf{\Gamma}\}.$$

Then F_T is a function on $\mathbf{\Gamma}\backslash\mathbf{G}$, as $F_T(\mathbf{g}) = F_T(\boldsymbol{\gamma}\mathbf{g})$ for all $\boldsymbol{\gamma} \in \mathbf{\Gamma}$. Also $F_T(\mathbf{e}) = N(V, T, \mathcal{O})$. Defining

$$\hat{F}_T(\mathbf{g}) = \frac{\mathrm{vol}(\mathbf{\Gamma}\backslash\mathbf{G})}{\mathrm{vol}(\mathbf{H} \cap \mathbf{\Gamma}\backslash\mathbf{H})\,\mathrm{vol}_{\mathbf{H}\backslash\mathbf{G}}(R_T)} F_T(\mathbf{g})$$

and using Theorem C2, Eskin, Mozes, and Shah showed (following the method of [DuRuSa]) that $\hat{F}_T(\mathbf{g}) \to 1$ weakly and pointwise on $\mathbf{\Gamma}\backslash\mathbf{G}$. In particular, $\hat{F}_T(\mathbf{e}) \to 1$. This yields Theorem C1.

6 Applications to Ergodic Theory and the "Rigidity" Phenomenon of Ad-unipotent Actions

The central problem of ergodic theory is that of classifying measure preserving (m.p.) transformations or flows up to isomorphism.

More precisely, let T and S be two m.p. transformations on probability spaces (X, μ_X) and (Y, μ_Y) respectively. We say that S is a *factor* of T if there is a m.p. ψ from X onto Y such that $\psi(T(x)) = S(\psi(x))$ for μ_X-almost every (a.e.) $x \in X$. If ψ is invertible, then T and S are called *isomorphic* and ψ is called an *isomorphism* between T and S. Similarly, one defines factors and isomorphisms of m.p. flows. One asks what m.p. transformations (or flows) are isomorphic? And what are the possible isomorphisms between T and S?

To approach this problem one looks for properties stable under isomorphisms. There are a number of dynamical properties of this kind, characterizing the degree of randomness of the system. There is also a *numerical* invariant of isomorphism called the *entropy*, which plays an important role in ergodic theory.

The definition of entropy will not be discussed, but we shall only mention that if an element $\mathbf{g}$ of a real Lie group $\mathbf{G}$ acts on $(\mathbf{\Gamma}\backslash\mathbf{G}, \nu_{\mathbf{G}})$, where $\nu_{\mathbf{G}}$ is the $\mathbf{G}$-invariant Borel probability measure on $\mathbf{\Gamma}\backslash\mathbf{G}$, then the entropy of this action is given by $\mathcal{E}(\mathbf{g}) = \sum\{\log|\lambda| : \lambda \text{ is an eigenvalue of } \mathrm{Ad}_{\mathbf{g}} \text{ with } |\lambda| > 1\}$, where the eigenvalues are counted with multiplicities. Thus if $\mathbf{u}$ is Ad-unipotent then $\mathcal{E}(\mathbf{u}) = 0$.

An element $\mathbf{g} \in \mathbf{G}$ is called Ad-semisimple if $\mathrm{Ad}_{\mathbf{g}}$ is diagonalizable over $\mathbb{C}$. The following theorem solves the isomorphism problem for Ad-semisimple actions.

THEOREM E1 (Ornstein, Weiss, Dani). *Let $\mathbf{G}_i$, $i = 1, 2$, be two real connected Lie groups. For each i let $\mathbf{\Gamma}_i$ be a lattice in $\mathbf{G}_i$, $\mathbf{g}^{(i)}$ an* Ad-*semisimple element of $\mathbf{G}_i$ with $\mathrm{Ad}_{\mathbf{g}^{(i)}}$ having at least one eigenvalue $|\lambda| \neq 1$. Suppose that $\mathbf{g}^{(i)}$ acts ergodically on $(M_i = \mathbf{\Gamma}_i\backslash\mathbf{G}_i, \nu_{\mathbf{G}_i})$, $i = 1, 2$. Then the actions of $\mathbf{g}^{(1)}$ and $\mathbf{g}^{(2)}$ are isomorphic if and only if $\mathcal{E}(\mathbf{g}^{(1)}) = \mathcal{E}(\mathbf{g}^{(2)})$.*

This theorem is proved by showing that the actions of $\mathbf{g}^{(1)}$ and $\mathbf{g}^{(2)}$ are isomorphic to Bernoulli shifts and then using Ornstein's isomorphism theorem, which states that Bernoulli shifts with the same entropy are isomorphic. Thus the isomorphism problem for actions of Ad-semisimple elements depends only on the entropy of these actions, hence only on the eigenvalues of $\mathrm{Ad}_{\mathbf{g}^{(1)}}$ and $\mathrm{Ad}_{\mathbf{g}^{(2)}}$. Neither $\mathbf{G}_1$, $\mathbf{G}_2$, nor $\mathbf{\Gamma}_1$, $\mathbf{\Gamma}_2$ play any significant role in the problem. Also one can show that there are uncountably many isomorphisms between isomorphic Ad-semisimple actions.

The following "rigidity" theorem, which can be deduced from our Theorem 1 demonstrates the profoundly different behavior of the actions of Ad-*unipotent* elements.

THEOREM E2 (Rigidity Theorem). *Let $\mathbf{G}_i$ be as above and let $\mathbf{\Gamma}_i$ be a lattice in $\mathbf{G}_i$ containing no nontrivial normal subgroups of $\mathbf{G}_i$, $i = 1, 2$. Let $\mathbf{u}^{(i)}$ be an* Ad-*unipotent element of $\mathbf{G}_i$, $i = 1, 2$. Suppose that the action of $\mathbf{u}^{(1)}$ on $(M_1, \nu_{\mathbf{G}_1})$ is ergodic and there is a m.p. $\psi : (M_1, \nu_{\mathbf{G}_1}) \to (M_2, \nu_{\mathbf{G}_2})$ such that $\psi(x\mathbf{u}^{(1)}) =$*

$\psi(x)\mathbf{u}^{(2)}$ for $\nu_{\mathbf{G}_1}$-a.e. $x \in M_1$. Then there is $\mathbf{c} \in \mathbf{G}_2$ and a surjective homomorphism $\alpha : \mathbf{G}_1 \to \mathbf{G}_2$ such that $\alpha(\mathbf{\Gamma}_1) \subset \mathbf{c}^{-1}\mathbf{\Gamma}_2\mathbf{c}$ and $\psi(\mathbf{\Gamma}_1\mathbf{h}) = \mathbf{\Gamma}_2\mathbf{c}\alpha(\mathbf{h})$ for $\nu_{\mathbf{G}_1}$-a.e. $\mathbf{\Gamma}_1\mathbf{h} \in M_1$. Also α is a local isomorphism whenever ψ is finite to one or $\mathbf{G}_1$ is simple and it is an isomorphism whenever ψ is one-to-one or $\mathbf{G}_1$ is simple with trivial center.

Note that $\mathcal{E}(\mathbf{u}^{(1)}) = \mathcal{E}(\mathbf{u}^{(2)}) = 0$. This theorem says in particular that if the actions of Ad-unipotent elements $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ are isomorphic then $\mathbf{G}_1$ must be isomorphic to $\mathbf{G}_2$ and $\mathbf{\Gamma}_1$ to $\mathbf{\Gamma}_2$.

The action of the unipotent group $\mathbf{H} = \{\mathbf{h}(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} : t \in \mathbb{R}\}$ on $(M = \mathbf{\Gamma}\backslash\mathbf{G}, \nu_{\mathbf{G}})$, $\mathbf{G} = SL(2,\mathbb{R})$ is called the horocycle flow on M and the action of the diagonal group $\mathbf{A} = \{\mathbf{a}(t) = \begin{bmatrix} e^t & \\ & e-t \end{bmatrix} : t \in \mathbb{R}\}$ the geodesic flow on M.

Theorem E2 for horocycle flows was proved in [Ra2] in 1981. Then using the method of [Ra2] and [Ra4] Witte [W1] extended it to any connected $\mathbf{G}_1, \mathbf{G}_2$ and Ad-unipotent $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}$ and furthermore to any mixing zero entropy affine maps [W2]. (Theorem E2 for nilpotent $\mathbf{G}_1, \mathbf{G}_2$ was proved earlier by Parry [P2].)

The proof in [Ra2] of the rigidity theorem for horocycle flows uses the polynomial divergence of horocycle orbits and the commutation relation with the geodesic flow. Generalizing this method, Feldman and Ornstein [FO] extended the rigidity theorem to horocycle flows on the unit tangent bundles of compact surfaces of variable negative curvature. Also generalizations to higher-dimensional hyperbolic space, to horospherical foliations and to geometrically finite groups were given by Flaminio [Fl], by Witte [W3], and by Flaminio and Spatzier [FlSp].

In fact, Theorem E2 is a consequence of a far stronger "Joinings Theorem" implied by Theorem 1. More specifically, let T and S be as above and let μ be a $T \times S$ invariant probability measure on $X \times Y$. Then μ is called a *joining* of T and S if $\mu(A \times Y) = \mu_X(A)$, $\mu(X \times B) = \mu_Y(B)$ for all measurable subsets $A \subset X$, $B \subset Y$. The joining $\mu_X \times \mu_Y$ is called the *trivial* joining. T and S are called *disjoint* if they have no nontrivial joinings.

It follows from Theorem 1 that every ergodic joining μ of the actions of Ad-unipotent elements $\mathbf{u}^{(1)}$ on $(M_1, \nu_{\mathbf{G}_1})$ and $\mathbf{u}^{(2)}$ on $(M_2, \nu_{\mathbf{G}_2})$ is algebraic. Thus $\mu(x\mathbf{\Lambda}(\mu)) = 1$ for some $x \in M_1 \times M_2$ (see Definition 2). Also $\mathbf{\Lambda}(\mu)$ is a subgroup of $\mathbf{G}_1 \times \mathbf{G}_2$ and the groups $\mathbf{\Lambda}_1(\mu)$ and $\mathbf{\Lambda}_2(\mu)$ defined by

$$\mathbf{\Lambda}_1(\mu) = \{\mathbf{h} \in \mathbf{G}_1 : (\mathbf{h}, \mathbf{e}) \in \mathbf{\Lambda}(\mu)\}$$
$$\mathbf{\Lambda}_2(\mu) = \{\mathbf{h} \in \mathbf{G}_2 : (\mathbf{e}, \mathbf{h}) \in \mathbf{\Lambda}(\mu)\}$$

are closed normal subgroups of $\mathbf{G}_1$ and $\mathbf{G}_2$ respectively. For $\mathbf{c} \in \mathbf{G}_2$ write $\mathbf{\Gamma}_2^{\mathbf{c}} = \{\gamma\mathbf{\Lambda}_2(\mu) : \gamma \in \mathbf{c}^{-1}\mathbf{\Gamma}_2\mathbf{c}\}$ and for $z \in M_1$ let

$$\xi_\mu(z) = \{y \in M_2 : (z, y) \in x\mathbf{\Lambda}(\mu)\}.$$

The set $\xi_\mu(z)$ is called the z-fiber of μ. We showed in [Ra9, Theorem 2] that there is $\mathbf{c} \in \mathbf{G}_2$ and a continuous surjective homomorphism $\alpha : \mathbf{G}_1 \to \mathbf{G}_2/\mathbf{\Lambda}_2(\mu)$ with kernel $\mathbf{\Lambda}_1(\mu)$, $\alpha(\mathbf{u}^{(1)}) = \mathbf{u}^{(2)}\mathbf{\Lambda}_2(\mu)$ such that

$$\xi_\mu(\mathbf{\Gamma}_1\mathbf{h}) = \{\mathbf{\Gamma}_2\mathbf{c}\boldsymbol{\beta}_i\alpha(\mathbf{h}) : i = 1, \ldots, n\} \tag{2}$$

for all $\mathbf{h} \in \mathbf{G}_1$, where the intersection $\mathbf{\Gamma}_0 = \alpha(\mathbf{\Gamma}_1) \cap \mathbf{\Gamma}_2^c$ is of finite index in $\alpha(\mathbf{\Gamma}_1)$ and in $\mathbf{\Gamma}_2^c$, $n = |\mathbf{\Gamma}_0 \backslash \alpha(\mathbf{\Gamma}_1)|$ and $\alpha(\mathbf{\Gamma}_1) = \{\mathbf{\Gamma}_0 \beta_i : i = 1, \dots, n\}$. This implies the following

THEOREM E3 [Ra9, Corollary 5] (The Joinings Theorem). *Every ergodic joining μ of the actions of* Ad-*unipotent elements $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ is algebraic and the fibers of μ are given by (2). If, in addition, $\mathbf{G}_i$ is simple, $i = 1, 2$, and μ is nontrivial, then every fiber of μ is finite and $\mathbf{G}_1$ and $\mathbf{G}_2$ are locally isomorphic.*

Thus if $\mathbf{G}_1$ and $\mathbf{G}_2$ are simple and not locally isomorphic, then the actions of $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ are disjoint.

The Joinings Theorem for horocycle flows was proved earlier in [Ra4, Theorem 6]. We showed there that if μ is a nontrivial ergodic joining of the horocycle flows $h_t^{(i)}$ on $M_i = \mathbf{\Gamma}_i \backslash SL(2, \mathbb{R})$, $h_t^{(i)}(x) = x\mathbf{h}(t)$, $x \in M_i$, $i = 1, 2$, then the flow $h_t^{(1)} \times h_t^{(2)}$ on $(M_1 \times M_2, \mu)$ is isomorphic to the horocycle flow on $\mathbf{\Gamma}_0 \backslash SL(2, \mathbb{R})$, where $\mathbf{\Gamma}_0$ is a subgroup of finite index in $\mathbf{\Gamma}_1$ and in $\mathbf{c}^{-1}\mathbf{\Gamma}_2\mathbf{c}$ for some $\mathbf{c} \in SL(2, \mathbb{R})$. This shows that the number of nonisomorphic ergodic joinings of the horocycle flows on M_1 and M_2 is at most countable and if $\mathbf{\Gamma}_1$ is uniform and $\mathbf{\Gamma}_2$ is not or if $\mathbf{\Gamma}_1$ is arithmetic and $\mathbf{\Gamma}_2$ is not then the horocycle flows are disjoint (see [Ra5] for more on this).

The central role in the proof of [Ra4, Theorem 6] is played by a dynamical property of horocycle flows, which we introduced in [Ra4, Definition 1] and called the H-property. It is a consequence of the polynomial divergence of horocycle orbits.

The H-property states that given $0 < \varepsilon < 1$ and $p, N > 0$ there are $\delta(\varepsilon, p, N)$, $\alpha(\varepsilon) \in (0, 1)$ such that if $d_{\mathbf{G}}(\mathbf{x}, \mathbf{e}) < \delta(\varepsilon, p, N)$ for some $\mathbf{x} \in \mathbf{G} = SL(2, \mathbb{R})$ and $\mathbf{x} \notin \mathbf{H}$ then there are $L, T > 0$ with $N < L < T$, $L \geq \alpha(\varepsilon)T$ such that either $d_{\mathbf{G}}(\mathbf{x}\mathbf{h}(t), \mathbf{h}(t + p)) \leq p\varepsilon$ for all $t \in [T - L, T]$ or $d_{\mathbf{G}}(\mathbf{x}\mathbf{h}(t), \mathbf{h}(t - p)) \leq p\varepsilon$ for all $t \in [T - L, T]$.

The H-property was proved in [Ra3, Lemma 2.1]. The latter proof also implies the following more general form of the H-property.

Given small $\theta, \varepsilon \in (0, 1)$ and $N > 1$ there are $\rho(\theta, N), \eta(\varepsilon) \in (0, 1)$ such that if $d_{\mathbf{G}}(\mathbf{x}, \mathbf{e}) < \rho(\theta, N)$ and $\mathbf{x} \notin \mathbf{N}_{\mathbf{G}}(\mathbf{H})$ then there exist $T > N$ and differentiable functions $\sigma(\mathbf{x}, s), \tau(\mathbf{x}, s) : [0, T] \to \mathbb{R}$, $\sigma(\mathbf{x}, 0) = \tau(\mathbf{x}, 0) = 0$ with $\sigma(\mathbf{x}, s)$ increasing in s such that

$$\begin{aligned} & d_{\mathbf{G}}(\mathbf{x}\mathbf{h}(\sigma(x, s)), \mathbf{h}(s)\mathbf{a}(\tau(x, s))) < C\theta T^{-1} \text{ for all } s \in [0, T] \\ & \max\{|\tau(\mathbf{x}, s)| : 0 \leq s \leq T\} = |\tau(\mathbf{x}, T)| = \theta \\ & |\tau(\mathbf{x}, s) - \tau(\mathbf{x}, T)| < \theta\varepsilon \text{ for all } s \in [(1 - \eta(\varepsilon))T, T], \end{aligned} \tag{*}$$

where $C > 0$ is a constant. Here $\mathbf{N}_{\mathbf{G}}(\mathbf{H})$ denotes the normalizer of $\mathbf{H}$ in $\mathbf{G} = SL(2, \mathbb{R})$ (it is generated by $\mathbf{A}$ and $\mathbf{H}$).

The first two relations in $(*)$ show, in particular, that for any $\mathbf{M} \subset \mathbf{G} - \mathbf{N}_{\mathbf{G}}(\mathbf{H})$ with $\mathbf{e} \in \overline{\mathbf{M}}$ the group generated by $\mathbf{N}_{\mathbf{G}}(\mathbf{H}) \cap \overline{\{\mathbf{h}_{-s}\mathbf{x}\mathbf{h}_t : \mathbf{x} \in \mathbf{M},\ s, t > 0\}}$ contains $\mathbf{A}$. (It also contains $\mathbf{H}$ by the H-property and hence $\mathbf{N}_{\mathbf{G}}(\mathbf{H})$.) A version of this fact for a more general case was later used by Margulis in [M1, Lemmas 5 and 8].

The H-property was generalized in [W1, the Ratner property] and property $(*)$ was generalized in [Ra8, Theorem 3.1], where it is called the R-property (see Section 3 for a description of the R-property). The latter property plays a crucial role in the proof of Theorem 1. (We showed in [Ra12] how to use property $(*)$ to prove Theorem 1 for $\mathbf{G} = SL(2,\mathbb{R})$.)

Theorem 1 allows one also to show that factors of Ad-unipotent actions on $(\mathbf{\Gamma}\backslash\mathbf{G}, \nu_{\mathbf{G}})$ have simple algebraic form. This was recently done by Witte [W4]. We showed earlier in [Ra3,4] that if S_t is a factor of the horocycle flow on $\mathbf{\Gamma}\backslash SL(2,\mathbb{R})$ then there is a lattice $\mathbf{\Gamma}'$ in $SL(2,\mathbb{R})$ such that $\mathbf{\Gamma} \subset \mathbf{\Gamma}'$ and S_t is isomorphic to the horocycle flow on $\mathbf{\Gamma}'\backslash SL(2,\mathbb{R})$. This implies that the number of nonisomorphic factors of the horocycle flow is finite and if $\mathbf{\Gamma}$ is maximal then there are no non-trivial factors.

It is a fact that there are uncountably many nonisomorphic ergodic joinings and factors of Ad-semisimple actions with positive entropy (this follows from Ornstein's theory of Bernoulli shifts). Also it was shown by Sinai and Bowen-Ruelle that the geodesic flow on $\mathbf{\Gamma}\backslash SL(2,\mathbb{R})$ possesses infinitely many ergodic invariant probability measures that are *not* algebraic. Also there are points x in $\mathbf{\Gamma}\backslash SL(2,\mathbb{R})$ for which closures of geodesic orbits are not manifolds. All these facts put Ad-semisimple actions in striking contrast with the rigid behavior of Ad-unipotent actions discussed in this section and given in Theorems 1, 3, 6, E2, and E3.

The rigidity theorem for the horocycle flows $h_t^{(i)}$ on $(M_i = \mathbf{\Gamma}_i\backslash\mathbf{G}_i, \nu_i)$, $\mathbf{G}_i = SL(2,\mathbb{R})$, $\nu_i = \nu_{\mathbf{G}_i}$, $i = 1,2$, says that if $h_t^{(1)}$ is isomorphic to $h_t^{(2)}$ then $\mathbf{\Gamma}_1$ is conjugate to $\mathbf{\Gamma}_2$. We ask: Can this "rigidity" be *destroyed* by time changes?

More specifically, let τ be a positive integrable function on M_1 with $\int_{M_1} \tau\, d\nu_1 = 1$. We say that h_t^τ is obtained from $h_t^{(1)}$ by the time change τ if $h_t^\tau(x) = h_{v(x,t)}^{(1)}(x)$ for all $x \in M_1$, $t \in \mathbb{R}$, where $v(x,t)$ is defined by $\int_0^{v(x,t)} \tau(x\mathbf{h}(s)ds) = t$. Then h_t^τ preserves the probability measure $\tau d\nu_1$ on M_1.

We ask: Is there a time change τ such that h_t^τ is isomorphic to $h_t^{(2)}$? If "yes" $h_t^{(1)}$ is called Kakutani equivalent to $h_t^{(2)}$.

Using the Feldman-Katok-Ornstein-Weiss theory of Kakutani equivalence (developed in the 1970s) we showed in [Ra0] that the answer to this question is affirmative. However, we also showed [Ra6] that even very mild smoothness conditions on τ cause the rigidity to persist. Namely, we say that τ is Hölder continuous in the direction of the rotation group

$$\mathbf{R} = \{\mathbf{r}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} : \theta \in [-\pi, \pi]\}$$

if $|\tau(x) - \tau(x\mathbf{r}(\theta))| \le C|\theta|^\alpha$ for some $C, \alpha > 0$ and all $x \in M_1$, $\theta \in [-\pi,\pi]$. We showed in [Ra6] that if h_t^τ is isomorphic to $h_t^{(2)}$ with τ being bounded, measurable, and Hölder continuous in the direction of $\mathbf{R}$, then $\mathbf{\Gamma}_1$ is conjugate to $\mathbf{\Gamma}_2$. Also all isomorphisms between h_t^τ and $h_t^{(2)}$ as well as factors and joinings of h_t^τ have an algebraic form [Ra7].

PROBLEM 1. Are Ad-unipotent ergodic flows on homogeneous spaces of a *general* Lie group $\mathbf{G}$ Kakutani equivalent? In particular, is the flow $h_t^{(1)} \times h_t^{(1)}$ acting on $(M_1 \times M_1, \nu_1 \times \nu_1)$ Kakutani equivalent to the flow $h_t^{(2)} \times h_t^{(2)}$ acting on $(M_2 \times M_2, \nu_2 \times \nu_2)$? (We showed in [Ra1] that $h_t \times h_t$ acting on $(M \times M, \nu \times \nu)$ is not Kakutani equivalent to h_t acting on (M, ν).)

PROBLEM 2. Do the rigidity properties discussed in this section hold for smoothly time-changed Ad-unipotent flows?

Theorem S1 above allows one to extend Theorems E2 and E3 and classify factors of Ad-unipotent flows in the $\mathbb{S}$-arithmetic setting discussed in Section 4. The latter flows represent measure preserving actions of the field $\mathbb{Q}_S$ (as an additive group) on $(\mathbf{\Gamma}\backslash\mathbf{G}_{\mathbb{S}}, \nu_{\mathbf{G}_{\mathbb{S}}})$, $s \in \mathbb{S}$, with $\mathbf{\Gamma}$ being a lattice in $\mathbf{G}_{\mathbb{S}}$ (see Section 4). It would be of interest to develop the ergodic theory of measure preserving actions of the p-adic field $\mathbb{Q}_p$ as an additive group and, in particular, to construct a theory of Kakutani equivalence for these actions. Applying such a theory to p-adic horocycle flows on $\Gamma\backslash SL(2, \mathbb{Q}_p)$ one can ask questions similar to those discussed in this section for the real case.

To conclude this section we mention that recently Starkov [St4] used Theorem 1 to give an affirmative answer to a question raised by Marcus in [Ma] (see also [M2]). Namely, he showed that if $\mathbf{\Gamma}$ is a lattice in a connected Lie group $\mathbf{G}$ and the action of a one-parameter subgroup $\mathbf{U} \subset \mathbf{G}$ on $(\mathbf{\Gamma}\backslash\mathbf{G}, \nu_{\mathbf{G}})$ is mixing, then it is mixing of all orders. Marcus [Ma] proved this result for semisimple $\mathbf{G}$ (see also [Mo1]). Starkov's argument exploits Marcus' result and a theorem of Witte [W2, Proposition 2.6] (cf. Corollary 1 in [St4]).

References

[AGH] L. Auslander, L. Green, and F. Hahn, *Flows on homogeneous spaces*, Ann. of Math. Stud. **53** (1963), Princeton Univ. Press, Princeton, NJ.

[BBo] F. Bien and A. Borel, *Sous-groupes épimorphiques de groupes algébriques linéaires I*, C.R. Acad. Sci. Paris **315** (1992).

[Bo] A. Borel, *Values of indefinite quadratic forms at integral points and flows on spaces of lattices*, Bull. Amer. Math. Soc. **32** (1995), 184–204.

[BoHC] A. Borel and Harish-Chandra, *Arithmetic subgroups of algebraic groups*, Ann. of Math. (2) **75** (1962), 485–535.

[BoPr] A. Borel and G. Prasad, *Values of isotropic quadratic forms at S-integral points*, Compositio Math. **83** (1992), 347–372.

[Bw] R. Bowen, *Weak mixing and unique ergodicity on homogeneous spaces*, Israel J. Math. **23** (1976), 267–273.

[D1] S. G. Dani, *Invariant measures and minimal sets of horospherical flows*, Invent. Math. **64** (1981), 357–385.

[D2] ______, *On orbits of unipotent flows on homogeneous spaces*, II, Ergodic Theory Dynamical Systems **6** (1986), 167–182.

[D3] ______, *Orbits of horospherical flows*, Duke Math. J. **53** (1986), 177–188.

[DM1] S. G. Dani and G. A. Margulis, *Values of quadratic forms at primitive integral points*, Invent. Math. **98** (1989), 405–425.

[DM2] ______, *Orbit closures of generic unipotent flows on homogeneous spaces of $SL(3, \mathbb{R})$*, Math. Ann. **286** (1990), 101–128.

[DM3] ______, *Values of quadratic forms at integral points: An elementary approach*, L'enseig. Math. **36** (1990), 143–174.

[DM4] ______, *Limit distributions of orbits of unipotent flows and values of quadratic forms*, Adv. in Sov. Math. **16** (1993), 91–137.

[DSm] S. G. Dani and J. Smillie, *Uniform distribution of horocycle orbits for Fuchsian groups*, Duke Math. J. **5** (1984), 185–194.

[DuRuSa] W. Duke, Z. Rudnick, and P. Sarnak, *Density of integer points on affine homogeneous varieties*, Duke Math. J. **71** (1993), 143–180.

[EPe] R. Ellis and W. Perrizo, *Unique ergodicity of flows on homogeneous spaces*, Israel J. Math. **29** (1978), 276–284.

[EsMc] A. Eskin and C. McMullen, *Mixing, counting and equidistribution in Lie groups*, Duke Math. J. **71** (1993), 181–209.

[EsMoS] A. Eskin, S. Mozes, and N. Shah, *Unipotent flows and counting lattice points on homogeneous varieties*, to appear in Ann. of Math.

[FO] J. Feldman and D. Ornstein, *Semirigidity of horocycle flows over compact surfaces of variable negative curvature*, Ergodic Theory Dynamical Systems **7** (1987), 49–72.

[Fl] L. Flaminio, *An extension of Ratner's rigidity theorem to n-dimensional hyperbolic space*, Ergodic Theory Dynamical Systems **7** (1987), 73–92.

[FlSp] L. Flaminio and R. Spatzier, *Geometrically finite groups, Patherson-Sullivan measures and Ratner's rigidity theorem*, Invent. Math. **99** (1990), 601–626.

[Fu1] H. Furstenberg, *Strict ergodicity and transformations of the torus*, Amer. J. Math. **83** (1961), 573–601.

[Fu2] ______, *The unique ergodicity of the horocycle flow*, in Recent Advances in Topological Dynamics, Lecture Notes in Math. **318**, Springer-Verlag, Berlin and New York (1972), 95–115.

[H] G.A. Hedlund, *Fuchsian groups and transitive horocycles*, Duke Math. J. **2** (1936), 530–542.

[L] E. Lesigne, *Théorèmes ergodiques pour une translation sur nilvariété*, Ergodic Theory Dynamical Systems **9** (1989), 115–126.

[Ma] B. Marcus, *The horocycle flow is mixing of all degrees*, Invent. Math. **46** (1978), 201–209.

[M1] G. A. Margulis, *Discrete subgroups and ergodic theory*, in Number Theory, Trace Formula and Discrete Groups, Symposium in honor of A. Selberg, Oslo 1987, Academic Press, New York (1989), 377–398.

[M2] ______, *Dynamical and ergodic properties of subgroup actions on homogeneous spaces with applications to number theory*, Proc. Internat. Congress Math. (1990), Kyoto, 193–215.

[MT1] G. A. Margulis and G. M. Tomanov, *Measure rigidity for algebraic groups over local fields*, C. R. Acad. Sci. Paris, t. 315, Série I (1992), 1221–1226.

[MT2] ______, *Invariant measures for actions of unipotent groups over local fields on homogeneous spaces*, Invent. Math. **116** (1994), 347–392.

[Mo1] S. Mozes, *Mixing of all orders of Lie groups actions*, Invent. Math. **107** (1992), 235–241.

[Mo2] ______, *Epimorphic subgroups and invariant measures*, Ergodic Theory Dynamical Systems **15** (1995), 1–4.

[MoS] S. Mozes and N. Shah, *On the space of ergodic invariant measures of unipotent flows*, Ergodic Theory Dynamical Systems **15** (1995), 149–159.

[P1] W. Parry, *Ergodic properties of affine transformations and flows on nilmanifolds*, Amer. J. Math. **91** (1969), 757–771.

[P2] ———, *Metric classification of ergodic nilflows and unipotent affines*, Amer. J. Math. **93** (1971), 819–828.

[Pr] G. Prasad, *Ratner's theorem in S-arithmetic setting*, in Workshop on Lie Groups, Ergodic Theory and Geometry, Math. Sci. Res. Inst. Publ. (1992), p. 53.

[R] M. S. Raghunathan, Discrete Subgroups of Lie Groups, Springer-Verlag, Berlin and New York, 1972.

[Ra0] M. Ratner, *Horocycle flows are loosely Bernoulli*, Israel J. Math. **31** (1978), 122–132.

[Ra1] ———, *The Cartesian square of the horocycle flow is not loosely Bernoulli*, Israel J. Math. **34** (1979), 72–96.

[Ra2] ———, *Rigidity of horocycle flows*, Ann. of Math. (2) **115** (1982), 597–614.

[Ra3] ———, *Factors of horocycle flows*, Ergodic Theory Dynamical Systems **2** (1982), 465–489.

[Ra4] ———, *Horocycle flows: Joinings and rigidity of products*, Ann. of Math. (2) **118** (1983), 277–313.

[Ra5] ———, *Ergodic theory in hyperbolic space*, Contemp. Math. **26** (1984), 302–334.

[Ra6] ———, *Rigidity of time changes for horocycle flows*, Acta Math. **156** (1986), 1–32.

[Ra7] ———, *Rigid reparametrizations and cohomology for horocycle flows*, Invent. Math. **88** (1987), 341–374.

[Ra8] ———, *Strict measure rigidity for unipotent subgroups of solvable groups*, Invent. Math. **101** (1990), 449–482.

[Ra9] ———, *On measure rigidity of unipotent subgroups of semisimple groups*, Acta Math. **165** (1990), 229–309.

[Ra10] ———, *On Raghunathan's measure conjecture*, Ann. of Math. (2) **134** (1991), 545–607.

[Ra11] ———, *Raghunathan's topological conjecture and distributions of unipotent flows*, Duke Math. J. **63** (1991), 235–280.

[Ra12] ———, *Raghunathan's conjectures for $SL(2,\mathbb{R})$*, Israel J. Math. **80** (1992), 1–31.

[Ra13] ———, *Raghunathan's conjectures for p-adic Lie groups*, International Mathematics Research Notices, no. 5 (1993), 141–146.

[Ra14] ———, *Invariant measures and orbit closures for unipotent actions on homogeneous spaces*, Geom. Functional Anal (GAFA) **4** (1994), 236–256.

[Ra15] ———, *Raghunathan's conjectures for cartesian products of real and p-adic Lie groups*, Duke Math. J. **77** (1995), 275–382.

[S1] N. Shah, *Uniformly distributed orbits of certain flows on homogeneous spaces*, Math. Ann. **289** (1991), 315–334.

[S2] ———, *Limit distributions of polynomial trajectories on homogeneous spaces*, Duke Math. J. **75** (1994), 711–732.

[St1] A. Starkov, *The reduction of the theory of homogeneous flows to the case of discrete isotropy subgroups*, Dokl. Akad. Nauk **301** (1988), 1328–1331 (Russian).

[St2] ———, *Solvable homogeneous flows*, Math. USSR-Sb. **62** (1989), 243–260.

[St3] ———, *Structure of orbits of homogeneous flows and Raghunathan's conjecture*, Russian Math: Surveys **45** (1990), 227–228.

[St4] ———, *On the mixing of all orders for homogeneous flows*, Dokl. Akad. Nauk **333** (1993), 442–445 (Russian).

[T] T. Tamagawa, *On discrete subgroups of p-adic algebraic groups*, in Arithmetic Algebraic Geometry, Shilling, Harper & Row, New York (1965), 11–17.

[V] W. Veech, *Unique ergodicity of horospherical flows*, Amer. J. Math. **99** (1977), 827–859.

[W1] D. Witte, *Rigidity of some translations on homogeneous spaces*, Invent. Math. **81** (1985), 1–27.

[W2] ———, *Zero entropy affine maps on homogeneous spaces*, Amer. J. Math. **109** (1987), 927–961.

[W3] ———, *Rigidity of horospherical foliations*, Ergodic Theory Dynamical Systems **9** (1989), 191–205.

[W4] ———, *Measurable quotients of unipotent translations on homogeneous spaces*, Trans. Amer. Math. Soc. **345** (1994), 577–594.